of discontinuity. The function may have values differing by a finite quantity on either side of such a curve; or its values at points along the curve may be discontinuous, or both of these kinds of discontinuity may be combined at the same curve. If L (σ), the total length of the curves at which the discontinuities surpass σ, be finite, the function can be integrated over the given space; since, if we draw curves parallel to the curves of discontinuity and at a distance d from them on either side, the area of the channel-like spaces thus obtained will be $2d\text{L}(\sigma)$, and will surpass the greatest sum of spaces, including the curves in any division of norm d. But the function may be integrable even if the total length of the curves of discontinuity is infinite; because an infinite number of contiguous curves may be enclosed in one and the same channel. And, provided that the curves can all be included in channels of which the length is L, and of which the breadth δ is comminuent with d, the condition that $\text{L} \times \delta$ should be comminuent with d, will suffice to ensure the integrability of the function.*

On the Higher Singularities of Plane Curves.

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[*Communicated (in part) Feb. 13th,* 1873; *supplemented by a Note, April 9th,* 1874; *and presented to the Society, Jan.* 1876.]

The ordinary singularities of a plane curve are its double points and double tangents, its stationary points and stationary tangents; or, as they have been also called, its nodes and links, its cusps and inflexions. The fundamental theorem, that any of the so-called higher singularities of a plane curve may be regarded as equivalent to a certain number of ordinary singularities of each of these four kinds, has been enunciated by Professor Cayley, who has also given a method for determining in every case the four indices δ, τ, κ, ι, proper to any given singularity.

Several enquiries, which appear to possess some interest, are suggested by this theorem. Among them we may mention the two following—

(1). It is important to prove that the indices of singularity, as defined by Professor Cayley, satisfy the equations of Plücker; and that the "genus" or "deficiency" of the plane curve is correctly given by these indices.

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* This Paper, though it was not read, was offered to the Society and accepted in the usual manner.

(2). It is also of interest to examine whether any given singularity can be actually formed by the coalescence of the ordinary singularities to which it is regarded as equivalent; in other words, whether a singularity of which the indices are δ, τ, κ, ι, and which is therefore to be regarded as equivalent to δ double points, τ double tangents, κ cusps, and ι inflexions, possesses a penultimate form, in which all these singularities exist, distinct from one another, but infinitely close together.

The present paper relates chiefly to the first of these enquiries; the second is reserved for a future communication.

1. Consider a plane curve C of order m and class n, defined by an equation $F(p, q) = 0$ between the parameters of two pencils, of which the corresponding rays intersect on C, and which are represented by equations of the form $p\,(QP) + (QR) = 0$, $q\,(PQ) + (PR) = 0$; P, Q, R denoting the three vertices of a triangle, $(PQ) = 0$, $(QR) = 0$, $(PR) = 0$ the equations of its sides. It is convenient to suppose that Q and P, the centres of the two pencils, have no speciality of position with regard to C; or, more precisely, that neither Q nor P lies on the curve, nor on any singular line appertaining to the curve. Under the general name of singular lines we include (1) lines joining two singular points, (2) singular tangents, (3) tangents at singular points, (4) tangents passing through a singular point; we shall also suppose that PQ is not a tangent to C, and does not pass through any singular point. Thus to every finite value of p there will correspond m finite values of q, and *vice versâ;* and, in particular, to any singular point on the curve there will correspond a finite pair of values of p and q. To an infinite value of q there will correspond m infinite values of p, and *vice versâ;* these answer to the m intersections of PQ with the curve, no two of which, by hypothesis, are coincident. We may, if we please, project the line PQ to an infinite distance, and regard p and q as Cartesian coordinates; we prefer, however, for our present purpose, to consider them as parametric ratios; *i. e.*, as purely numerical quantities (real or complex).

