Some problems of „Partitio numerorum": II. Proof that every large number is the sum of at most 21 biquadrates.

By

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1. Introduction.

1.1. This memoir is essentially a sequel to one which we published recently in the Göttinger Nachrichten [1]). It could not in any case be intelligible to a reader unacquainted with our earlier memoir; and we shall therefore quote formulae from the latter without further explanation.

In the memoir referred to we laid the foundations of our new method for the solution of Waring's Problem, carrying our analysis just so far as was necessary for the proof of Hilbert's Theorem, the fundamental existence theorem for the numbers $g(k)$ and $G(k)$. Here our object is to find the best possible inequality for the particular number $G(4)$. A good deal of our analysis, however, is valid for a general k, and will be useful to us when we proceed to the corresponding general problem. It will be found that the special interest of the case $k = 4$ is quite sufficient to justify its consideration in a separate memoir.

1.2. It is known that

$$19 \leqq g(4) \leqq 37, \qquad 16 \leqq G(4) \leqq 37,$$

these inequalities, from left to right, being due to Waring, Wieferich, Kempner, and Wieferich respectively. For detailed references we may refer to the dissertations of Kempner [2]) and of Baer [3]). We need men-

[1]) G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio numerorum'; I: A new solution of Waring's Problem, Göttinger Nachrichten 1920, S. 33–54. We shall refer to this memoir as W. P.

[2]) A. J. Kempner, Über das Waringsche Problem und einige Verallgemeinerungen, Inaugural-Dissertation, Göttingen 1912.

[3]) W. S. Baer, Beiträge zum Waringschen Problem, Inaugural-Dissertation, Göttingen 1913.

tion only that the deepest result, viz. $g(4) \leqq 37$, was obtained in 1909 by Wieferich, whose analysis is a refinement upon that by which Landau, in 1907, had proved that $g(4) \leqq 38$. Here we shall prove nothing concerning $g(4)$; but we shall improve the upper bound for $G(4)$ very notably, by proving

Theorem A: $G(4) \leqq 21$.

2. A sharpening of our earlier analysis.

2. 1. In § 9. 2. of W. P. we proved that, assuming always

$$s \geqq 2K + 1 = 2^k + 1,$$

$$(2.11) \qquad \begin{aligned} r_{k,s}(n) &= C n^{s\alpha-1} S + O(n^{s\alpha\varkappa+\varepsilon}) + O(n^{s\alpha+\alpha\varkappa-\alpha-1+\varepsilon}) \\ &= \varrho_{k,s}(n) + O(n^{s\alpha\varkappa+\varepsilon}) + O(n^{s\alpha+\alpha\varkappa-\alpha-1+\varepsilon}), \end{aligned}$$

where

$$S = \sum \left(\frac{S_{p,q}}{q}\right)^s e_q(-np).$$

It will be necessary now to replace the term $O(n^{s\alpha\varkappa+\varepsilon})$ by a term of lower order[4]).

2. 2. It will be found, on an examination of the analysis of W. P., that the critical error term $O(n^{s\alpha\varkappa+\varepsilon})$ arises in two places only. All other errors are of lower order than that of the dominant factor $n^{s\alpha-1}$, either independently of the value of s, or at any rate when $s \geqq 2K+1$. The two critical errors arise as follows.

In the first place we have

$$S_2 = \sum \frac{1}{2\pi i} \int_{\mathfrak{m}} \frac{(f(x))^s}{x^{n+1}} \, dx = O(n^{s\alpha\varkappa+\varepsilon}),$$

where $\mathfrak{m}$ is a typical minor arc of the Farey dissection.

Secondly, when we consider the corresponding sum connected with the major arcs, we are confronted by a sum $\sum \sigma_{p,q}$, where

$$\sigma_{p,q} = \int_{\mathfrak{M}} \frac{|\Phi|^s}{|x|^{n+1}} \, |dx|$$

and $\mathfrak{M}$ is a typical major arc of the dissection, and we write

$$\sum \sigma_{p,q} = O(n^{s\alpha\varkappa+\varepsilon}).$$

It will be observed that these two errors arise in exactly the same way. The upper bounds are obtained by substituting in the integrals the crude approximations, $f = O(n^{\alpha\varkappa+\varepsilon})$ on a minor arc and $\Phi = O(n^{\alpha\varkappa+\varepsilon})$

[4]) The formula (2. 11) would lead only to $G(4) \leqq 33$, in itself a new result.

on a major arc. Here we refine on our previous argument by the use of a single new idea. This idea consists in an appropriate use of a known result, viz. that the number of positive integral solutions of the equation $x^k + y^k = n$ is $O(n^\varepsilon)$ for every $k > 1$ [5]), or, as we may express it in our notation,

$$r_{k,2}(n) = O(n^\varepsilon).$$

2.3. We have

$$\sum_{\nu=0}^{n} r_{k,2}(\nu) = O\left(\iint\limits_{|x|^k+|y|^k \leq n} dx\,dy\right) = O(n^{2a}).$$

