

On the Invariants of a Certain Orthogonal Transformation, with special reference to the Theory of the Strains and Stresses of an Elastic Solid. By W. J. C. SHARP, M.A.

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I. If a, b, c, f, g, h are transformed according to the same law as $x^2, y^2, z^2, yz, zx, xy$; x, y , and z being the rectangular Cartesian coordinates of a point, transformed without change of origin; they will have a system of invariants entirely unaffected by the transformation—viz.,

- (i.) $a + b + c \equiv D$ say,
- (ii.) $ab + bc + ca - f^2 - g^2 - h^2 \equiv S$,
- (iii.) $abc + 2fgh - af^2 - bg^2 - ch^2 \equiv \Delta$.

This is easily proved from the equations of transformation; for, since there are six equations of condition connecting the direction cosines, and six more expressing the new values of a, b, c , &c. in terms of the old, it is evident that there ought to be three resultants; whilst, from the symmetry of the relations which express the direct and inverse transformations, each of these must be a symmetrical function of a, b, c , &c., and a', b', c' , &c.; and, as these relations will usually express some property of the system, independent of the particular axes, these symmetrical relations will usually be reducible to the form $\phi(a, b, c \dots) = \phi(a', b', c' \dots)$, i.e., to invariants.

From the number of equations it appears that $(a, b, c \dots \nabla x^2, y^2 \dots) \equiv U$ ought to have two independent covariants.

The two simplest of these are

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy \equiv U',$$

where A, B , &c. are the first minors of the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

which is the polar reciprocal of U , and is the S mentioned in Dr. Salmon's *Geometry of Three Dimensions*, p. 44; and $x^2 + y^2 + z^2$ itself, or, perhaps better, the covariant F of this and U , viz.,

$$(B + C)x^2 + (C + A)y^2 + (A + B)z^2 - 2Fyz - 2Gzx - 2Hxy \equiv U'',$$

which satisfies the equation

$$U'' \equiv D(x^2 + y^2 + z^2) - U',$$

and may be derived from U' as Poinot's *Momental Ellipsoid* is from

$$\Sigma mx^2 \cdot \xi^2 + \dots + 2\Sigma mys \cdot \eta\xi \dots = c,$$

or the strain ellipsoid from the elongation quadric.

The relation between U and U' is that between two conjugate cones, or that between the momental ellipsoid and that of gyration, or that between the two expressions for the work of an electric current (Clerk-Maxwell's *Electricity and Magnetism*, Vol. I., p. 385) when the conditions there mentioned are fulfilled.

II. If u, v, w and u', v', w' be any two sets of quantities cogredient with xyz , the equations $\frac{u}{u'} = \frac{v}{v'} = \frac{w}{w'}$, which express the parallelism of the vectors $ui + vj + wk$ and $u'i + v'j + w'k$, will be unaltered by transformation.

The following are some examples of equations of the kind:—

$$\dagger \frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2},$$

the equations to the right line through (x_1, y_1, z_1) (x_2, y_2, z_2) .

$$\dagger \frac{x-\xi}{\frac{dx}{ds}} = \frac{y-\eta}{\frac{dy}{ds}} = \frac{z-\zeta}{\frac{dz}{ds}},$$

the equations to the tangent to a curve in space at (x, y, z) .

$$\dagger \frac{x-\xi}{\frac{dU}{dx}} = \frac{y-\eta}{\frac{dU}{dy}} = \frac{z-\zeta}{\frac{dU}{dz}},$$

those to the normal to the surface $U = 0$ at (x, y, z) .

$$\dagger \frac{\frac{d^2x}{ds^2}}{\frac{dU}{dx}} = \frac{\frac{d^2y}{ds^2}}{\frac{dU}{dy}} = \frac{\frac{d^2z}{ds^2}}{\frac{dU}{dz}},$$

the differential equations to geodesics on $U = 0$.

$$\frac{\xi - \frac{N\Sigma Y - M\Sigma Z}{R^2}}{\Sigma X} = \frac{\eta - \frac{L\Sigma Z - N\Sigma X}{R^2}}{\Sigma Y} = \frac{\zeta - \frac{M\Sigma X - L\Sigma Y}{R^2}}{\Sigma Z},$$

the equations to the central axis in Statics.

$$\dagger \frac{\frac{dz}{dZ} - \frac{dY}{dZ}}{\frac{dy}{dZ} - \frac{dX}{dZ}} = \frac{\frac{dX}{dZ} - \frac{dZ}{dX}}{\frac{dX}{dZ} - \frac{dZ}{dX}} = \frac{\frac{dY}{dZ} - \frac{dX}{dZ}}{\frac{dY}{dZ} - \frac{dX}{dZ}},$$

† Quantities marked † are also unaltered by change of origin.

the differential equations to the curve of intersection of the surfaces of equal pressure and of equal density in Hydrostatics.

$$\dagger \frac{d\xi}{u} = \frac{d\eta}{v} = \frac{d\zeta}{w},$$

those to the lines of motion in Hydrokinetics; and

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3},$$

where ω_1 , ω_2 , ω_3 are the angular velocities about the axes, the equations to the instantaneous axis when a rigid body is acted upon by no forces.

III. Two special classes of functions of the kind denoted by a , b , c &c., naturally suggest themselves first for consideration.

Those in which a , b , &c., are proportional to u^2 , v^2 , w^2 , vw , wu , uv ; u , v , and w being cogredient with x , y , and z respectively. In this case the related quadric represents a pair of parallel planes, and the invariants S and Δ vanish identically, while D becomes $u^2 + v^2 + w^2$, the square of the tensor of the vector $ui + vj + wk$.

The following are some examples:—

$$x^2 + y^2 + z^2,$$

the square of the distance of (x, y, z) from the origin.

$$\dagger (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2,$$

that of the distance between (x, y, z) and (x_1, y_1, z_1) .

$$\dagger (dx)^2 + (dy)^2 + (dz)^2,$$

the square of the element of the arc of a curve.

$$\dagger \left(\frac{d^2x}{ds^2} \right)^2 + \left(\frac{d^2y}{ds^2} \right)^2 + \left(\frac{d^2z}{ds^2} \right)^2,$$

the square of the curvature.

$$\dagger \left(\frac{dy}{dt} \cdot \frac{d^2z}{dt^2} - \frac{dz}{dt} \cdot \frac{d^2y}{dt^2} \right)^2 + \left(\frac{dz}{dt} \cdot \frac{d^2x}{dt^2} - \frac{dx}{dt} \cdot \frac{d^2z}{dt^2} \right)^2 + \left(\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2} \right)^2,$$

an important expression in Solid Geometry.

$$\dagger X^2 + Y^2 + Z^2,$$

the square of the form of which X , Y , and Z are the components.

$$\dagger (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$

that of the resultant of a set of forces.

$$(Yz - Zy)^2 + (Zx - Xz)^2 + (Xy - Yx)^2,$$

or more generally

$$\{\Sigma(Yz - Zy)\}^2 + \{\Sigma(Zx - Xz)\}^2 + \{\Sigma(Xy - Yx)\}^2,$$

the square of the resultant moment.

$$+ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2,$$

the velocity squared.

$$+ \Sigma \frac{m}{2} \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right\},$$

the *vis viva*.

$$+ \left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2,$$

the square of the resultant acceleration.

$$(\omega_1)^2 + (\omega_2)^2 + (\omega_3)^2,$$

that of the resultant angular velocity.

IV. Then the case in which a, b, c , &c. are proportional to $uu', vv', ww', \frac{1}{2}(vw' + v'w), \frac{1}{2}(wu' + w'u),$ and $\frac{1}{2}(uv' + u'v)$; u, v, w and u', v', w' being cogredient to x, y, z . In this case the auxiliary quadric

$$(ux + vy + wz)(u'x + v'y + w'z) = c$$

is a cylinder. The invariant Δ vanishes identically, D takes the form of $uu' + vv' + ww'$, and S that of $(vw' - v'w)^2 + (wu' - w'u)^2 + (uv' - u'v)^2$.

If u, v, w and u', v', w' be the components of the vectors α, β , the last two invariants are $-S \cdot \alpha\beta$ and $T^2 \cdot V\alpha\beta$.

If $v_1w_2 - v_2w_1, w_1u_2 - w_2u_1, u_1v_2 - u_2v_1$ be substituted for u', v' , and w' in the value of D , this takes the important form

$$\begin{vmatrix} u, & v, & w \\ u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \end{vmatrix}^2 \quad \text{or} \quad -S \cdot \alpha\beta\gamma,$$

if $\alpha = ui + vj + wk, \beta = u_1i + v_1j + w_1k,$

and $\gamma = u_2i + v_2j + w_2k;$

and hence the Jacobian of any three functions of x, y , and z is an invariant.

The following are examples of these forms:—

$$xx' + yy' + zz',$$

or $rr' \cos \theta$, where r and r' are the distances of (x, y, z) and (x', y', z') from the origin and θ the angle between them.