2. Let $f(q)$ be the discriminant of $F(p, q) = 0$, considered as an equation of the order m in p; we may suppose the coefficient of p^m, which is certainly different from zero, to be unity. The first polar of P with regard to C is $\frac{dF}{dp} = 0$, and $f(q)$ is the resultant of the elimination of p from F and $\frac{dF}{dp}$, so that the roots of $f(q) = 0$ are the parameters of the lines drawn from P to the points of intersection of C with the first polar of P. Attending to the suppositions which have been made as to the situation of P and Q relatively to the curve C, we infer (α) that $f(q)$ has no infinite roots, and is therefore of the

full order $m(m-1)$ in q; (β) that $f(q)$ has n, and only n, non-multiple roots q'; (γ) that for each of these n roots q' the equation $F(p, q') = 0$ acquires two equal roots p', its remaining roots being all different from p', and from one another; (δ) that q' is not a multiple root of the equation $F(p', q) = 0$. The n sets (p', q') give the n points of contact of tangents from P; the remaining factor of $f(q)$, viz., $f_1(q) = f(q) \div \Pi(q-q')$, consists exclusively of multiple factors, and appertains to the singular points of the curve. The index of its order, *i. e.*, $m(m-1)-n$, we may term the *total discriminantal index of the singular points of the curve.* Let q_0 be a root of $f_1(q) = 0$ of multiplicity ν; the equation $F(p, q_0) = 0$ has but one multiple root; let this be p_0, and let its multiplicity be μ; then (p_0, q_0) is a singular point O on the curve, of which the *order* (*i. e.*, the least number of points in which it is cut by any straight line passing through it) is μ, and of which ν may be termed the *discriminantal index.* It is evident that the number of singular points is equal to the number of unequal roots of $f_1(q) = 0$, and that the total discriminantal index is equal to the sum of the discriminantal indices of the separate singular points. We shall presently (Art. 8) see that the discriminantal index of a singular point can in general be further subdivided into parts, appertaining respectively to the different branches of the curve which pass through the point, taken singly, and in pairs.

3. It is a well-known theorem of Cauchy, that so long as the analytical modulus of $q-q_0$ is less than the least of the modules of any of the quantities q_1-q_0, where q_1 is any root of $f(q) = 0$ other than q_0, the m roots of the equation $F(p, q) = 0$ are developable in convergent series of the form

$$(A) \ldots\ldots p-p_0 = A + A_0(q-q_0) + A_1(q-q_0)^{\alpha_1} + A_2(q-q_0)^{\alpha_2} + \ldots\ldots,$$

the exponents $\alpha_1, \alpha_2, \ldots\ldots$ being rational and positive numbers, which satisfy the inequalities $1 < \alpha_1 < \alpha_2 < \ldots\ldots$ Of the equations (A), $m-\mu$ give the values of p corresponding to the $m-\mu$ points not in the vicinity of O, in which C is cut by the line (q). The series in the right hand members of these $m-\mu$ equations we shall designate by $\bar{A}_1, \bar{A}_2, \ldots \bar{A}_{m-\mu}$: we observe that in them the quantities A are all different from one another and from zero; because (q_0), not being a singular line, intersects C in $m-\mu$ points, which are different from one another and from O; also, in these equations, the exponents $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots$ are integral. In the remaining μ equations, which give the developments appertaining to the branches of C that pass through O, the quantities A are all equal to zero: these equations divide themselves into groups of conjugate equations, the equations of any one group being of the type

$$p-p_0 = B_0(q-q_0) + B_1\,\omega^{\beta_1}(q-q_0)^{\frac{\beta_1}{\lambda}} + \ldots\ldots,$$

where the numerators β are positive, integral, and increasing; Δ is less than β_1, and is the least common denominator of the fractional exponents; ω is any root of $\omega^\Delta = 1$: so that, if we use one and the same value of the radical in all the Δ equations of the group, they will differ from one another only by containing different values of ω; each of the μ equations defines a branch of the curve passing through O. If $\Delta = 1$, the branch is linear or of order 1; if $\Delta > 1$, the Δ conjugate equations are regarded by Professor Cayley as defining Δ partial branches forming a single superlinear branch of order Δ; in every case the sum of the orders of the branches is equal to the order of the point, *i.e.*, $\Sigma\Delta = \mu$. The coefficients B_0 are all different from zero, and the indices $\frac{\beta_1}{\Delta}$ are all greater than unity, because neither (p_0) nor (q_0) is one of the tangents at O; but these coefficients and indices are not necessarily different in two developments belonging to two different linear or superlinear branches—indeed any two such developments may coincide for any finite number of terms; and to ascertain the true nature of any singular point it is indispensable to continue the developments until they all become different from one another. The series in the right-hand members of the μ equations we denote by $\overline{B}_1, \overline{B}_2, \overline{B}_3 \ldots\ldots \overline{B}_\mu$.