Hence

$$\sum \int\limits_{\mathfrak{m}} |f(x)|^4 |dx| \leqq \int\limits_{\Gamma} |f(x)|^4 |dx| \leqq \int\limits_{0}^{2\pi} |f(Re^{i\theta})|^4 d\theta$$

$$= O\left(\sum (r_{k,2}(\nu))^2 R^{2\nu}\right).$$

Now

$$\sum_{0}^{n} (r_{k,2}(\nu))^2 = O\left(\sum_{0}^{n} r_{k,2}(\nu) \times \underset{\nu \leq n}{\mathrm{Max}}\, r_{k,2}(\nu)\right) \quad O(n^{2a+\varepsilon});$$

and so, since $R = 1 - \frac{1}{n}$,

$$\sum \int\limits_{\mathfrak{m}} |f(x)|^4 |dx| = O(n^{2a+\varepsilon}).$$

Hence

$$S_2 = O\left(\sum \int\limits_{\mathfrak{m}} |f(x)|^4 |f(x)|^{s-4} |dx|\right)$$

$$= O\left(n^{(s-4)a\varkappa+\varepsilon} \sum \int\limits_{\mathfrak{m}} |f(x)|^4 |dx|\right) = O(n^{(s-4)a\varkappa+2a+\varepsilon}).$$

Again, we have

$$\sum \int\limits_{\mathfrak{M}} |\Phi|^4 |dx| = \sum \int\limits_{\mathfrak{M}} |f-\varphi|^4 |dx| = O\left(\sum \int\limits_{\mathfrak{M}} |f|^4 |dx|\right) + O\left(\sum \int\limits_{\mathfrak{M}} |\varphi|^4 |dx|\right)$$

$$= O(n^{2a+\varepsilon}) + O\left(\sum \int\limits_{\mathfrak{M}} |\varphi|^4 |dx|\right),$$

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[5]) For a formal proof of this result see D. Cauer, Neue Anwendungen der Pfeifferschen Methode zur Abschätzung zahlentheoretischer Funktionen, Inaugural-Dissertation, Göttingen 1914, S. 38. For $k = 2$ (when the result includes *a fortiori* the corresponding results for 4, 6, ...) see E. Landau, Über die Anzahl der Gitterpunkte in gewissen Bereichen, Göttinger Nachrichten, 1912, S. 750.

$$\sum \int_{\mathfrak{M}} |\varphi|^4 |dx| = O\Big(\sum_{q=1}^{n^a} \sum_p \int_{-\theta_0}^{\theta_0} \Big|\frac{S_{p,q}}{q}\Big|^4 |\nu - i\theta|^{-4a} d\theta\Big)$$

$$= O\Big(\sum_q \sum_p q^{-4(1-\varkappa)+\varepsilon} \nu^{-4a} \theta_0\Big) = O\Big(\sum_q q \cdot q^{-4(1-\varkappa)+\varepsilon} \cdot n^{4a} \cdot q^{-1} n^{a-1}\Big)$$

$$= \begin{cases} O\Big(n^{5a-1+\varepsilon} \sum_q 1\Big) = O(n^{6a-1+\varepsilon}) = O(n^{2a+\varepsilon}) & (k \geqq 4) \\ O\Big(n^{5a-1} \sum_q q^{-1+\varepsilon}\Big) = O(n^{5a-1+\varepsilon}) = O(n^{2a+\varepsilon}) & (k = 3). \end{cases}$$

Thus

$$\sum \int_{\mathfrak{M}} |\Phi|^4 |dx| = O(n^{2a+\varepsilon}) \qquad (k > 2),$$

$$\sum \sigma_{p,q} = \sum \int_{\mathfrak{M}} \frac{|\Phi|^s}{|x|^{n+1}} |dx| = O\Big(\sum \int_{\mathfrak{M}} |\Phi|^s |dx|\Big)$$

$$= O\Big(\sum \int_{\mathfrak{M}} |\Phi|^{s-4} |\Phi|^4 |dx|\Big)$$

$$= O\Big(n^{(s-4)a\varkappa+\varepsilon} \sum \int_{\mathfrak{M}} |\Phi|^4 |dx|\Big) = O(n^{(s-4)a\varkappa+2a+\varepsilon}).$$

2. 4. The argument of W. P. showed that

$$r_{k,s}(n) = C n^{sa-1} S + O(n^\lambda),$$

where $\lambda < sa - 1$ if $sa\varkappa < sa - 1$, i. e. if $s > kK = k 2^{k-1}$. It is now clear that this result holds if only

$$(s-4)a\varkappa + 2a < sa - 1,$$

i. e. if

$$s > (k-2)K + 4.$$

For $k = 4$ these inequalities reduce to $s > 32$ and $s > 20$ respectively, so that the improvement is very substantial. And if only we can establish the existence of a positive constant σ such that

$$|S| > \sigma$$

when $s \geqq 21$, we shall have proved not only Theorem A but the more precise theorem

Theorem B: $r_{4,s}(n) \sim C n^{\frac{1}{4}s-1} S \qquad (s \geqq 21)$.