* The invariance of this determinant may also be easily proved by squaring.

$$(ys' - s'x)^2 + (zx' - s'x)^2 + (xy' - s'y)^2,$$

or $r^2 \sin^2 \theta$, four times the square of the area of the triangle of which these are sides.

$$\dagger \begin{vmatrix} x_1 - x, & y_1 - y, & z_1 - z \\ x_2 - x, & y_2 - y, & z_2 - z \\ x_3 - x, & y_3 - y, & z_3 - z \end{vmatrix},$$

six times the volume of the tetrahedron whose vertices are (x, y, z) (x_1, y_1, z_1) , &c.,—the quantity which, equated to zero, gives the equation to the plane through the three points (x_1, y_1, z_1) , &c.

$$\dagger (x - \xi) \frac{dU}{dx} + (y - \eta) \frac{dU}{dy} + (z - \zeta) \frac{dU}{dz},$$

the left-hand side of the equation to the tangent to the surface $U = 0$ at (x, y, z) .

$$\dagger (x - \xi) \frac{ds}{dx} + (y - \eta) \frac{ds}{dy} + (z - \zeta) \frac{ds}{dz},$$

that of the equation to the normal plane to a curve at (x, y, z) ;

$$\dagger (dy \cdot d^2x - dx \cdot d^2y) (x - \xi) + (dz \cdot d^2x - dx \cdot d^2z) (y - \eta) \\ + (dx \cdot d^2y - dy \cdot d^2x) (z - \zeta),$$

that of the equation to the osculating plane.

$$\begin{vmatrix} x, & y, & z \\ U_1, & U_2, & U_3 \\ U'_1, & U'_2, & U'_3 \end{vmatrix} = 0,$$

the expression which is put to give the equation to the principal planes of the quadric $U = 0$ (Salmon, *Geometry of Three Dimensions*, p. 44).

$$\begin{vmatrix} dx, & dy, & dz \\ \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ d \frac{dU}{dx}, & d \frac{dU}{dy}, & d \frac{dU}{dz} \end{vmatrix},$$

the left-hand side of the differential equation to the lines of curvature upon the surface $U = 0$.

The reciprocal of the radius of torsion

$$\dagger k^2 \cdot \begin{vmatrix} \frac{dx}{dt}, & \frac{dy}{dt}, & \frac{dz}{dt} \\ \frac{d^2x}{dt^2}, & \frac{d^2y}{dt^2}, & \frac{d^2z}{dt^2} \\ \frac{d^3x}{dt^3}, & \frac{d^3y}{dt^3}, & \frac{d^3z}{dt^3} \end{vmatrix},$$

where

$$\frac{1}{k^2} = \left(\frac{dy}{dt} \cdot \frac{d^2x}{dt^2} - \frac{dx}{dt} \cdot \frac{d^2y}{dt^2} \right)^2 + \left(\frac{dz}{dt} \cdot \frac{d^2x}{dt^2} - \frac{dx}{dt} \cdot \frac{d^2z}{dt^2} \right)^2 + \left(\frac{dz}{dt} \cdot \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2z}{dt^2} \right)^2,$$

which is itself an invariant.

$$\frac{\frac{dU}{dx} \cdot \frac{dV}{dx} + \frac{dU}{dy} \cdot \frac{dV}{dy} + \frac{dU}{dz} \cdot \frac{dV}{dz}}{\sqrt{\left\{ \left(\frac{dU}{dx} \right)^2 + \left(\frac{dU}{dy} \right)^2 + \left(\frac{dU}{dz} \right)^2 \right\}} \sqrt{\left\{ \left(\frac{dV}{dx} \right)^2 + \left(\frac{dV}{dy} \right)^2 + \left(\frac{dV}{dz} \right)^2 \right\}}},$$

the cosine of the angle at which the two surfaces $U = 0$, $V = 0$ intersect.

$$\begin{vmatrix} U_1 & U_2 & U_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

the left-hand side of the partial differential equation to a family of surfaces (Salmon's *Geometry of Three Dimensions*, p. 340). [If X, Y, Z be the components of a force and (x, y, z) its point of application.]

$$Xx + Yy + Zz,$$

the work done against the force from the origin to the point of application.

Its variation, $X \cdot \delta x + Y \cdot \delta y + Z \cdot \delta z$,
the virtual work.

$$\int_a^b (X dx + Y dy + Z dz),$$

the work from A to B .

The same expressions with accelerations

$$\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right)$$

substituted for the forces X, Y, Z .

The expression

$$\begin{vmatrix} X & Y & Z \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \\ \frac{d^2x}{dt^2} & \frac{d^2y}{dt^2} & \frac{d^2z}{dt^2} \end{vmatrix},$$

in the theory of inextensible strings.

$$\int m \left(X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} \right) ds,$$

and $\left\{ \left(\int mX ds \right)^2 + \left(\int mY ds \right)^2 + \left(\int mZ ds \right)^2 \right\}^{\frac{1}{2}},$

two expressions for the tension; and

$$\begin{vmatrix} T \frac{d^2x}{ds^2} + mX, & \frac{dx}{ds}, & \frac{du}{ds} \\ T \frac{d^2y}{ds^2} + mY, & \frac{dy}{ds}, & \frac{du}{dy} \\ T \frac{d^2z}{ds^2} + mZ, & \frac{dz}{ds}, & \frac{du}{dz} \end{vmatrix}$$

(see Minchin's *Statics*, p. 351).

The quantity $u\omega_1 + v\omega_2 + w\omega_3$, the vanishing of which is a condition that the motion of a rigid body should be a mere rotation.

The functions

$$u \, dx + v \, dy + w \, dz,$$

$$u \frac{d\rho}{dx} + v \frac{d\rho}{dy} + w \frac{d\rho}{dz},$$

$$u \frac{dx}{ds} + v \frac{dy}{ds} + w \frac{dz}{ds},$$

in Hydrostatics.

$$\text{The expression} \quad \frac{dV}{dx} \cdot \frac{dx}{dh} + \frac{dV}{dy} \cdot \frac{dy}{dh} + \frac{dV}{dz} \cdot \frac{dz}{dh}$$

in the theory of the Potential (Clerk-Maxwell's *Electricity, &c.*, Vol. I., p. 180).

The numerous expressions for work in the same book. The quantity

$$\Pi = \frac{1}{r^3} \begin{vmatrix} \xi - x, & \eta - y, & \zeta - z \\ \frac{d\xi}{d\sigma}, & \frac{d\eta}{d\sigma}, & \frac{d\zeta}{d\sigma} \\ \frac{dx}{ds}, & \frac{dy}{ds}, & \frac{dz}{ds} \end{vmatrix}$$

(Vol. II., p. 39), and

$$X \left(\frac{dZ}{dy} - \frac{dY}{dz} \right) + Y \left(\frac{dX}{dz} - \frac{dZ}{dx} \right) + Z \left(\frac{dY}{dx} - \frac{dX}{dy} \right),$$

the vanishing of which is the condition that $X \, dx + Y \, dy + Z \, dz$ should become an exact differential when multiplied by a factor.

V. Turning now to the general case of functions which transform like $a, b, c, &c.$ One very important set is that in which a, b, c, f, g, h are proportional to $\Sigma Xx, \Sigma Yy, \Sigma Zz, \frac{1}{2}\Sigma (Yz + Zy), \frac{1}{2}\Sigma (Zx + Xz), \frac{1}{2}\Sigma (Xy + Yx)$ respectively. $X_1, X_2, &c., Y_1, Y_2, &c., Z_1, Z_2, &c.$, being cogredient with x, y , and z . In this case $\Sigma Xx + \Sigma Yy + \Sigma Zz$ is the in-

variant D ; whilst

$$\{\Sigma(Yz - Zy)\}^2 + \{\Sigma(Zx - Xz)\}^2 + \{\Sigma(Xy - Yx)\}^2$$

is an invariant, by (III.)*

(i.) If X_1, Y_1, Z_1 , &c. be forces, and (x_1, y_1, z_1) their points of application;

$$\Sigma(X\delta x + Y\delta y + Z\delta z),$$

the variation of $\Sigma(Xx + Yy + Zz)$, the virtual work;

$$(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$

the square of the resultant of forces equal to the given forces and all acting at one point;

$$\{\Sigma(Yz - Zy)\}^2 + \{\Sigma(Zx - Xz)\}^2 + \{\Sigma(Xy - Yx)\}^2,$$

the square of the moment of the couple produced by fixing the origin, —are all invariants.

$L \cdot \Sigma X + M \cdot \Sigma Y + N \cdot \Sigma Z$ is also an invariant, and therefore

$$\frac{L \cdot \Sigma X + M \cdot \Sigma Y + N \cdot \Sigma Z}{\sqrt{\{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2\}}}$$

the principal moment. The equations to the central axis are invariable (III.), and the pitch of the screw along this (Dr. Ball's *Theory of*

$$\text{Screws, p. 5)} = \frac{L \cdot \Sigma X + M \cdot \Sigma Y + N \cdot \Sigma Z}{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2} \text{ also.}$$

If this vanish, the system of forces is reducible to a single resultant.

(ii.) If accelerations $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ be substituted for X, Y, Z , &c.,

the expressions $\Sigma m \left\{ \frac{d^2x}{dt^2} \cdot \frac{dx}{dt} + \frac{d^2y}{dt^2} \cdot \frac{dy}{dt} + \frac{d^2z}{dt^2} \cdot \frac{dz}{dt} \right\}$

and $\Sigma m \left(X \frac{dx}{dt} + Y \frac{dy}{dt} + Z \frac{dz}{dt} \right)$

are invariants as well as those on both sides of the equation of *vis viva*,

$$\Sigma m \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right\} = C + \Sigma m \int (Xdx + Ydy + Zdz),$$

which they serve to establish.