4. The series $\overline{A}$ and $\overline{B}$ of the preceding article are *absolutely* convergent within the assigned limits; *i.e.*, any one of these series would continue to be convergent within these limits if its terms were replaced by their analytical modules. For the multiplication of two absolutely convergent series we have the theorem:—

"If the product of two given absolutely convergent series, proceeding by ascending powers of a variable, be arranged in a series proceeding in the same manner, this series is absolutely convergent for all values of the variable for which the given series are absolutely convergent, and its sum is equal to the product of the sums of the given series." (Cauchy, "Analyse Algébrique," cap. vi.)

Multiplying together the m series $p-p_0-\overline{A}$, and $p-p_0-\overline{B}$, we obtain, by virtue of this theorem, the equation

$$F(p, q) = \Pi(p-p_0-\overline{A}) \times \Pi(p-p_0-\overline{B}).$$

This equation is an identity; *i.e.*, if the multiplication be actually effected on the right-hand side, all powers of $q-q_0$ above the m^{th} will disappear, and the terms that remain will be precisely the terms of $F(p_0+\overline{p-p_0}, q_0+\overline{q-q_0})$ or $F(p, q)$. But an arithmetical equality between the two sides of the equation subsists only so long as the analytical modulus of $q-q_0$ does not surpass the limit assigned in Cauchy's theorem (Art. 3). Subject to the same limitation, $f(q)$ is equal to the product of the squares of the differences of every two of the series $\overline{A}$ and $\overline{B}$.

5. The number of the intersections at any point O of two branches of the same curve, or of different curves, which pass through the point, and which are there represented by equations of the form

$$p^{(1)}-p_0 = B_0^{(1)}(q-q_0)+ \ldots\ldots$$
$$p^{(2)}-p_0 = B_0^{(2)}(q-q_0)+ \ldots\ldots$$

is defined by Professor Cayley to be the number which expresses the order of evanescence of $p^{(1)}-p^{(2)}$, *i.e.*, the integral or fractional exponent λ for which $\frac{p^{(1)}-p^{(2)}}{(q-q_0)^\lambda}$ λ has a finite limit, when $q-q_0$ is diminished without limit. We may justify this definition by proving that, whenever two curves C_1 and C_2 have a multiple intersection at any point, its multiplicity is correctly obtained by adding together the numbers (as thus defined) of the intersections of each branch of C_1 by each branch of C_2. If we suppose (as we may do) that the points P and Q have no speciality of position with regard to the curves C_1 and C_2 considered as one curve, the resultant $\Phi(q)$ of the equations $C_1(p, q) = 0$ and $C_2(p, q) = 0$ is of the order $m_1 \times m_2$; and if μ_1 branches of C_1 and μ_2 branches of C_2 pass through O, we shall have, for C_1, $m_1-\mu_1$ equations $\overline{A}^{(1)}$, and μ_1 equations $\overline{B}^{(1)}$; and similarly, for C_2, $m_2-\mu_2$ equatious $\overline{A}^{(2)}$, and μ_2 equations $\overline{B}^{(2)}$. Denoting by $\Pi.(\overline{B}^{(1)}-\overline{B}^{(2)})$ the product of the $\mu_1\times\mu_2$ differences obtained by subtracting them in succession, each series $\overline{B}^{(2)}$ from each series $\overline{B}^{(1)}$, and by λ the number of intersections, as above defined, of any one branch of C_1 by any one branch of C_2, we see that the limit of $\Pi(\overline{B}^{(1)}-\overline{B}^{(2)}) \div (q-q_0)^{\Sigma\lambda}$ is finite. But $\Phi(q) = \Pi(\overline{B}^{(1)}-\overline{B}^{(2)})$, the sign of multiplication now extending to all the $m_1 \times m_2$ differences obtained by considering the m_1 series $\overline{A}^{(1)}$ and $\overline{B}^{(1)}$, and the m_2 series $\overline{A}_2$ and $\overline{B}_2$; and of these $m_1 \times m_2$ differences, none, except the $\mu_1 \times \mu_2$ differences already considered, are evanescent with $q-q_0$ (for the hypothesis that P and Q have no speciality of position with regard to the system of the two curves C_1 and C_2 implies that none of the constants $A^{(1)}$ can be equal to any of the constants $A^{(2)}$). Hence $\Sigma\lambda$ is the multiplicity of the factor $q-q_0$ in $\Phi(q)$; *i.e.*, since (p_0, q_0) is the only intersection of C_1 and C_2 which lies on (q_0), $\Sigma\lambda$ is the multiplicity of that intersection.