The proof of this theorem is in fact reduced to a discussion of the singular series S.

3. Factorization of the singular series.

3.1. The discussion of the singular series is greatly simplified by the following fundamental lemma.

Lemma 1. *If*

$$A_q = \sum_p \left(\frac{S_{p,q}}{q}\right)^s e_q(-np),$$

so that

$$S = 1 + A_2 + A_3 + A_4 + \ldots = \sum A_q$$

if $s \geqq 2K+1$, *and if* $(q, q') = 1$, *then*

$$A_{qq'} = A_q A_{q'};$$

and

$$S = \chi_2 \chi_3 \chi_5 \ldots = \Pi \chi_\pi,$$

where π *is prime*[6]) *and*

$$\chi_\pi = 1 + A_\pi + A_{\pi^2} + A_{\pi^3} + \ldots$$

We have

$$S_{pq'^{k-1},q} S_{pq^{k-1},q'} = \sum_h e\left(\frac{h^k p q'^{k-1}}{q}\right) \sum_{h'} e\left(\frac{h'^k p q^{k-1}}{q'}\right),$$

where h and h' describe complete systems of residues to moduli q and q'. But

$$e\left(\frac{h^k p q'^{k-1}}{q} + \frac{h'^k p q^{k-1}}{q'}\right) = e\left(\frac{\mathfrak{h}^k p}{q q'}\right),$$

where $\mathfrak{h} = hq' + h'q$; and, since $(q, q') = 1$, $\mathfrak{h}$ describes a complete system of residues to modulus qq'. Hence

$$(3.11) \qquad S_{p,qq'} = S_{pq'^{k-1},q} S_{pq^{k-1},q'}.$$

Next we observe that, if p describes a complete system of residues *prime* to modulus q, and p' a similar system for modulus q', then $\mathfrak{p} = pq' + p'q$ describes a similar system for modulus qq'. Also

$$S_{\mathfrak{p}q'^{k-1},q} = \sum_h e\left(\frac{(pq'+p'q)\, q'^{k-1} h^k}{q}\right) = \sum_h e\left(\frac{p\,(q'h)^k}{q}\right) = \sum_h e\left(\frac{ph^k}{q}\right) = S_{p,q}.$$

Hence

$$\begin{aligned}(qq')^s A_q A_{q'} &= \sum_{p,p'} (S_{p,q})^s (S_{p',q'})^s e\left(-n\left(\frac{p}{q} + \frac{p'}{q'}\right)\right)\\ &= \sum_{\mathfrak{p}} (S_{\mathfrak{p}q'^{k-1},q})^s (S_{\mathfrak{p}q^{k-1},q'})^s e\left(-\frac{n\mathfrak{p}}{qq'}\right)\\ &= \sum_{\mathfrak{p}} (S_{\mathfrak{p},qq'})^s e_{qq'}(-n\mathfrak{p}) = (qq')^s A_{qq'};\end{aligned}$$

which proves the lemma.

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[6]) The symbol π is used in this sense down to the end of 5.2, after which it is used in the ordinary sense.

4. Rules for the calculation of A_{π^ν}.

4.1. Lemma 1 is true for any value of k. The lemmas which follow are also true generally, provided only that k is not divisible by π. Thus when $k=4$ they hold for $\pi > 2$.

The sum A_{π^ν} involves the argument n, and we might write $A_{\pi^\nu} = A_{\pi^\nu}(n)$. When, as will sometimes happen, n is replaced by another argument, this argument will be shown explicitly.

Lemma 2. *If* $(\pi, k) = 1$, $\alpha > 0$, $0 < \mu < k$, *then*

$$A_{\pi^{\alpha k+\mu}} = 0$$

or

$$A_{\pi^{\alpha k+\mu}} = \pi^{\alpha(k-s)} A_{\pi^\mu}\left(\frac{n}{\pi^{\alpha k}}\right),$$

according as n is not or is a multiple of $\pi^{\alpha k}$.