So also are $\Sigma m \left(\frac{d^2x}{dt^2} \delta x + \frac{d^2y}{dt^2} \delta y + \frac{d^2z}{dt^2} \delta z \right),$

* This expression

$= -4S + 4 \{ \Sigma Xx \cdot \Sigma Yy + \Sigma Yy \cdot \Sigma Zz + \Sigma Zz \cdot \Sigma Xx - \Sigma Xy \cdot \Sigma Yx - \Sigma Yz \cdot \Sigma Zx - \Sigma Xz \cdot \Sigma Yy \},$
so that the quantity in the bracket is also an invariant.

or

$$\Sigma \left(\frac{d}{dt} \cdot \frac{dT}{d\theta} - \frac{dT}{d\theta} \right) \frac{1}{2} \delta \theta,$$

and

$$\Sigma m (X\delta x + Y\delta y + Z\delta z), \text{ or } \delta U,$$

the expressions involved in Lagrange's equations.

The generalized components of force, acceleration, and momentum,

$$\begin{aligned} \Sigma m \left(X \frac{dx}{dq_r} + Y \frac{dy}{dq_r} + Z \frac{dz}{dq_r} \right), \\ \Sigma m \left(\frac{d^2x}{dt^2} \cdot \frac{dx}{dq_r} + \frac{d^2y}{dt^2} \cdot \frac{dy}{dq_r} + \frac{d^2z}{dt^2} \cdot \frac{dz}{dq_r} \right), \\ \Sigma m \left(\frac{dx}{dt} \cdot \frac{dx}{dq_r} + \frac{dy}{dt} \cdot \frac{dy}{dq_r} + \frac{dz}{dt} \cdot \frac{dz}{dq_r} \right), \end{aligned}$$

and the action
$$\int \Sigma m \left(\frac{dx}{dt} dx + \frac{dy}{dt} dy + \frac{dz}{dt} dz \right),$$

are also invariants.

Again, in the case where a, b, c , &c., are replaced by P, Q, R, U, V, W , the components of stress, the quadric $U=c$ is the stress quadric, and $U'=0$ the cone of shearing stress. The invariants are

$$\begin{aligned} P+Q+R &= D, \\ PQ+QR+RP-U^2-V^2-W^2 &= S, \\ PQR+2UVW-PU^2-QV^2-RW^2 &= \Delta. \end{aligned}$$

If the system be equivalent to a shearing stress, D and Δ vanish, and $-S$ is the square of the shearing stress.

D and S are the S and Σ in Clapeyron's Theorem (Minchin's *Statics*, p. 515), and the expression

$$\frac{1}{2\mu} \int (P^2 + Q^2 + R^2 + 2U^2 + 2V^2 + 2W^2) dw,$$

is equivalent to
$$\frac{1}{2\mu} \int (D^2 - 2S) dw.$$

The consideration of another important class of functions, which includes the components of strain, will come more conveniently after that of the combinants of two sets of functions.

VI. When there are two sets of quantities a, b, c , &c., and a', b', c' , &c., both transforming as before, the invariants D^2-2S and Δ give rise to three combinants

$$\begin{aligned} aa' + bb' + cc' + 2ff' + 2gg' + 2hh' &\equiv E \text{ say,} \\ a'bc + ab'c + abc' + 2f'gh + 2fg'h + 2fgh' \\ - a'f^2 - b'g^2 - c'h^2 - 2aff' - 2bgg' - 2chh' &\equiv \Theta, \end{aligned}$$

and
$$ab'c' + a'b'c + 2fg'h' + 2f'gh' + 2f'g'h - af'^2 - bg'^2 - ch'^2 - 2a'f'f - 2b'g'g - 2c'h'h = \Theta'.$$

If $a', b', \&c.$, be proportional to $u^2, v^2, \&c.$, as in (III.),

$$E = au^2 + bv^2 + cw^2 + 2fuv + 2gwu + 2huv,$$

a form which shows that the result of putting any quantities co-gredient with x, y, z for these in U , is unchanged by transformation. The same substitution gives

$$\Theta = Au^2 + Bv^2 + Cw^2 + 2Fuv + 2Gwu + 2Huv,$$

where $A, B, C, \&c.$ have the same meaning as in (I.) So that θ is the result of writing u, v, w for x, y, z in U' . With these values of $a', b', \&c.$, Θ' vanishes identically, or, if $a', b', \&c.$ be proportional to $uu', vv', \&c.$, as in (IV.),

$$\begin{aligned} E &= auu' + bvv' + cww' + f(vw' + v'w) + g(wu' + w'u) + h(uv' + u'v) \\ &= (au' + hv' + fw')u + (hu' + bv' + fw')v + (gu' + fv' + cw')w, \end{aligned}$$

and $\Theta = (Au' + Hv' + Fw')u + (Hu' + Bv' + Fw')v + (Gu' + Fv' + Cw')w$,

$$\begin{aligned} \text{and } -4\Theta' &= a(vw' - v'w)^2 + b(wu' - w'u)^2 + c(uv' - u'v)^2 \\ &\quad + 2f(wu' - w'u)(uv' - u'v) + 2g(uv' - u'v)(vw' - v'w) \\ &\quad + 2h(vw' - v'w)(wu' - w'u), \end{aligned}$$

the result of writing $vw' - v'w, \&c.$ for x, y, z in U . So that this is, in fact, a case of the value of E obtained before. Examples of most of these combinants occur in the theory of Quadrics.

Again, if $a, b, c, \&c.$ be the components of strain, $a', b', c', \&c.$, those of stress, $E dw$ is the element of the work done in strain. With the same suppositions, the expression

$$a'da + b'db + c'dc + 2f'df + 2g'dg + 2h'dh,*$$

a quantity employed by Sir W. Thomson, and which he has shown must be an exact differential, is a form of the combinant E . The two quantities $A\omega_1^2 + B\omega_2^2 + C\omega_3^2$ and $A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2$, A, B , and C being the principal moments of inertia, which occur in the solution of Euler's equations of motion in the case where there are no impressed forces, are both instances of E . The quantities $a, b, c, \&c.$, and $a', b', c', \&c.$, being, for the first, $A, B, C, -F, -G, -H$, when these expressions represent the moments and products of inertia, and $\omega_1^2, \omega_2^2, \omega_3^2, \omega_1\omega_2, \&c.$, where $\omega_1, \omega_2, \omega_3$ are the angular velocities about the axes. So that

$$E = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2F\omega_1\omega_2 - 2G\omega_2\omega_3 - 2H\omega_1\omega_3,$$

which reduces to $A\omega_1^2 + B\omega_2^2 + C\omega_3^2$,

* I am indebted for my acquaintance with this function to a remark made by Professor C. Niven.

when the axes are principal. For the second, they are ω_1^2 , &c., as before, and the coefficients of a covariant U''' of the momental ellipsoid U , which may be obtained as follows.

Let U' denote the ellipsoid of gyration

$$(BC - F^2)x^2 + (CA - G^2)y^2 + (AB - H^2)z^2 - 2(GH + AF)yz \\ - 2(HF + BG)zx - 2(FG + CH)xy.$$

Then $U''' = (A + B + C)U + U'$

$$- (AB + BC + CA - F^2 - G^2 - H^2)(x^2 + y^2 + z^2) \\ \equiv (A^2 + G^2 + H^2)x^2 + (B^2 + F^2 + H^2)y^2 + (C^2 + G^2 + F^2)z^2 \\ - 2(2AF + BF + CF + GH)yz \\ - 2(2BG + CG + AG + HF)zx \\ - 2(2CH + AH + BH + FG)xy;$$

so that

$$E = (A^2 + G^2 + H^2)\omega_1^2 + \&c.,$$

$$- 2(2AF + BF + CF + GH)\omega_1\omega_2 - \&c.,$$

which reduces to $A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2$ when the axes are principal. The equations $U''' = \lambda^6$ and $U = \epsilon^4$ determine the polhode. The next section will furnish some further examples of the combinants.

VII. The special class already mentioned is that in which a, b, c, f, g, h , are proportional to $\frac{du}{dx}, \frac{dv}{dy}, \frac{dw}{dz}, \frac{1}{2}\left(\frac{dv}{dz} + \frac{dw}{dy}\right), \frac{1}{2}\left(\frac{dw}{dx} + \frac{du}{dz}\right), \frac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right)$ respectively, u, v, w being quantities cogredient with x, y , and z . In this case it appears from the number of equations that there ought to be three additional invariants. These may be derived as follows. Since $\frac{dv}{dy} - \frac{dv}{dz}, \frac{du}{dz} - \frac{dw}{dx}, \frac{dv}{dx} - \frac{du}{dy}$ are cogredient with x, y, z ,

$$\left(\frac{dw}{dy} - \frac{dv}{dz}\right)^2 + \left(\frac{du}{dz} - \frac{dv}{dx}\right)^2 + \left(\frac{dv}{dx} - \frac{du}{dy}\right)^2 = \Omega^2 \text{ say,}$$

is an invariant.