If we regard the equation $F(p, q) = 0$ as determining a correspondence of points on a line, the coincidences of corresponding points (except indeed the coincidences $p = q = \infty$) answer in number and multiplicity to the intersections of C by the straight line $p = q$. We are thus led to a theorem given by M. Zeuthen (Bulletin des Sciences Mathématiques, Vol. V., p. 186).

6. As it is only the hypothesis that the points P and Q have no

speciality of position with regard to C, which gives us a right to assert that every one of the developments $\overline{\mathrm{B}}$ contains a term linear in $q-q_0$, and no term in which the exponent of $q-q_0$ is less than unity, it is worth while to see how far the results of the preceding article can be depended on when this hypothesis is dispensed with. It will be found that if (p_0) is one of the tangents at O, *i.e.*, if one, or more, of the coefficients B_0 is zero, the discriminantal index of the point is still equal to the order of evanescence of $\Pi\,(\overline{\mathrm{B}}_i-\overline{\mathrm{B}}_j)^2$. But this conclusion would no longer hold, if (q_0) were one of the tangents at O. In this case the developments appertaining to the branches to which (q_0) is a tangent would contain powers of $q-q_0$ inferior to unity; and the order of evanescence of $\Pi\,(\overline{\mathrm{B}}_i-\overline{\mathrm{B}}_j)^2$ would exceed the discriminantal index of O by r, if $n-r$ is the number of tangents other than (q_0) which can be drawn to the curve from P. But the multiplicity of the intersection of two different curves, at a point which is singular for one or both of them is correctly obtained by the process of Art. 5, even when the developments contain positive powers of $(q-q_0)$ inferior to unity. Thus, in the curve $p-p_0=(q-q_0)^{\frac{1}{a}}$, a being an integer, the order of evanescence of $\Pi[\overline{\mathrm{B}}_i-\overline{\mathrm{B}}_j]^2$ is $a-1$, whereas the point is not a singular point at all, and has consequently a discriminantal index equal to zero: its tangent (q_0) however is a singular tangent, and counts as $a-1$ tangents drawn from P. On the other hand, if we consider the two curves $(p-p_0)=(q-q_0)^{\frac{1}{a}}$, $p-p_0=(q-q_0)^{\frac{1}{b}}$, in which a and b are both integers and $b<a$, the order of evanescence of $\Pi\,(\overline{\mathrm{B}}_i-\overline{\mathrm{B}}_j)^2$ is b; and this is the multiplicity of the intersection at (p_0, q_0).

7. We can now prove that the discriminantal index of the singular point O is equal to twice the number of the intersections of C by itself at that point; and, again, that this discriminantal index is equal to the number of the intersections at the same point of C by its first polar with regard to any point not having a special position. For (1), considering the μ equations $\overline{\mathrm{B}}$, we see that twice the number of intersections of C by itself at the point (p_0, q_0), is the order of evanescence of $\Pi\,(\overline{\mathrm{B}}_i-\overline{\mathrm{B}}_j)^2$, the sign of multiplication extending to all the $\frac{1}{2}\mu\,(\mu-1)$ differences; or, observing that $f(q)\div\Pi\,(\overline{\mathrm{B}}_i-\overline{\mathrm{B}}_j)^2$ is a product of $\frac{1}{2}m\,(m-1)-\frac{1}{2}\mu\,(\mu-1)$ squared differences, none of which vanish with $q-q_0$, twice the number of intersections of C by itself at the point (p_0, q_0) is equal to the order of evanescence of $f(q)$ with $q-q_0$, *i.e.*, to the discriminantal index ν. And (2), since the polar of P is $\frac{d\mathrm{F}}{dp}=0$, and since the resultant of $\mathrm{F}=0$ and $\frac{d\mathrm{F}}{dp}=0$ is $f(q)$, we infer (Art. 5)

that ν is the number of intersections at (p_0, q_0) of C by its polar with regard to P.