(1) We have first

$$S_{p,\pi^{\alpha k+\mu}} = \sum_h e\left(\frac{h^k p}{\pi^{\alpha k+\mu}}\right).$$

We write

$$h = \pi^{\alpha k+\mu-1} z + h' \qquad (0 \leqq z < \pi,\ 0 \leqq h' < \pi^{\alpha k+\mu-1}),$$

and we obtain

$$S_{p,\pi^{\alpha k+\mu}} = \sum_{h'} \sum_z e\left(\frac{p h'^k}{\pi^{\alpha k+\mu}} + \frac{k p h'^{k-1} z}{\pi}\right).$$

The sum with respect to z vanishes unless h' is divisible by π, i. e. $h' = \pi h_1$, where $0 \leqq h_1 < \pi^{\alpha k+\mu-2}$; in this case the sum is π. Observing that this range of variation of h_1 is, to modulus $\pi^{(\alpha-1)k+\mu}$, equivalent to π^{k-2} descriptions of the range $0 \leqq h_1 < \pi^{(\alpha-1)k+\mu}$, we obtain

$$S_{p,\pi^{\alpha k+\mu}} = \pi \cdot \pi^{k-2} \sum_{h_1} e\left(\frac{p h_1^k}{\pi^{(\alpha-1)k+\mu}}\right) = \pi^{k-1} S_{p,\pi^{(\alpha-1)k+\mu}}.$$

It should be observed that the preceding argument is valid even when $\mu = 0$. We obtain in fact

$$S_{p,\pi^{\alpha k}} = \pi^{\alpha(k-1)} \qquad (\alpha > 0) \tag{4.11}$$

and otherwise

$$S_{p,\pi^{\alpha k+\mu}} = \pi^{\alpha(k-1)} S_{p,\pi^\mu}. \tag{4.12}$$

(2) We have now

$$A_{\pi^{\alpha k+\mu}} = \sum_p \left(\frac{S_{p,\pi^{\alpha k+\mu}}}{\pi^{\alpha k+\mu}}\right)^s e\left(-\frac{np}{\pi^{\alpha k+\mu}}\right).$$

We write

$$p = \pi^{\mu} z + p' \qquad (\mu > 0),$$

where $0 \leqq z < \pi^{\alpha k}$ and p' is less than π^{μ} and not divisible by π. We have then, by (4.12),

$$S_{p, \pi^{\alpha k+\mu}} = \pi^{\alpha(k-1)} S_{p, \pi^{\mu}} = \pi^{\alpha(k-1)} S_{p', \pi^{\mu}};$$

and

$$A_{\pi^{\alpha k+\mu}} = \pi^{-\alpha s} \sum_{p'} \sum_{z} \left(\frac{S_{p', \pi^{\mu}}}{\pi^{\mu}}\right)^{s} e\left(-\frac{nz}{\pi^{\alpha k}}, -\frac{np'}{\pi^{\alpha k+\mu}}\right).$$

The sum with respect to z vanishes unless n is divisible by $\pi^{\alpha k}$. If however $n = \pi^{\alpha k} \nu$, where ν is an integer, we have

$$A_{\pi^{\alpha k+\mu}} = \pi^{\alpha(k-s)} \sum_{p'} \left(\frac{S_{p', \pi^{\mu}}}{\pi^{\mu}}\right)^{s} e\left(-\frac{\nu p'}{\pi^{\mu}}\right) = \pi^{\alpha(k-s)} A_{\pi^{\mu}}\left(\frac{n}{\pi^{\alpha k}}\right).$$

4.2. Lemma 3. *If* $(\pi, k) = 1$, $\alpha > 0$,

$$A_{\pi^{\alpha k}} = 0 \qquad (n \not\equiv 0 \pmod{\pi^{\alpha k-1}}),$$
$$A_{\pi^{\alpha k}} = -\pi^{\alpha(k-s)-1} \qquad (n \equiv 0 \pmod{\pi^{\alpha k-1}}, n \not\equiv 0 \pmod{\pi^{\alpha k}}),$$
$$A_{\pi^{\alpha k}} = (\pi - 1)\pi^{\alpha(k-s)-1} \qquad (n \equiv 0 \pmod{\pi^{\alpha k}}).$$

By (4.11), we have

$$S_{p, \pi^{\alpha k}} = \pi^{\alpha(k-1)}, \qquad A_{\pi^{\alpha k}} = \pi^{-\alpha s} \sum_{p} e\left(-\frac{np}{\pi^{\alpha k}}\right).$$

Writing

$$p = \pi z + p' \qquad (0 \leqq z < \pi^{\alpha k-1},\ 0 < p' < \pi),$$

we obtain

$$A_{\pi^{\alpha k}} = \pi^{-\alpha s} \sum_{p'} \sum_{z} e\left(-\frac{nz}{\pi^{\alpha k-1}} - \frac{np'}{\pi^{\alpha k}}\right).$$

The sum with respect to z is zero unless n is a multiple of $\pi^{\alpha k-1}$. If $n = \pi^{\alpha k-1}\nu$, we have

$$A_{\pi^{\alpha k}} = \pi^{\alpha(k-s)-1} \sum_{p'} e\left(-\frac{\nu p'}{\pi}\right).$$

The last sum is -1 or $\pi - 1$, according as ν is not or is divisible by π. This proves the lemma.