This combined with S gives

$$\frac{du}{dx} \cdot \frac{dv}{dy} + \frac{dv}{dy} \cdot \frac{dw}{dz} + \frac{dw}{dz} \cdot \frac{du}{dx} \\ - \frac{du}{dy} \cdot \frac{dv}{dx} - \frac{dv}{dz} \cdot \frac{dw}{dy} - \frac{dw}{dx} \cdot \frac{du}{dz} = \frac{4S + \Omega^2}{4}.$$

Then the combinant E of

$$\left(\frac{du}{dx}, \frac{dv}{dy}, \dots, \chi^x(x, y, z)\right)^2$$

and

$$\left\{ \left(\frac{dw}{dy} - \frac{dv}{dz}, \frac{du}{dz} - \frac{dv}{dx}, \frac{dv}{dx} - \frac{du}{dy}\right) \chi^x(x, y, z) \right\}^2,$$

$$\begin{aligned}
\text{or} \quad & \frac{du}{dx} \left(\frac{dw}{dy} - \frac{dv}{dz} \right)^2 + \frac{dv}{dy} \left(\frac{du}{dz} - \frac{dw}{dx} \right)^2 + \frac{dw}{dz} \left(\frac{dv}{dx} - \frac{du}{dy} \right)^2 \\
& + \left(\frac{dv}{dy} + \frac{dw}{dz} \right) \left(\frac{du}{dz} - \frac{dw}{dx} \right) \left(\frac{dv}{dx} - \frac{du}{dy} \right) \\
& + \left(\frac{du}{dz} + \frac{dw}{dx} \right) \left(\frac{dv}{dx} - \frac{du}{dy} \right) \left(\frac{dw}{dy} - \frac{dv}{dz} \right) \\
& + \left(\frac{dv}{dx} + \frac{du}{dy} \right) \left(\frac{dw}{dy} - \frac{dv}{dz} \right) \left(\frac{du}{dz} - \frac{dw}{dx} \right) = H \text{ say,}
\end{aligned}$$

which combined with Δ gives

$$\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{vmatrix} = \Delta + \frac{H}{4},$$

the Jacobian of u, v , and w .

The third additional invariant is the combinant Θ of the same func-

$$\begin{aligned}
\text{tions} \quad & \left(\frac{dw}{dy} - \frac{dv}{dz} \right)^2 \frac{dv}{dy} \cdot \frac{dw}{dz} + \left(\frac{du}{dz} - \frac{dw}{dx} \right)^2 \frac{dw}{dz} \cdot \frac{du}{dx} \\
& + \left(\frac{dv}{dx} - \frac{du}{dy} \right)^2 \frac{du}{dx} \cdot \frac{dv}{dy} - \frac{du}{dz} \left(\frac{dw}{dy} + \frac{dv}{dz} \right) \left(\frac{du}{dz} - \frac{dw}{dx} \right) \left(\frac{dv}{dx} - \frac{du}{dy} \right) \\
& - \frac{dv}{dy} \left(\frac{du}{dz} + \frac{dw}{dx} \right) \left(\frac{dv}{dx} - \frac{du}{dy} \right) \left(\frac{dw}{dy} - \frac{dv}{dz} \right) \\
& - \frac{dw}{dz} \left(\frac{dv}{dx} + \frac{du}{dy} \right) \left(\frac{dw}{dy} - \frac{dv}{dz} \right) \left(\frac{du}{dz} - \frac{dw}{dx} \right) \\
& - \frac{1}{4} \left\{ \left(\frac{dw^2}{dy^2} - \frac{dv^2}{dz^2} \right)^2 + \left(\frac{du^2}{dz^2} - \frac{dw^2}{dx^2} \right)^2 + \left(\frac{dv^2}{dx^2} - \frac{du^2}{dy^2} \right)^2 \right. \\
& \quad - 2 \left(\frac{dw^2}{dy^2} - \frac{dv^2}{dz^2} \right) \left(\frac{du^2}{dz^2} - \frac{dw^2}{dx^2} \right) - 2 \left(\frac{du^2}{dz^2} - \frac{dw^2}{dx^2} \right) \left(\frac{dv^2}{dx^2} - \frac{du^2}{dy^2} \right) \\
& \quad \left. - 2 \left(\frac{dv^2}{dx^2} - \frac{du^2}{dy^2} \right) \left(\frac{dw^2}{dy^2} - \frac{dv^2}{dz^2} \right) \right\} = Z \text{ say.}
\end{aligned}$$

H and Z both vanish if Ω does.

Whenever two vectors whose components are u, v, w and x, y, z , are so related that one of them is a linear and vector function of the other, the nine quantities $\frac{du}{dx}, \frac{du}{dy}$, &c. are the coefficients in the expressions for u, v, w in terms of x, y , and z , and the six a, b, c , &c. (with the values given above) are those in the value of the invariant $ux + vy + wz$, in terms of x, y , and z . The strains of a solid or fluid body are one

important set of functions of this kind. In this case

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = D, \text{ the dilatation,}$$

is the coefficient of ρ in the differential equation of continuity, $\Delta + \frac{H}{4}$ being that in the integral equation. Ω is the resultant rotation, and $D, S + \frac{\Omega^2}{4}, \Delta + \frac{H}{4}$ are the coefficients in the equation which determines the lines of no rotation. $\Omega = 0$ is the condition for steady motion, *i.e.*, the condition that the invariant

$$u dx + v dy + w dz$$

should be an exact differential. This involves the vanishing of H and Z .

In the case of a strain, $\Omega = 0$ is the condition that it should be pure.

$D = 0$ and $\Delta = 0$ that it should be equivalent to a shear, and in this case $-S$ is the square of the shear.

$\Delta = 0$ that there should be two planes of no elongation. And this, together with $S = 0$, that the strain should be equivalent to a simple elongation.

When

$$u = \frac{dF}{dx}, \quad v = \frac{dF}{dy}, \quad w = \frac{dF}{dz},$$

$$D = \frac{d^2 F}{dx^2} + \frac{d^2 F}{dy^2} + \frac{d^2 F}{dz^2},$$

the important quantity on the left-hand side of Lagrange's and Poisson's equations.

$$S = \frac{d^2 F}{dx^2} \cdot \frac{d^2 F}{dy^2} + \frac{d^2 F}{dy^2} \cdot \frac{d^2 F}{dz^2} + \frac{d^2 F}{dz^2} \cdot \frac{d^2 F}{dx^2} \\ - \left(\frac{d^2 F}{dx dy} \right)^2 - \left(\frac{d^2 F}{dy dz} \right)^2 - \left(\frac{d^2 F}{dz dx} \right)^2,$$

and Δ is the Hessian of F .

The class of linear and vector functions of vectors includes several important quantities which appear in the theory of Electricity and Magnetism—as the Electromotive Intensity which is a function of the Electric displacement (Clerk Maxwell's *Electricity and Magnetism*, Vol. i., p. 136); the Electromotive Force of the Current (p. 383); the Magnetisation of the Magnetic Force (Vol. ii., p. 60). In every case the six invariant relations hold among the coefficients.

If

$$u = u_1 x + u_2 y + u_3 z, \quad v = v_1 x + v_2 y + v_3 z,$$

$$w = w_1 x + w_2 y + w_3 z,$$

be the equations connecting the vectors, when referred to any rectangular axes, the coefficients corresponding to any others with same

origin may be obtained by transforming

$$(u_1 x + u_2 y + u_3 z) u + (v_1 x + v_2 y + v_3 z) v + (w_1 x + w_2 y + w_3 z) w,$$

u, v, w and x, y, z being transformed by the same transformation. In all cases, the coefficients of such a function will have the six invariants given above.

VIII. The invariants may also be made use of in other ways. Thus—

(a.) To transform the equation $U = \mu$ to the form

$$H(x^2 + y^2 + z^2) + \lambda x^2 + 2kyz = \mu$$

(a form suggested by Thomson and Tait as a canonical one for strains, &c.)

$$\text{Here} \quad 3H + \lambda = D, \quad 3H^2 + 2H\lambda - k^2 = S,$$

$$\text{and} \quad H^2 + H^2\lambda - (H + \lambda)k^2 = \Delta.$$

$$\text{Hence} \quad (H + \lambda)^3 - D(H + \lambda)^2 + S(H + \lambda) - \Delta = 0.$$

So that $H + \lambda$ is a root of the discriminating cubic; and, therefore equal to one of the principal coefficients (the squares of the reciprocals of the principal axes). Let these be A, B , and C .

$$\text{If } H + \lambda = A, \quad H = \frac{B+C}{2}, \quad \lambda = A - \frac{B+C}{2}, \quad \text{and } k^2 = \left(\frac{B-C}{2}\right)^2.$$

If λ is to be positive, A must be greater than $\frac{B+C}{2}$ which may be secured by supposing $\frac{1}{\sqrt{A}}$ the least axis.

Comparing the resulting form of the equation

$$Ax^2 + \frac{B+C}{2}(y^2 + z^2) \pm (B-C)y'z' = \mu$$

with $Ax^2 + By^2 + Cz^2 = \mu$, it is evident that the axis of x is unchanged; and if $y' = \lambda y + \mu z$, $z' = \mu y - \lambda z$ be the equations between the new and old coordinates, $\lambda = \mu = \frac{1}{\sqrt{2}}$. So that the new axes of y and z bisect the angles between the old.