8. Considering a superlinear branch, of which the component branches are defined by the Δ equations

$$(\Delta) \;\ldots\ldots\; p-p_0 = B_0(q-q_0)+B_1\,\omega^{\beta_1}(q-q_0)^{\frac{\beta_1}{\Delta}}+\ldots,$$

let Δ_1 be the greatest common divisor of Δ and β_1; and if γ_1 is the first of the numbers β which is not divisible by Δ_1, let Δ_2 be the greatest common divisor of Δ_1 and γ_1; if, again, γ_2 is the first of the numbers β which is not divisible by Δ_2, let Δ_3 be the greatest common divisor of Δ_2 and γ_2, and so on continually. Since the numbers β have no common divisor with Δ, we shall at last arrive in the series $\Delta_1, \Delta_2, \Delta_3 \ldots\ldots$ at a term equal to unity, when the series will terminate: and twice the number of the intersections of the superlinear branch by itself will be expressed by the formula

$$2N = \gamma(\Delta-\Delta_1)+\gamma_1(\Delta_1-\Delta_2)+\gamma_2(\Delta_2-\Delta_3)+\ldots\ldots,$$

in which γ is written for β_1. For if ω denote any given root of the equation $\omega^\Delta = 1$, of its remaining roots x there are $\Delta_i - 1$ which verify the equations $x^\gamma = \omega^\gamma$, $x^{\gamma_1} = \omega^{\gamma_1}$, $\ldots\ldots$ $x^{\gamma_{i-1}} = \omega^{\gamma_{i-1}}$; because Δ_i is the greatest common divisor of $\Delta, \gamma, \gamma_1 \ldots \gamma_{i-1}$: similarly there are $\Delta_{i+1}-1$ roots other that ω_1 which verify the same equations, and in addition the equation $x^{\gamma_i} = \omega^{\gamma_i}$. Thus, of the $\Delta(\Delta-1)$ differences obtained by subtracting each of the series (Δ) in turn from every other, there are $\Delta(\Delta_i-\Delta_{i+1})$ which are of the order $\frac{\gamma_i}{\Delta}$; *i.e.*, $2N = \Sigma\gamma_i(\Delta_i-\Delta_{i+1})$. The value of N depends, therefore, not on every exponent in the series (Δ), but only on certain *critical* exponents $\frac{\gamma}{\Delta}$, in the denominators of which, when reduced to their lowest terms, a new factor appears for the first time. The number 2N, which is the "discriminantal index" of the superlinear branch is, not itself necessarily even, but the difference $2N-(\Delta-1)$ is always even, since we have

$$2N-(\Delta-1) = (\gamma-1)(\Delta-\Delta_1)+(\gamma_1-1)(\Delta_1-\Delta_2)+\ldots,$$

and in this expression, if Δ is uneven, so also are $\Delta_1, \Delta_2 \ldots\ldots$; if Δ is even, let Δ_i be the first of the numbers $\Delta_1, \Delta_2 \ldots$ which is uneven; then γ_{i-1} is uneven, and so are all the subsequent numbers $\Delta_{i+1}, \Delta_{i+2}, \ldots\ldots$ In either case, therefore, every term in the expression of $2N-(\Delta-1)$ is even.

Again, if two superlinear branches of the orders Δ and Δ' have the same tangent, let $(q-q_0)^h$ be the lowest power of $q-q_0$ which has not the same coefficient B in the two sets of series (Δ) and (Δ'): it may, of course, in one of these sets have a zero coefficient. Then the terms of lower exponent are common to the two sets; and if the exponents be reduced to their least common denominator, these initial terms will be of the form

$$B_0(q-q_0)+B'\theta^{a_1}(q-q)^{\frac{a_1}{d}}+B\theta^{a_2}(q-q)^{\frac{a_2}{d}}\ldots\ldots+B_i\theta^{a_i}(q-q_0)^{\frac{a_i}{d}},$$

where d is a common divisor of Δ and Δ', θ is any root of $\theta^d=1$, and $\frac{a_i}{d}$ is the exponent next inferior to h. The number of intersections of the two superlinear branches is then

$$N'=h\frac{\Delta\Delta'}{d}+\frac{\Delta\Delta'}{d^2}[\sigma(d-d_1)+\sigma_1(d_1-d_2)+\ldots\ldots],$$