4.3. Lemma 4. *If* $(\pi, k) = 1$, $1 < \mu < k$,

$$A_{\pi^{\mu}} = 0 \qquad (n \not\equiv 0 \pmod{\pi^{\mu-1}}),$$
$$A_{\pi^{\mu}} = -\pi^{\mu-s-1} \qquad (n \equiv 0 \pmod{\pi^{\mu-1}},\ n \not\equiv 0 \pmod{\pi^{\mu}}),$$
$$A_{\pi^{\mu}} = (\pi - 1)\pi^{\mu-s-1} \qquad (n \equiv 0 \pmod{\pi^{\mu}}).$$

(1) In the equation

$$S_{p, \pi^{\mu}} = \sum_{h} e\left(\frac{ph^{k}}{\pi^{\mu}}\right)$$

we write

$$h = \pi^{\mu-1} z + h' \quad (0 \leqq z < \pi,\ 0 \leqq h' < \pi^{\mu-1});$$

and we obtain

$$S_{p,\pi^\mu} = \sum_{h'} \sum_{z} e\left(\frac{p h'^k}{\pi^\mu} + \frac{k p h'^{k-1} z}{\pi}\right).$$

The sum with respect to z vanishes unless $h' \equiv 0 \pmod{\pi}$, or unless $h' = \pi h_1$, where $0 \leqq h_1 < \pi^{\mu-2}$. In this case the exponential is unity (since $\mu < k$), and we obtain

$$S_{p,\pi^\mu} = \pi^{\mu-1}.$$

(2) We have thus

$$A_{\pi^\mu} = \pi^{-s} \sum_{p} e\left(-\frac{np}{\pi^\mu}\right).$$

Writing

$$p = \pi z + p' \quad (0 \leqq z < \pi^{\mu-1},\ 0 < p' < \pi)$$

we obtain

$$A_{\pi^\mu} = \pi^{-s} \sum_{p'} \sum_{z} e\left(-\frac{np'}{\pi^\mu} - \frac{nz}{\pi^{\mu-1}}\right),$$

and the sum with respect to z vanishes unless $n = \pi^{\mu-1}\nu$, where ν is an integer. In this case

$$A_{\pi^\mu} = \pi^{\mu-s-1} \sum_{p'} e\left(-\frac{\nu p'}{\pi}\right),$$

and the sum is -1 or $\pi - 1$, according as ν is not or is divisible by π.

5. The form of χ_π $(k = 4, \pi > 2)$.

5.1. We now suppose $k = 4$, so that all the results of § 4 hold for $\pi > 2$. Taking first the case $\mu = 0$, we have

$$|A_{\pi^{\alpha k}}| \leqq \pi^{\alpha(k-s)},$$

by Lemma 3.

Next, if $\mu = 1$, we have $|A_\pi| < \pi$ and so

$$|A_{\pi^{\alpha k+1}}| < \pi^{\alpha(k-s)+1},$$

by Lemma 2.

Finally, if $1 < \mu < k$, we have

$$|A_{\pi^\mu}| < \pi^{\mu-s},$$

by Lemma 4, and

$$|A_{\pi^{\alpha k+\mu}}| < \pi^{\alpha(k-s)+\mu-s},$$

by Lemma 2.

Thus the terms of χ_π may be exhibited in the form[7])

$$\begin{aligned} &1 + A_\pi + [\pi^{2-s}] + [\pi^{3-s}] + \dots + [\pi^{k-1-s}] \\ &+ \pi^{k-s}([1] + [\pi] + [\pi^{2-s}] + [\pi^{3-s}] + \dots + [\pi^{k-1-s}]) \\ &+ \pi^{2(k-s)}([1] + [\pi] + [\pi^{2-s}] + [\pi^{3-s}] + \dots + [\pi^{k-1-s}]) \\ &+ \dots \dots \dots \dots \dots \dots \dots \dots \end{aligned}$$

where $[x]$ denotes a number whose modulus is less than x. Hence (provided only $s > k$) we have

$$\chi_\pi = 1 + A_\pi + B_\pi,$$

where

$$|B_\pi| < \frac{\pi^{k-s} - \pi^{2-s}}{\pi - 1} + \frac{\pi^{k-s}}{1-\pi^{k-s}}\left(1 + \pi + \frac{\pi^{k-s} - \pi^{2-s}}{\pi - 1}\right).$$

Taking now $k = 4$, $s > 20$, we have

$$|B_\pi| < \pi^{-17} + 2\pi^{-17}(1 + \pi + \pi^{-17}) < 7\pi^{-16} < \pi^{-14},$$

(5.11) $$\chi_\pi = 1 + A_\pi + [\pi^{-14}].$$

5.2. When we come to consider A_π, it is necessary to distinguish different cases.

Suppose first that π is of the form $4m + 3$. Then the residues of h^4 to modulus π are the same as those of h^2, and $S_{p,\pi}$ reduces to an ordinary Gaussian sum. Thus $|S_{p,\pi}| \leqq \sqrt{\pi}$ and

$$|A_\pi| < \pi^{1-\frac{1}{2}s} < \pi^{-9},$$

$$\chi_\pi = 1 + [\pi^{-9}] + [\pi^{-14}] = 1 + [\pi^{-8}] \qquad (\pi = 4m + 3).$$