(b.) If $U = \mu$ be a spheroid, the discriminating cubic must have equal roots, and therefore

$$(DS - 9\Delta)^2 = 4(D^2 - 3S)(S^2 - 3D\Delta).$$

If U can be reduced to $x^2 + \lambda(y^2 + z^2)$, $D = 1 + 2\lambda$, $S = 2\lambda + \lambda^2$, $\Delta = \lambda^3$, and therefore

$$4(S + 1) = (D + 1)^2 = 4(D + \Delta), \quad \text{and } \lambda = \frac{D-1}{2}.$$

(c.) The conditions that two quadrics, the coefficients in the equations of which are a, b, c , &c., and a', b', c' , &c., should be similar, may be obtained from the consideration that in this case the roots of $t^3 - Dt^2 + St - \Delta = 0$ must each be the same multiple of the corresponding root of $t^3 - D't^2 + S't - \Delta' = 0$, and they are

$$\frac{D}{D'} = \sqrt{\frac{S}{S'}} = \sqrt[3]{\frac{\Delta}{\Delta'}}.$$

(d.) It is shown in Salmon's *Geometry of Three Dimensions*, p. 155, that $D = 0$ is the condition that it should be possible to draw three rectangular asymptotic lines to the surface; and $S = 0$, that it should be possible to draw three rectangular asymptotic planes. This follows from the previous statement, since S is the D of U' , the polar reciprocal. $\Delta = 0$ is, as was pointed out in (IV.), the condition that $U = \mu$ should be a cylinder, and, since the roots of the discriminating cubic are all real,

$$4(D^3 - 3S)(S^2 - 3D\Delta) - (DS - 9\Delta)^2,$$

its discriminant with the sign changed, is positive.

IX. The invariants may also be used to show that every motion of a rigid body of which one point is fixed, is a rotation about an axis; and that the most general motion of a rigid body may be reduced to a twist about a screw. (Ball's *Theory of Screws*, p. 3.)

If a rigid body have one point fixed, and x, y, z ; ξ, η, ζ be the old and new coordinates of any other point in it referred to the same rectangular axes, with the fixed point as origin. Then

$$\xi = l_1 x + m_1 y + n_1 z, \quad \eta = l_2 x + m_2 y + n_2 z,$$

$$\zeta = l_3 x + m_3 y + n_3 z,$$

l_1, l_2, l_3 ; m_1, m_2, m_3 ; n_1, n_2, n_3 being the direction cosines of the three lines (fixed in the body), which originally coincided with the axes. Then $x\xi + y\eta + z\zeta$ is not altered by transformation to other axes with the same origin; and its equivalent

$$l_1 x^2 + m_2 y^2 + n_3 z^2 + (n_2 + m_3) yz + (l_3 + n_1) zx + (m_1 + l_2) xy$$

is also unaltered. It is easy to show that the invariants of this expression satisfy the conditions (VIII. b)

$$4(S+1) = (D+1)^2 = 4(\Delta+D),$$

and therefore the expression can be reduced, by a proper choice of axes, to the form

$$x^2 + \lambda (y^2 + z^2), \text{ where } \lambda = \frac{D-1}{2}.$$

Therefore, if x', y', z' and ξ, η, ζ be the old and new coordinates of the point whose original coordinates were x, y, z .

$$\xi = x', \quad \eta = \lambda y' - \mu z', \quad \zeta = \mu y' + \lambda z',$$

where $\lambda^2 + \mu^2 = 1$. So that the motion is reduced to a rotation about the axis of x' , through an angle

$$= \cos^{-1} \left(\frac{D-1}{2} \right).$$

If $x' = a_1 x + a_2 y + a_3 z$ and therefore $\xi = a_1 \xi + a_2 \eta + a_3 \zeta$, so that a_1, a_2, a_3 are the direction cosines of the axis of rotation; the equation $x' = \xi$ leads to

$$(l_1 - 1) a_1 + l_2 a_2 + l_3 a_3 = 0,$$

$$m_1 a_1 + (m_2 - 1) a_2 + m_3 a_3 = 0,$$

$$n_1 a_1 + n_2 a_2 + (n_3 - 1) a_3 = 0,$$

three equations which are consistent with each other, and which determine $a_1 : a_2 : a_3$; and the equations to the axis may be put in the form

$$(l_1 - 1) x + l_2 y + l_3 z = 0,$$

$$m_1 x + (m_2 - 1) y + m_3 z = 0,$$

$$n_1 x + n_2 y + (n_3 - 1) z = 0,$$

any two of which determine it, or

$$(n_2 + m_2) x = (l_2 + n_1) y = (m_1 + l_3) z.$$

Again, if x, y, z and ξ, η, ζ be the original and final coordinates of any point in a rigid body referred to the same fixed axes; the most general motion may be denoted by

$$\xi = u + l_1 x + m_1 y + n_1 z, \quad \eta = v + l_2 x + m_2 y + n_2 z,$$

$$\zeta = w + l_3 x + m_3 y + n_3 z,$$

l_1 , &c. having the same signification as before, and u, v, w being the displacements of the point originally at the origin parallel to the axes. Then the invariable quantity

$$x(\xi - u) + y(\eta - v) + z(\zeta - w)$$

may be reduced as above; so that, x', y', z' and ξ, η, ζ being the original and final coordinates referred to the new axes, which are determined exactly as before,

$$\xi = u' + x', \quad \eta = v' + \lambda y' - \mu z', \quad \zeta = w' + \mu y' + \lambda z';$$

and, by changing the origin to $(0, k, l)$, if x'', y'', z'' and ξ'', η'', ζ'' be the new coordinates,

$$\xi'' = u' + x'',$$

$$\eta'' = v' + \lambda (y'' + k) - \mu (z'' + l) = \lambda y'' - \mu z'',$$

$$\zeta'' = w' + \mu (y'' + k) + \lambda (z'' + l) = \mu y'' + \lambda z''.$$

If k and l be determined by the equations

$$\left. \begin{aligned} \lambda k - \mu l + v' &= 0 \\ \mu k + \lambda l + w' &= 0 \end{aligned} \right\} \text{ or } \left\{ \begin{aligned} k &= -(\lambda v' + \mu w') \\ l &= \mu v' - \lambda w' \end{aligned} \right.$$

this proves that the motion can always be reduced to a twist about a screw, whose pitch

$$= \frac{u'}{\cos^{-1}\left(\frac{D-1}{2}\right)}, \text{ where } u' = \frac{1}{\sqrt{-S}} \{(n_1 + m_1)u + (l_1 + n_1)v + (m_1 + l_1)w\},$$

and the axis of which is parallel to the line

$$(n_1 + m_1)x = (l_1 + n_1)y = (m_1 + l_1)z$$

through the point (x', y', z') , where

$$x' = 0, \quad y' = k, \quad z' = l.$$

X. Whatever may be the correct expression of stress in terms of strain, the invariants of stress and the combinants of strain and stress will be expressible in terms of the invariants of strain.

If, then, it be assumed that the stresses are linear functions of the strains (i.e., that these last are small), these conditions may be applied to find relations among the coefficients.

Let \bar{P} , \bar{Q} , \bar{R} , \bar{U} , \bar{V} , \bar{W} denote the stresses (chief stresses, say) which correspond to the principal strains A , B , C , and let

$$\bar{P} = p_1 A + p_2 B + p_3 C,$$

$$\bar{Q} = q_1 A + q_2 B + q_3 C,$$

$$\bar{R} = r_1 A + r_2 B + r_3 C,$$

$$\bar{U} = u_1 A + u_2 B + u_3 C,$$

$$\bar{V} = v_1 A + v_2 B + v_3 C,$$

$$\bar{W} = w_1 A + w_2 B + w_3 C.*$$

Then, denoting the invariants of stress by D' , S' , and Δ' , those of strain by D , S , and Δ , and the combinants by E , Θ , and Θ' ,—

$$D' = \alpha D \dots \dots \dots (1),$$

$$S' = \beta D^2 + \gamma S \dots \dots \dots (2),$$

$$\Delta' = \delta D^3 + \epsilon DS + \zeta \Delta \dots \dots \dots (3),$$

$$E = \beta' D^2 + \gamma' S \dots \dots \dots (4),$$

$$\Theta = \delta' D^3 + \epsilon' DS + \zeta' \Delta \dots \dots \dots (5),$$

$$\Theta' = \delta'' D^3 + \epsilon'' DS + \zeta'' \Delta \dots \dots \dots (6),$$

* The eighteen coefficients p_1 , p_2 , p_3 , q_1 , &c., together with three additional quantities to determine the direction of the principal axes, make up twenty-one quantities upon which the general values of the coefficients in the expressions for stress in terms of strain depend.

a series of equations which give

$$p_1 + q_1 + r_1 = p_2 + q_2 + r_2 = p_3 + q_3 + r_3, \text{ from (1),}$$

$$p_1 = q_2 = r_3, \quad p_2 + q_1 = p_3 + r_1 = q_3 + r_2, \text{ from (4),}$$

and from (5),

$$*p_2 = p_3 = q_1 = q_3 = r_1 = r_2,$$

a set of values which will satisfy all the equations when $\bar{U} = 0$, $\bar{V} = 0$, $\bar{W} = 0$, that is, when the principal axes of strain coincide with those of stress. In this case, if $p_1 = \&c. = k$, $p_2 = p_3 = \&c. = \mu$,

$$\bar{P} = kA + \mu(B + C),$$

$$\bar{Q} = kB + \mu(C + A),$$

$$\bar{R} = kC + \mu(A + B);$$

and hence, by transformation to any axes,

$$P = ka + \mu(b + c), \quad Q = kb + \mu(c + a), \quad R = kc + \mu(a + b),$$

$$U = (k - \mu)f, \quad V = (k - \mu)g, \quad W = (k - \mu)h;$$

in which expressions for the stresses the coefficients are independent of the direction of the axes. It follows that, in this case, the body is isotropic.† The mechanical meaning of the constants may be deduced from the following considerations:—

(i.) If the strain be such that $a + b + c = 0$, there is no expansion, and the strain is wholly distortion. In this case,

$$P = (k - \mu)a, \quad Q = (k - \mu)b, \quad R = (k - \mu)c, \quad \&c.,$$

$$\text{and} \quad E = (k - \mu)[a^2 + b^2 + c^2 + 2f^2 + 2g^2 + 2h^2]$$

$$= (k - \mu)[D^2 - 2S],$$

and $k - \mu$ must be the resistance to distortion.

