Note on the Zeros of the Spherical Harmonic $P_n^{-m}(\mu)$ *By* H. M. Macdonald. Received and read May 9th, 1901.

In a former communication* to the Society it was proved that the values of n for which $P_n^{-m}(\cos\theta)$ vanishes, where m is a real positive quantity, decrease as θ increases from 0 to π. The object of the following note is to point out some properties of the function depending on this.

It is known† that when θ is very near to π the values of n for which $P_n^{-m}(\cos\theta)$ vanishes differ from $m+k$, where k is a positive integer, by a very small quantity; when $\theta = \frac{\pi}{2}$ the values of n are $m+2k+1$, where k is a positive integer; and when θ is very small the values of n tend to increase indefinitely. Hence, supposing the family of curves $P_n^{-m}(\cos\theta) = 0$ to be drawn, $n\ (\equiv x)$ being the abscissa, and $\theta\ (\equiv y)$ the ordinate, any one of them (say the k-th) starts from a point not on the line $y = \pi$, but indefinitely near to the point $x = m+k$ on it, bends steadily towards the right, cuts the line $y = \frac{\pi}{2}$ at the point $x = m+2k+1$, and then approaches the line $y = 0$ asymptotically, there being no point of inflexion on the curve. The following properties of the zeros of $P_n^{-m}(z)$ considered as a function of z, where n and m are real positive quantities, are immediate consequences. The number of the real values of z lying between -1 and 1 for which $P_n^{-m}(z)$ vanishes is the greatest integer less than $n-m+1$; for this is the number of the curves cut by the line $x = n$. If the number of these zeros is even, $2s$, half of them are positive and the other half negative; for the extreme curve on the right cut by $x = n$ is the one which starts from a point near to the point $x = m+2s-1$ on the line $y = \pi$, and cuts the line $y = \frac{\pi}{2}$ where $x = m+4s-1$; further the line $x = n$ cuts $y = \frac{\pi}{2}$ between $x = m+2s-1$ and $x = m+2s$; so that s curves are crossed by

* *Proceedings*, Vol. xxxi., p. 277.
† *Loc. cit.*

$x = n$ above $y = \frac{\pi}{2}$ and s below. If the number of these zeros is odd, $2s+1$, there are s positive zeros and $s+1$ negative zeros, except when $n-m$ is an odd integer, in which case $z = 0$ is a zero, and there are in addition s positive and s negative zeros. For in this case the extreme curve on the right cut by $x = n$ is the one which starts near to $x = m+2s$ on $y = \pi$ and cuts $y = \frac{\pi}{2}$ where $x = m+4s+1$, and the straight line $x = n$ cuts $y = \frac{\pi}{2}$ between $x = m+2s$ and $x = m+2s+1$, except when $n-m = 2s+1$, in which case it cuts it at the point $x = m+2s+1$; so that, when $n-m \neq 2s+1$, $s+1$ curves are crossed by the line $x = n$ above $y = \frac{\pi}{2}$, and s below, and, when $n-m = 2s+1$, s curves are crossed by $x = n$ above $y = \frac{\pi}{2}$, s below, and one on it.

Thursday, June 13th, 1901.

Dr. HOBSON, F.R.S., President, in the Chair.

Twelve members present.

After the ballot had been taken, the President announced that the following gentlemen had been elected honorary members: — Prof. Ulisse Dini, of Pisa; Prof. Georg Cantor, of Halle; and Prof. David Hilbert, of Göttingen.

Mr. Arthur William Conway, B.A. Corpus Christi College, Oxford, was elected a member of the Society.

The following papers were communicated:—

The Theory of Cauchy's Principal Values (ii.), by Mr. G. H. Hardy.

On the general Form of Three Rational Cubes whose Sum is a Cube," by Prof. Steggall.

Invariants of Curves on the same Surface, in the neighbourhood of a common Tangent Line, by Mr. T. Stuart (communicated by Dr. J. Larmor).