Next, suppose π of the form $4m + 1$. Then[8])

$$|S_{p,\pi}| < 3\sqrt{\pi}$$

and

$$|A_\pi| < \pi\left(\frac{3}{\sqrt{\pi}}\right)^s.$$

If $\pi = 17$, $|A_{17}| < 17\left(\frac{3}{4}\right)^{21} < \frac{1}{10}$,

(5.21) $$\chi_{17} = 1 + [\tfrac{1}{10}] + [17^{-14}] = 1 + [\tfrac{1}{9}].$$

If $\pi \geqq 29 > 27$, $3^{21} < \pi^7$, $\left(\frac{3}{\sqrt{\pi}}\right)^{21} < \pi^{-3\cdot 5}$, $|A_\pi| < \pi^{-2\cdot 5}$,

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[7]) It should be observed that, owing to the vanishing of $A_{\pi^{ak}}$ and $A_{\pi^{ak+\mu}}$ when n does not satisfy certain congruence conditions, χ_π is in all cases a *finite* series; but this is irrelevant for our argument.

[8]) See H. Weber, Lehrbuch der Algebra, Bd. 1, S. 584. In Weber's notation, $S_{p,\pi}$ is one of the numbers

$$\zeta = 4\eta + 1 = \sqrt{n} + (i, \eta) + (-i, \eta).$$

and

(5.22) $\chi_\pi - 1 + [\pi^{-2\cdot 5}] + [\pi^{-14}] = 1 + [\pi^{-2}] \quad (\pi = 4m + 1 \geqq 29).$

From Lemma 1, (5.11), (5.21), and (5.22), it follows that

$$S = \chi_2 \chi_5 \chi_{13} \left(1 + [\tfrac{1}{9}]\right) \prod_{\pi = 4m+1 \geqq 29} (1 + [\pi^{-2}]) \prod_{\pi = 4m+3} (1 + [\pi^{-8}]).$$

Thus *in order to establish our conclusion when* $s = 21$, *it is only necessary to show that*

$$|\chi_2| > \sigma > 0, \quad |\chi_5| > \sigma, \quad |\chi_{13}| > \sigma.$$

5.3. We find by direct calculation that [9])

$$S_{1,5} = 1 + 4e^{\frac{2\pi i}{5}}, \quad S_{2,5} = 1 + 4e^{\frac{4\pi i}{5}}, \quad \ldots, \quad \ldots,$$

$$|S_{p,5}| \leqq \sqrt{17 + 8\cos\frac{2\pi}{5}} = \sqrt{15 + 2\sqrt{5}}.$$

It is however (as we have to consider the case of 13 also) more convenient to proceed as follows. The numbers $S_{p,5}$ are the roots of the equation [10])

$$(\zeta^2 + 15)^2 = 20(\zeta - 1)^2,$$

from which

$$\zeta = \pm\sqrt{5} \pm i\sqrt{10 \pm 2\sqrt{5}},$$

$$|\zeta|^2 \leqq 15 + 2\sqrt{5} < 19{\cdot}473,$$

$$|S_{p,5}| = |\zeta| < 4{\cdot}413,$$

$$|A_5| < 4({\cdot}8826)^{21} < {\cdot}291,$$

$$\chi_5 = 1 + [{\cdot}291] + [5^{-14}] = 1 + [{\cdot}3],$$

$$|\chi_5| > {\cdot}7 = \sigma.$$

5.4. Similarly the various values of $S_{p,13}$ are the roots of

$$(\zeta^2 + 39)^2 = 52(\zeta - 3)^2,$$

from which

$$\zeta = \pm\sqrt{13} + i\sqrt{26 + 6\sqrt{13}},$$

$$|\zeta|^2 \leqq 39 + 6\sqrt{13} < 60{\cdot}7,$$

$$|S_{p,13}| = |\zeta| < 7{\cdot}8,$$

$$|A_{13}| < 12({\cdot}6)^{21} < {\cdot}002,$$

so that

$$|\chi_{13}| > \sigma.$$

[9]) From this point onwards π is used in the ordinary sense.

[10]) Weber, l. c., p. 584.

The proof of Theorems A *and* B *is thus reduced to a proof that* $|\chi_2| > \sigma$.