(ii.) If the strain be a uniform dilatation, there will be no distortion.

In this case $E = 3(k + 2\mu)a^2 = (k + 2\mu)S$, and $k + 2\mu$ is the resistance to elongation,—which are in fact the known results in the case of isotropic bodies.

Returning to the general case, when the principal axes of strain and stress do not coincide, if, as before, k be put for p_1, q_2 , or r_3 , and μ for p_2, p_3 ,

* Hence $\bar{P}dA + \bar{Q}dB + \&c.$, is a perfect differential, and as this expression is an invariant, $Pda + Qdb + \&c.$, is always so, as remarked by Sir W. Thomson.

† This is the converse of Cauchy's assumption that the axes of strain and stress coincide in an isotropic body.

$q_1, q_2, r_1, \text{ or } r_2;$

$$\begin{aligned} S &= \overline{P} \cdot \overline{Q} + \overline{Q} \cdot \overline{R} + \overline{R} \cdot \overline{P} - \overline{U}^2 - \overline{V}^2 - \overline{W}^2 \\ &= (2k\mu + \mu^2 - u_1^2 - v_1^2 - w_1^2) A^2 \\ &\quad + (2k\mu + \mu^2 - u_2^2 - v_2^2 - w_2^2) B^2 + (2k\mu + \mu^2 - u_3^2 - v_3^2 - w_3^2) C^2 \\ &\quad + (k^2 + 2k\mu + 3\mu^2 - 2u_1u_2 - 2v_1v_2 - 2w_1w_2) A \cdot B \\ &\quad + (k^2 + 2k\mu + 3\mu^2 - 2u_2u_3 - 2v_2v_3 - 2w_2w_3) B \cdot C \\ &\quad + (k^2 + 2k\mu + 3\mu^2 - 2u_3u_1 - 2v_3v_1 - 2w_3w_1) C \cdot A, \end{aligned}$$

which, by (2), leads to

$$\begin{aligned} u_1^2 + v_1^2 + w_1^2 &= u_2^2 + v_2^2 + w_2^2 = u_3^2 + v_3^2 + w_3^2, \\ u_1u_2 + v_1v_2 + w_1w_2 &= u_2u_3 + v_2v_3 + w_2w_3 = u_3u_1 + v_3v_1 + w_3w_1, \\ \Theta' &= A\overline{Q}\overline{R} + B\overline{R}\overline{P} + C\overline{P}\overline{Q} - A\overline{U}^2 - B\overline{V}^2 - C\overline{W}^2 \\ &= A^2(\mu^2 - u_1^2) + B^2(\mu^2 - v_2^2) + C(\mu^2 - w_3^2) \\ &\quad + A^2B(\mu^2 + 2k\mu - 2u_1u_2 - v_1^2) + AB^2(\mu^2 + 2k\mu - 2v_1v_2 - u_2^2) \\ &\quad + A^2C(\mu^2 + 2k\mu - 2u_1u_3 - w_1^2) + \&c. \\ &\quad + ABC(3k^2 + 3\mu^2 - 2u_2u_3 - 2v_2v_1 - 2w_1w_2), \end{aligned}$$

and therefore, by (5),

$$\begin{aligned} u_1^2 &= v_2^2 = w_3^2 \quad \text{and} \quad 2u_1u_2 + v_1^2 = 2v_1v_2 + u_2^2 \\ &= 2u_1u_3 + w_1^2 = 2w_1w_2 + u_3^2 = 2v_2v_3 + w_3^2 = 2w_2w_3 + v_3^2, \end{aligned}$$

and

$$\begin{aligned} \Delta' &= \overline{P}\overline{Q}\overline{R} + 2\overline{U}\overline{V}\overline{W} - \overline{P}\overline{U}^2 - \overline{Q}\overline{V}^2 - \overline{R}\overline{W}^2 \\ &= A^2\{k\mu^2 + 2u_1v_1w_1 - ku_1^2 - \mu v_1^2 - \mu w_1^2\} \\ &\quad + B^2\{k\mu^2 + 2u_2v_2w_2 - kv_2^2 - \mu w_2^2 - \mu u_2^2\} \\ &\quad + C^2\{k\mu^2 + 2u_3v_3w_3 - kw_3^2 - \mu u_3^2 - \mu v_3^2\} \\ &\quad + AB\{k^2\mu + k\mu^2 + \mu^2 + 2u_1v_1w_2 + 2u_2v_1w_1 + 2u_1v_2w_1 \\ &\quad \quad - 2ku_1u_2 - \mu u_1^2 - kv_1^2 - 2\mu v_1v_2 - \mu w_1^2 - 2\mu w_1w_2\} \\ &\quad + AB^2\{k^2\mu + k\mu^2 + \mu^2 + 2u_2v_2w_1 + 2u_1v_2w_3 + 2u_2v_1w_2 \\ &\quad \quad - 2kv_1v_2 - \mu v_2^2 - ku_2^2 - 2\mu u_1u_2 - \mu w_2^2 - 2\mu w_1w_2\} \\ &\quad + A^2C\{k^2\mu + k\mu^2 + \mu^2 + 2u_1v_1w_3 + 2u_2v_1w_1 + 2u_1v_2w_1 \\ &\quad \quad - 2ku_1u_2 - \mu u_1^2 - kw_1^2 - 2\mu w_1w_2 - \mu v_1^2 - 2\mu v_1v_2\} \\ &\quad + \&c. \\ &\quad + ABC\{k^2 + 3k\mu^2 + 2\mu^2 + 2u_1v_1w_2 + 2u_2v_1w_3 + 2u_2v_2w_1 \\ &\quad \quad + 2u_1v_2w_3 + 2u_2v_3w_1 + 2u_3v_2w_1 - 2ku_2u_3 - 2\mu u_1u_2 - 2\mu u_2u_1 \\ &\quad \quad - 2kv_3v_1 - 2\mu v_1v_2 - 2\mu v_2v_3 - 2kw_1w_2 - 2\mu w_2w_1 - 2\mu w_3w_2\}, \end{aligned}$$

which, by (3), using the previous equations among the u 's, v 's, and

w 's, shows that

$$\begin{aligned} u_2 v_1 w_1 + u_1 v_2 w_1 + u_1 v_1 w_2 &= u_2 v_1 w_1 + u_1 v_2 w_1 + u_1 v_1 w_2 \\ &= u_1 v_2 w_2 + u_2 v_1 w_2 + u_2 v_2 w_1 = \&c., \end{aligned}$$

and $u_1 v_1 w_1 = u_2 v_2 w_2 = u_3 v_3 w_3,$

and therefore $(u_1 - u_2)(v_1 - v_2)(w_1 - w_2) = 0,$

$$(u_1 - u_2)(v_1 - v_3)(w_1 - w_3) = 0,$$

$$(u_2 - u_3)(v_2 - v_3)(w_2 - w_3) = 0,$$

and ultimately $u_1^2 = v_2^2 = w_3^2 = \lambda^2$ suppose,

$$u_2^2 = u_3^2 = v_1^2 = v_3^2 = w_1^2 = w_2^2 = \nu^2,$$

all the u 's, all the v 's, and all the w 's being respectively of the same sign.

Assuming that they all have the positive sign, the values of the chief strains are

$$\bar{P} = kA + \mu(B + C),$$

$$\bar{Q} = kB + \mu(C + A),$$

$$\bar{R} = kC + \mu(A + B),$$

$$\bar{U} = \lambda A + \nu(B + C),$$

$$\bar{V} = \lambda B + \nu(C + A),$$

$$\bar{W} = \lambda C + \nu(A + B),$$

and those of the invariants and combinants are

$$D' = (k + 2\mu) D,$$

$$\begin{aligned} S' &= (2k\mu + \mu^2 - \lambda^2 - \nu^2) D^2 \\ &\quad + \{(k - \mu)^2 + 2(\lambda - \nu)^2\} S, \end{aligned}$$

$$\begin{aligned} \Delta' &= (k\mu^2 - k\lambda^2 + 2\lambda\nu^2 - 2\mu\nu^2) D^3 \\ &\quad + (k^2\mu - 2k\mu^2 + \mu^3 + 3k\lambda^2 - 2k\lambda\nu - k\nu^2 + 2\lambda^2\nu \\ &\quad - 4\lambda\nu^2 + 2\nu^3 - 2\mu\lambda^2 - 2\mu\lambda\nu + 4\mu\nu^2) DS \\ &\quad + (k^3 - 3k^2\mu + 3k\mu^2 - \mu^3 - 3k\lambda^2 + 6k\lambda\nu - 3k\nu^2 \\ &\quad + 2\lambda^3 - 6\lambda^2\nu + 6\lambda\nu^2 - 2\nu^3 + 6\mu\lambda^2 - 6\mu\lambda\nu) \Delta, \end{aligned}$$

$$E = kD^2 - 2(k - \mu) S,$$

$$\Theta = 3(k - \mu) \Delta + \mu DS,$$

$$\begin{aligned} \Theta' &= (\mu^2 - \lambda^2) D^3 + (2k\mu - 2\mu^2 + 3\lambda^2 - 2\lambda\nu - \nu^2) DS \\ &\quad + 3\{(k - \mu)^2 - (\lambda - \nu)^2\} \Delta. \end{aligned}$$