Dr. Macaulay made two short impromptu communications, and Lt.-Col. Allan Cunningham made an impromptu communication about Euler's *Idoneal Numbers*. If I denote one of these numbers, they have the property that, if an odd number N be expressible in *only one way* in the form $N = mx^2 + ny^2$, wherein $mn = I$ and mx^2 is prime to ny^2, then N must be either a prime or the *square of a prime*. Euler gives a list of sixty-five idoneals of which the highest is 1848, and states that there are no more under 4000. Col. Cunningham has extended the search, and finds that there are no more under 50,000; this work has been verified by the Rev. J. Cullen.

The following presents were made to the Library :—

"Educational Times," June, 1901.

"Indian Engineering," Vol. xxix., Nos. 16–20, April 20–May 18, 1901.

Durán-Loriga, Juán J.—"Charles Hermite," 8vo pamphlet; Coruña, 1901.

Lemoine, E.—

"La géométrographie dans l'espace ou stéréométrographie" (*Comptes Rendus*); Paris, 1900.

"Suite de téorèmes et de résultats concernant la géométrie du triangle," 8vo pamphlet; Paris, 1900.

"Note sur deux nouvèles décompositions des nombres entiers," 8vo pamphlet; Paris, 1900.

"Comparaison géométrografique de diverses constructions d'un même problème," 8vo pamphlet; Paris, 1900.

"Géométrografie dans l'espace," 8vo pamphlet; Paris, 1900.

"Mémoires de la section mathématique de la Société des Naturalistes de la Nouvelle-Russie," Tome xix.; Odessa, 1899.

"Publications of the United States Naval Observatory," Series 2, Vol. i., 4to; Washington, 1900. "Sun, Moon, Planets, and Miscellaneous Stars," 1894–1899.

The following exchanges were received :—

"Supplemento al Periodico di Matematica," Anno iv., Fasc. 7, Maggio 1901; Livorno.

"Proceedings of the Royal Society," Vol. lxviii., Nos. 444, 445; 1901.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxv., Hefte 5, 6; Leipzig, 1901.

"Rendiconti del Circolo Matematico di Palermo," Tomo xv., Fasc. 1, 2; 1901.

"Bulletin de la Société Mathématique de France," Tome xxix., Fasc. 2; Paris, 1901.

"Bulletin of the American Mathematical Society," Series 2, Vol. vii., No. 8, May, 1901; New York.

"Bulletin des Sciences Mathématiques," Tome xxv., Mars 1901; Paris.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Série 3, Vol. VII., Fasc. 4, Aprile 1901; Napoli.

"Journal für die reine und angewandte Mathematik," Bd. CXXIII., Heft 3; Berlin, 1901.

"Annali di Matematica," Serie 3, Tomo v., Fasc. 3, 4; Milano, 1901.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. x., Fasc. 8–10; Roma, 1901.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang XLV., Hefte 3, 4; 1900.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," 1–22; 1901.

The Theory of Cauchy's Principal Values. (Second Paper: *The use of Principal Values in some of the Double Limit Problems of the Integral Calculus.*) *By* G. H. HARDY. Read and received June 13th, 1901.

Principal Values depending on a Parameter.

1. If $f(x, \alpha)$ is a function of the two variables x, α, which for certain values of α possesses a convergent integral from $x = a$ to $x = A$,

$$I(\alpha) = \int_a^A f(x, \alpha)\, dx$$

is a function of α defined for those values of α. We may suppose a, A independent of α; for, if they depended on α, we could make the substitution

$$x = a + (A - a)\, y,$$

and so obtain an integral with the constant limits 0, 1.

We suppose further that the values of α for which $I(\alpha)$ is defined are infinite in number, and form a *closed set* S; and that α_0 is a limiting point of the set. Then the general double limit problem of the integral calculus is: *To determine the relations between*

$$I(\alpha_0) = \int_a^A f(x, \alpha_0)\, dx$$

and the limits of indetermination of $I(\alpha)$ *for* $\alpha = \alpha_0$.

It is not difficult to show that we may without loss of generality