6. Discussion of χ_2.

6. 1. The arguments of § 3 fail when $\pi = 2$, and it is necessary to go back to the definitions of $A_2, A_4, \ldots$ The first step is to calculate the sums $S_{p,2^\nu}$. We find by direct calculation that

$$S_{p,2} = 0,\ S_{p,4} = 2(1+e^{\frac{1}{2}p\pi i}),\ S_{p,8} = 4(1+e^{\frac{1}{4}p\pi i}),\ S_{p,16} = 8(1+e^{\frac{1}{8}p\pi i}).$$

If $\nu \geqq 5$,

$$S_{p,2^\nu} = \sum_h e\left(\frac{ph^4}{2^\nu}\right) \qquad (0 \leqq h < 2^\nu).$$

Writing

$$h = 2^{\nu-3}z + h' \quad (0 \leqq z < 8,\ 0 \leqq h' < 2^{\nu-3}),$$

we obtain

$$S_{p,2^\nu} = \sum_{h'}\sum_z e\left(\frac{ph'^4}{2^\nu} + \frac{ph'^3 z}{2}\right);$$

and the sum with respect to z is zero unless h' is even. Supposing $h' = 2h_1$, so that $0 \leqq h_1 < 2^{\nu-4}$, we obtain

$$S_{p,2^\nu} = 8\sum_{h_1} e\left(\frac{ph_1^4}{2^{\nu-4}}\right) = 8S_{p,2^{\nu-4}}.$$

Thus

$$S_{p,2^{4\alpha+\mu}} = 2^{3\alpha}S_{p,2^\mu} \qquad (\alpha > 0,\ 0 < \mu \leqq 4).$$

6. 2. Observing that $A_2 = 0$, we write

$$\begin{aligned}\chi_2 &= 1 + (A_4 + A_8 + A_{16}) + (A_{64} + A_{128} + A_{256}) + \ldots\\ &= 1 + (A_4 + A_8 + A_{16}) + B.\end{aligned}$$

For $\alpha > 0,\ 2 \leqq \mu \leqq 4$, we have

$$|A_{2^{4\alpha+\mu}}| = \left|\sum_p \left(\frac{S_{p,2^\mu}}{2^{\alpha+\mu}}\right)^s e\left(-\frac{np}{2^{4\alpha+\mu}}\right)\right| < 2^{4\alpha+\mu}(2^{-\alpha})^s = 2^{\mu-(s-4)\alpha},$$

$$|A_{2^{4\alpha+2}} + A_{2^{4\alpha+3}} + A_{2^{4\alpha+4}}| < 28\cdot 2^{-17\alpha},$$

$$|B| < 28\,\frac{2^{-17}}{1-2^{-17}} < 2^{-12} < \cdot 00025,$$

$$\chi_2 = 1 + A_4 + A_8 + A_{16} + [\cdot 00025].$$

6. 3. Of the terms A_4, A_8, A_{16} the most important is the last. We have

$$A_{16} = \sum_{p=1,3,\ldots,15}\left(\cos\frac{p\pi}{16}\right)^{21}\exp\left(\frac{21\,p\pi i}{16} - \frac{np\pi i}{16}\right) = \mathfrak{A}_1 + \mathfrak{A}_3 + \mathfrak{A}_5 + \mathfrak{A}_7,$$

where $\mathfrak{A}_1$ is given by $p=1, 15$, $\mathfrak{A}_3$ by $p=3, 13$, and so on. And

$$\left(\cos\frac{\pi}{16}\right)^{21} = -\left(\cos\frac{15\pi}{16}\right)^{21} = \cdot 665\,350 + [3\cdot 10^{-6}],$$

$$\mathfrak{A}_1 = -(1\cdot 3307 + [10^{-4}])\cos\left(\frac{5\pi}{16} - \frac{n\pi}{8}\right);$$

$$\left(\cos\frac{3\pi}{16}\right)^{21} = -\left(\cos\frac{13\pi}{16}\right)^{21} = \cdot 020\,736 + [10^{-6}],$$

$$\mathfrak{A}_3 = (\cdot 0415 + [10^{-4}])\cos\left(\frac{\pi}{16} + \frac{3n\pi}{8}\right);$$

$$\left(\cos\frac{5\pi}{16}\right)^{21} = -\left(\cos\frac{11\pi}{16}\right)^{21} = [5\cdot 10^{-6}],$$

and so also for $\left(\cos\frac{7\pi}{16}\right)^{21}$. Thus

$$\mathfrak{A}_5 + \mathfrak{A}_7 = [2\cdot 10^{-5}].$$

Similarly we may write

$$A_8 = \sum_{1, 3, 5, 7}\left(\cos\frac{p\pi}{8}\right)^{21}\exp\left(\frac{21p\pi i}{8} - \frac{np\pi i}{4}\right) = \mathfrak{A}_1' + \mathfrak{A}_3',$$

where $\mathfrak{A}_1'$ is given by $p=1, 7$, and so on: and

$$\left(\cos\frac{\pi}{8}\right)^{21} = -\left(\cos\frac{3\pi}{8}\right)^{21} = \cdot 189\,636 + [10^{-6}],$$

$$\mathfrak{A}_1' = -(\cdot 3793 + [10^{-4}])\cos\left(\frac{3\pi}{8} + \frac{n\pi}{4}\right);$$

$$\left(\cos\frac{3\pi}{8}\right)^{21} = -\left(\cos\frac{5\pi}{8}\right)^{21} = [10^{-8}],$$

$$\mathfrak{A}_3' = [10^{-7}].$$

Finally

$$A_4 = \sum_{1, 3}\left(\cos\frac{p\pi}{4}\right)^{21}\exp\left(\frac{21p\pi i}{4} - \frac{n\pi i}{2}\right) = [2^{-9\cdot 5}] = [\cdot 0014].$$