XI. The same method is no longer practically available, when the axes are not principal. For, since the quadric of which the coefficients are a , b , &c., has principal axes, there will be a set of quantities of the

nature of covariants; that is, if $l_1 m_1 n_1$, $l_2 m_2 n_2$, $l_3 m_3 n_3$ be the direction cosines of the principal axes,

$$F = 0 = a l_3 l_2 + b m_2 m_3 + c n_2 n_3 + f (m_2 n_2 + m_3 n_3) \\ + g (n_2 l_2 + n_3 l_3) + h (l_2 m_2 + l_3 m_3),$$

$$G = 0 = a l_3 l_1 + b m_2 m_1 + c n_2 n_1 + f (m_1 n_2 + m_2 n_1) \\ + g (n_2 l_1 + n_1 l_2) + h (l_2 m_1 + l_1 m_2),$$

$$H = 0 = a l_2 l_1 + b m_1 m_2 + c n_1 n_2 + f (m_1 n_2 + m_2 n_1) \\ + g (n_1 l_2 + n_2 l_1) + h (l_1 m_2 + l_2 m_1),$$

as well as the expressions corresponding to these and derived from the covariant quadrics, which are, however, evidently not independent relations like those above. These all vanished identically when the axes were principal, and did not require to be noticed, but in any other case functions of the values of F , G , and H , must be added to the expressions for the components of stress. This would introduce an unmanageable number of terms into the equations, which, without altering the expressions for the invariants and combinants of stress, would affect the coefficients of the various strain components in the values of those of stress.

The coefficients for any axes may, however, be determined as follows, in terms of k , μ , λ , and ν , the constants already employed, and the direction cosines of the principal axes of strain with respect to those to which the strain is actually referred, and which involve three additional constants. Let these be $(l_1 m_1 n_1)$, &c., so that $(l_1 l_2 l_3)$, $(m_1 m_2 m_3)$, $(n_1 n_2 n_3)$ are the direction cosines of the axes referred to the principal axes. Then

$$P = \bar{P} l_1^2 + \bar{Q} l_2^2 + \bar{R} l_3^2 + \bar{U} l_2 l_3 + 2 \bar{V} l_2 l_1 + 2 \bar{W} l_1 l_3 \\ = A (k l_1^2 + \mu l_2^2 + \mu l_3^2 + 2 \lambda l_2 l_3 + 2 \nu l_2 l_1 + 2 \nu l_1 l_3) \\ + B (\mu l_1^2 + k l_2^2 + \mu l_3^2 + 2 \nu l_2 l_3 + 2 \lambda l_2 l_1 + 2 \nu l_1 l_2) \\ + C (\mu l_1^2 + \mu l_2^2 + k l_3^2 + 2 \nu l_2 l_3 + 2 \nu l_2 l_1 + 2 \lambda l_1 l_2) \\ \equiv P_1 A + P_2 B + P_3 C \text{ suppose} \\ = P_1 (a l_1^2 + b m_1^2 + c n_1^2 + 2 f m_1 n_1 + 2 g n_1 l_1 + 2 h l_1 m_1) \\ + P_2 (a l_2^2 + b m_2^2 + c n_2^2 + 2 f m_2 n_2 + 2 g n_2 l_2 + 2 h l_2 m_2) \\ + P_3 (a l_3^2 + b m_3^2 + c n_3^2 + 2 f m_3 n_3 + 2 g n_3 l_3 + 2 h l_3 m_3) \\ = a (P_1 l_1^2 + P_2 l_2^2 + P_3 l_3^2) + b (P_1 m_1^2 + P_2 m_2^2 + P_3 m_3^2) \\ + c (P_1 n_1^2 + P_2 n_2^2 + P_3 n_3^2) + 2 f (P_1 m_1 n_1 + P_2 m_2 n_2 + P_3 m_3 n_3) \\ + 2 g (P_1 n_1 l_1 + P_2 n_2 l_2 + P_3 n_3 l_3) + 2 h (P_1 l_1 m_1 + P_2 l_2 m_2 + P_3 l_3 m_3),$$

with similar expressions for Q and R .

$$\begin{aligned}
U &= \overline{P}m_1n_1 + \overline{Q}m_2n_2 + \overline{R}m_3n_3 + \overline{U}(m_2n_2 + m_3n_3) \\
&\quad + \overline{V}(m_2n_1 + m_1n_2) + \overline{W}(m_1n_2 + m_2n_1) \\
&= A \{ km_1n_1 + \mu m_2n_2 + \mu m_3n_3 + \lambda (m_2n_2 + m_3n_3) \\
&\quad + \nu (m_2n_1 + m_1n_2) + \nu (m_1n_2 + m_2n_1) \} \\
&\quad + B \{ \mu m_1n_1 + k m_2n_2 + \mu m_3n_3 + \nu (m_2n_2 + m_3n_3) \\
&\quad + \lambda (m_2n_1 + m_1n_2) + \nu (m_1n_2 + m_2n_1) \} \\
&\quad + C \{ \mu m_1n_1 + \mu m_2n_2 + k m_3n_3 + \nu (m_2n_2 + m_3n_3) \\
&\quad + \nu (m_2n_1 + m_1n_2) + \lambda (m_1n_2 + m_2n_1) \} \\
&\equiv U_1A + U_2B + U_3C \text{ suppose} \\
&= a (U_1l_1^2 + U_2l_2^2 + U_3l_3^2) + b (U_1m_1^2 + U_2m_2^2 + U_3m_3^2) \\
&\quad + c (U_1n_1^2 + U_2n_2^2 + U_3n_3^2) + 2f (U_1m_1n_1 + U_2m_2n_2 + U_3m_3n_3) \\
&\quad + 2g (U_1n_1l_1 + U_2n_2l_2 + U_3n_3l_3) + 2h (U_1l_1m_1 + U_2l_2m_2 + U_3l_3m_3),
\end{aligned}$$

with similar values for V and W .

In the particular case when the strain is expressed in the form suggested by Thomson and Tait (see VIII. a),

$$\begin{aligned}
l_1 &= 1, \quad l_2 = 0, \quad l_3 = 0; \quad m_1 = 0, \quad m_2 = \frac{1}{\sqrt{2}}, \quad m_3 = \frac{1}{\sqrt{2}}, \\
n_1 &= 0, \quad n_2 = \frac{1}{\sqrt{2}}, \quad n_3 = -\frac{1}{\sqrt{2}}.
\end{aligned}$$

If a, b, c , and f be the components of the strain, and A, B, C the principal elongations,

$$a = A, \quad b = c = \frac{B+C}{2}, \quad f = \frac{B-C}{2},$$

$$\text{and therefore} \quad A = a, \quad B = \frac{b+c+2f}{2}, \quad C = \frac{b+c-2f}{2},$$

$$\text{therefore} \quad P_1 = k, \quad P_2 = \mu = P_3,$$

$$\text{therefore} \quad P = kA + \mu(B+C) = ka + \mu(b+c) = ka + 2\mu b.$$

$$Q_1 = \mu + \lambda, \quad Q_1 = \frac{k+\mu+2\nu}{2} = Q_3,$$

$$\text{therefore} \quad Q = (\mu + \lambda)A + \frac{k+\mu+2\nu}{2}(B+C)$$

$$= (\mu + \lambda)a + \frac{k+\mu+2\nu}{2}(b+c) = (\mu + \lambda)a + (k + \mu + 2\nu)b.$$

$$R_1 = \mu - \lambda, \quad R_2 = \frac{k+\mu-2\nu}{2} = R_3,$$

$$\begin{aligned}\text{therefore } R &= (\mu - \lambda) A + \frac{k + \mu - 2\nu}{2} (B + C) \\ &= (\mu - \lambda) a + \frac{k + \mu - 2\nu}{2} (b + c) = (\mu - \lambda) a + (k + \mu - 2\nu) b.\end{aligned}$$

$$U_1 = 0, \quad U_2 = \frac{k - \mu}{2} = -U_3,$$

$$\text{therefore} \quad U = \frac{k - \mu}{2} (B - C) = (k - \mu) f.$$

$$V_1 = 0, \quad V_2 = \frac{\nu - \lambda}{\sqrt{2}} = -V_3,$$

$$\text{therefore} \quad V = \frac{\nu - \lambda}{\sqrt{2}} (B - C) = (\nu - \lambda) f \sqrt{2}.$$

$$W_1 = \nu \sqrt{2}, \quad W_2 = \frac{\nu + \lambda}{\sqrt{2}} = W_3,$$

$$\begin{aligned}\text{therefore} \quad W &= \nu \sqrt{2} A + \frac{\nu + \lambda}{\sqrt{2}} (B + C) \\ &= \nu \sqrt{2} a + (\nu + \lambda) \sqrt{2} b.\end{aligned}$$

XII. As in the case of the simpler functions of section VII., these transformations may be effected by means of a complex function.

For, if in

$$\begin{aligned}& (kx^2 + \mu y^2 + \mu z^2 + 2\lambda yz + 2\nu zx + 2\nu xy) \xi^2 \\ & + (\mu x^2 + ky^2 + \mu z^2 + 2\nu yz + 2\lambda zx + 2\nu xy) \eta^2 \\ & + (\mu x^2 + \mu y^2 + kx^2 + 2\nu yz + 2\nu zx + 2\lambda xy) \zeta^2\end{aligned}$$

the coordinates (x, y, z) be transformed from axes parallel to the principal axes to others, the result will be (if $x = l_1 x' + m_1 y' + n_1 z'$, &c.)

$$\begin{aligned}& (P_1 \xi^2 + P_2 \eta^2 + P_3 \zeta^2) x'^2 + (Q_1 \xi^2 + Q_2 \eta^2 + Q_3 \zeta^2) y'^2 \\ & + (R_1 \xi^2 + R_2 \eta^2 + R_3 \zeta^2) z'^2 + 2 (U_1 \xi^2 + U_2 \eta^2 + U_3 \zeta^2) y' z' \\ & + 2 (V_1 \xi^2 + V_2 \eta^2 + V_3 \zeta^2) z' x' + 2 (W_1 \xi^2 + W_2 \eta^2 + W_3 \zeta^2) x' y';\end{aligned}$$

and if this be again transformed by the equations $\xi = l_1 \xi' + m_1 \eta' + n_1 \zeta'$, &c., the result is

$$\begin{aligned}& \{ (P_1 l_1^2 + P_2 l_2^2 + P_3 l_3^2) x'^2 + (Q_1 l_1^2 + Q_2 l_2^2 + Q_3 l_3^2) y'^2 \\ & + (R_1 l_1^2 + R_2 l_2^2 + R_3 l_3^2) z'^2 + 2 (U_1 l_1^2 + U_2 l_2^2 + U_3 l_3^2) y' z' \\ & + 2 (V_1 l_1^2 + V_2 l_2^2 + V_3 l_3^2) z' x' + 2 (W_1 l_1^2 + W_2 l_2^2 + W_3 l_3^2) x' y' \} \xi'^2 \\ & + \&c. \\ & + 2 \{ (P_1 m_1 n_1 + P_2 m_2 n_2 + P_3 m_3 n_3) x'^2 + (Q_1 m_1 n_1 + Q_2 m_2 n_2 + Q_3 m_3 n_3) y'^2 \\ & + (R_1 m_1 n_1 + R_2 m_2 n_2 + R_3 m_3 n_3) z'^2 + 2 (U_1 m_1 n_1 + U_2 m_2 n_2 + U_3 m_3 n_3) y' z' \\ & + 2 (V_1 m_1 n_1 + V_2 m_2 n_2 + V_3 m_3 n_3) z' x' \\ & + 2 (W_1 m_1 n_1 + W_2 m_2 n_2 + W_3 m_3 n_3) x' y' \} \eta' \zeta' \\ & + \&c.,\end{aligned}$$

where P_1 , &c. have the same values as in the last article; so that this is

$$\begin{aligned} & (p_1x^2 + q_1y^2 + r_1z^2 + 2u_1y'z' + 2v_1z'x' + 2w_1x'y') \xi'^2 \\ & + (p_2x^2 + q_2y^2 + \&c.) \eta'^2 + (p_3x^2 + q_3y^2 + \&c.) \zeta'^2 \\ & + 2(p_4x^2 + q_4y^2 + r_4z^2 + 2u_4y'z' + 2v_4z'x' + 2w_4x'y') \eta'\zeta' \\ & + \&c., \end{aligned}$$

$$\begin{aligned} \text{or } & (p_1\eta'^2 + p_2\eta'^2 + p_3\zeta'^2 + 2p_4\eta'\zeta' + 2p_5\zeta'\xi' + 2p_6\xi'\eta') x'^2 \\ & + (q_1\xi'^2 + \&c.) y'^2 + (r_1\zeta'^2 + \&c.) z'^2 \\ & + 2(u_1\xi'^2 + u_2\eta'^2 + u_3\zeta'^2 + 2u_4\eta'\zeta' + 2u_5\zeta'\xi' + 2u_6\xi'\eta') y'z' \\ & + 2(v_1\xi'^2 + \&c.) z'x' + 2(w_1\xi'^2 + \&c.) x'y', \end{aligned}$$

$$\begin{aligned} \text{where } P &= p_1a + p_2b + p_3c + 2p_4f + 2p_5g + 2p_6h, \\ &\quad \&c. \qquad \qquad \&c. \qquad \qquad \&c. \end{aligned}$$

This method of transformation applies whatever initial values of the coefficients are adopted. Also, since $Pda + \&c.$ is an exact differential, it follows that the determinant

$$\begin{vmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

must be symmetrical, as it is with the values already found for the principal coefficients.

[Since the paper, from which the above is expanded, was read before the Society, my attention has been drawn to the fact that Professor C. Niven had, in a paper read to the Royal Society of Edinburgh (*Transactions*, Vol. 27, p. 473), anticipated me in noticing the existence of the invariants D , S , and Δ , and some of the consequences of this.]

In the Recess, Cartes-de-Visite likenesses of Miss C. A. Scott, and of Profs. Donkin and Rankine (the last two presented by Mr. R. F. Scott), were received.

The following books were also presented to the library:—

"Annali di Matematica," Serie ii^a, Tomo xi., Fasc. 1^o, Luglio, 1882.

"American Journal of Mathematics," Vol. iv., No. 4, Dec. 1881; Baltimore, 1882.

"Di alcuni Integrali Formati dagl' Integrali Ellittici e di qualche loro applicazione," Nota di G. Torelli; Napoli, 1873.

"Intorno agl' Integrali Ellittici considerati come funzioni del modulo," per G. Torelli. (No date.)

"Sulle funzioni simmetriche complete e semplice," per G. Torelli. (No date.)

"Sopra alcune proprietà numeriche," Memoria per G. Torelli; Napoli, 1878.

"Sui Determinanti Circolanti," Nota di G. Torelli.

"A Treatise on the Theory of Determinants, with Graduated Lists of Exercises, for use in Colleges and Schools," by Thomas Muir, M.A., F.R.S.E.; London, 1882 (from the Author).

- "Archiv for Mathematisk og Naturvidenskab" (S. Lie, W. Müller, and G. O. Sars), Sjette Bind, Tredie Hefte, Fjerde Hefte ; Kristiania, 1882.
- "Transactions of the Connecticut Academy of Arts and Sciences," Vol. iv., Pt. 2, Vol. v., Pt. 2 ; Newhaven, 1882.
- "Annual Report of the Board of Regents of the Smithsonian Institution for the year 1880 ;" Washington, 1881.
- "Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti," Tomo ottavo, Serie quinta, Dispensa prima, seconda, terza, quarta, quinta, sesta ; Venezia, 1881-82.
- "Bulletin de la Société Mathématique de France," Tome ii., 1, 2, 3 ; v., 2 ; vii., 5 ; viii., 2, 3, 4, 6 ; ix., 1, 2, 4 ; x., 1, 2, 4.
- "Transactions of the Royal Irish Academy—Science," Vol. xxviii., vi. (Oct., 1881), vii. (Nov., 1881), viii., ix. (April, 1882), x. (June, 1882).
- "Proceedings of the Royal Irish Academy—Science," Vol. iii., Series ii., No. 7 (Dec., 1881), No. 8 (May, 1882). "Polite Literature and Antiquities," Vol. ii., Series ii., No. 3 (Dec., 1881).
- Carr's "Synopsis of Results in Pure and Applied Mathematics," Vol. i. and §§ viii. and ix. ; Hodgson, 1882.
- "Crelle," 92 Band, 3^{te}, 4^{te} Heft.
- "Educational Times," July, 1882.
- "Bulletin des Sciences Mathématiques et Astronomiques," Tome vi., Janvier, 1882.
- "Atti della R. Accademia dei Lincei—Transunti," Vol. vi., Fasc. 11^o, Aprile 1882.
- "Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xvii., 1^{me}, 2^{me} Livr. ; Harlem, 1882.
- "The Proceedings of the Royal Society," Vol. xxxiv., No. 220.
- "Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," i.—xvii. ; Berlin, 1882.
- "Grundriss einer elementar-geometrischen Kegelschnittslehre," von H. G. Zeuthen ; Leipzig, 1882.
- "Bulletin Astronomique et Météorologique de l'Observatoire Impérial de Rio de Janeiro," Janvier (No. 1), Févr. (No. 2), Mars (No. 3) ; Rio, 1882.
- "The Physical Society of London—Proceedings," Vol. v., Pt. i., January to June, 1882 ; London, 1882.
- "Beiblätter zu den Annalen der Physik und Chemie," Band vi., Stück vi. ; Leipzig, 1882.
- "Jahrbuch über die Fortschritte der Mathematik," zwölfter Band, Jahrgang 1880, Heft 1 ; Berlin, 1882.

APPENDIX.

THE joint paper by Messrs. Jenkins and Merrifield (p. 3) was withdrawn, in consequence of its having been pointed out that the authors had been anticipated in their result by M. Lejeune-Dirichlet, in a paper "Ueber die Bestimmung . . . in der Zahlen Theorie" (see the Berlin *Abhandlungen*, for 1849, pp. 77, &c. ; the *Monatsbericht* for 1838 contains also a paper on the same subject by the same writer).