Collecting our results, we may write

$$\chi_2 = 1 - 1\cdot 3307\cos\left(\frac{5\pi}{16} - \frac{n\pi}{8}\right) + \cdot 0415\cos\left(\frac{\pi}{16} + \frac{3n\pi}{8}\right)$$
$$- \cdot 3793\cos\left(\frac{3\pi}{8} + \frac{n\pi}{4}\right) + 3[\cdot 0001] + [\cdot 0017],$$

and the total possible error is $[\cdot 002]$.

6. 4. We have now to verify that the sum of the first four terms of χ_2 is in all cases greater than $\cdot 002$. It is easy to see that the least favourable cases are those in which $\cos\left(\frac{5\pi}{16} - \frac{n\pi}{8}\right)$ has its greatest possible value, viz. $\cos\frac{\pi}{16}$. This happens when $n\equiv 2, 3 \pmod{16}$. We have then

$$\begin{aligned}\chi_2 &= 1 - 1{\cdot}3307\cos\frac{\pi}{16} - {\cdot}0415\cos\frac{3\pi}{16} + {\cdot}3793\cos\frac{\pi}{8} + [{\cdot}002]\\ &= 1 - 1{\cdot}3051 - {\cdot}0345 + {\cdot}3504 + 3\,[{\cdot}0001] + [{\cdot}002]\\ &= {\cdot}0108 + [{\cdot}0023] > {\cdot}0085 = \sigma > 0.\end{aligned}$$

It will easily be verified that, when n has any other residue to modulus 16, the margin is much greater.

7. Conclusion.

7. 1. We have now proved Theorem B when $s = 21$, and Theorem A is an obvious corollary. It is not immediately obvious that, if Theorem B is true for $s = 21$, it is also true for $s > 21$. All our arguments are valid for $s \geqq 21$, except those of §§ 6. 3—6. 4; but the numerical discussion of these two paragraphs has, strictly, to be repeated for each value of s in question. Our own calculations refer only to the cases $s = 21, 31, 33$, in which we have, at various times, been particularly interested. No point of principle is involved, and the calculations in other cases may be left to anyone who may be sufficiently interested in the matter to make them[11]).

It is evident that we may, with the help of the singular series, study as closely as we wish the variations of $r_{4,s}(n)$ as n assumes various residues to modulus 16. It is clear, for example, that the numbers $16m + 2$ and $16m + 3$ are, to put it roughly, less readily expressible by 21 biquadrates than any other numbers, and something like 200 times less readily expressible than the numbers $16m + 10$ and $16m + 11$.

There is no difficulty in applying the methods of this paper to the proof that

$$G(k) \leqq (k-2)\,2^{k-1} + 5 \tag{7.11}$$

for any *particular* value of k, as for example 3, 5, 6 or 7. We find thus that $G(3) \leqq 9$, $G(5) \leqq 53$, $G(6) \leqq 133$, and $G(7) \leqq 325$. The first of these inequalities is not new[12]), and in fact Landau has proved that $G(3) \leqq 8$: but the numbers 53, 133, 325 compare very favourably with the 58, 478, 3806 at present known. The proof that (7. 11) is true *generally*, however, presents certain algebraical difficulties, of complication rather than of principle, and we must postpone it to a later memoir. We have not indeed worked out this proof in detail, the analysis which we possess carrying us only so far as the less favourable inequality

$$G(k) \leqq k\,2^{k-1} + 1$$

indicated by our earlier researches.

[11]) See however the following note of Herr Ostrowski.

[12]) The accompanying asymptotic formula is of course new.

We conclude with one final remark. It might well be supposed that the proof of (7. 11) for (say) $k = 7$ or 13 would be more difficult than for $k = 4$. This is not so; the proof for $k = 4$ is, in essentials, more delicate and critical than for any other value of k. The fact is that *it is only for* $k = 4$ *that our inequality expresses something near the ultimate truth.* It is known that $G(4) \geqq 16$, and, the difference between 16 and 21 is comparatively small: this corresponds to the facts that *the critical factor of* § 6 *nearly vanishes in the least favourable case,* and that there is a term in χ_2 which is sometimes actually greater than the leading term 1. When k is larger, our value is much too high, and the singular series tends (for such values of s as are contemplated in our analysis) to be dominated completely by its leading term.

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