the numbers $\sigma, \sigma_1, \sigma_2, \ldots\ldots d_1, d_2, \ldots\ldots$ (of which in particular $\sigma=a_1$) being determined from the series of exponents $\frac{a}{d}$, in the same way that the numbers $\gamma, \gamma_1, \ldots\ldots \Delta_1, \Delta_2, \ldots\ldots$ were determined from the series of exponents $\frac{\beta}{\Delta}$. For, if θ represent a given root of the equation $\theta^d=1$, the $\frac{\Delta}{d}$ roots of the equation $\omega^\Delta=1$, which satisfy the equation $\omega^{\frac{\Delta}{d}}=\theta$, will give the same initial terms; and we may thus divide the equations (Δ) into $\frac{\Delta}{d}$ groups, each containing d equations; the equations of the same group differing from one another by containing different values of θ, but the different groups not differing from one another, so far as the initial terms are concerned. Similarly we may divide the equations (Δ') into $\frac{\Delta'}{d}$ groups. Considering only one group of each set, we find (by the same reasoning as before) for the order of the product of the $d\times d$-differences obtained from them, the expression

$$hd+\sigma(d-d_1)+\sigma_1(d_1-d_2)+\ldots\ldots,$$

the additional term hd appearing because we have now to take into account the d differences in which all the initial terms vanish: the result, multiplied by $\frac{\Delta}{d}\times\frac{\Delta'}{d}$, gives the value of N'.

Lastly, when two superlinear branches have not the same tangent, the number of their intersections is evidently $N''=\Delta\Delta'$. By means of these formulæ the discriminantal indices of the branches at any singular point, taken by themselves or in pairs, may always be obtained as soon as the developments appertaining to the branches have been found. The sum of these separate discriminantal indices is of course the discriminantal index of the point, or $\nu=2\Sigma N+2\Sigma N'+2\Sigma N''$.

9. Every singular point of a plane curve is regarded by Professor Cayley as being equivalent in a certain manner to δ common nodes and

κ common cusps; and, correlatively, every singular tangent as equivalent to τ double tangents and ι inflexional tangents. For any superlinear branch of order Δ passing through a singular point, the *cuspidal index* κ is by definition $\Delta-1$; thus, for a linear branch $\kappa = 0$. The cuspidal index of a singular point is the sum of the cuspidal indices of the several superlinear branches passing through it; so that, for any singular point, $\kappa = \Sigma(\Delta-1) = \mu-\lambda$, if μ is the order (Art. 2) of the point, and λ the number of distinct linear, or superlinear, branches passing through it. The *nodal index* δ for a singular point, and for its branches, taken singly or in pairs, is defined, not directly, but by equating $2\delta+3\kappa$ to the discriminantal index; thus, for any superlinear branch of order Δ, we have

$$2\delta = (\gamma-3)(\Delta-\Delta_1)+(\gamma_1-3)(\Delta_1-\Delta_2)+\ldots\ldots,$$

which is always even (Art. 8), and positive, except when $\gamma = 3$, $\Delta = 2$, in which case $\delta = 0$, and the superlinear branch is a common cusp.

For τ and ι we have correlative definitions.

10. Adopting these definitions, we have now to prove that the numbers $\Sigma\delta$, $\Sigma\kappa$, $\Sigma\tau$, $\Sigma\iota$ (the summations extending to all the singularities of the curve) satisfy the equations of Plücker, and further that the deficiency of the curve is correctly given by the formula

$$H = \tfrac{1}{2}(m-1)(m-2)-\Sigma\delta-\Sigma\kappa.$$

It is sufficient to establish the four equations,

$$\text{(i.)} \ldots\ldots n = m(m-1)-2\Sigma\delta-3\Sigma\kappa,$$

$$\text{(ii.)} \ldots\ldots m = n(n-1)-2\Sigma\tau-3\Sigma\iota,$$

$$\text{(iii.)} \ldots\ldots H = \tfrac{1}{2}(m-1)(m-2)-\Sigma\delta-\Sigma\kappa,$$

$$\text{(iv.)} \ldots\ldots H = \tfrac{1}{2}(n-1)(n-2)-\Sigma\tau-\Sigma\iota,$$

because the three equations (i.), (ii.), and (iii.) = (iv.) are equivalent to the six equations of Plücker. But the equation (i.) has been already proved; for we have found (Art. 2) that $n = m(m-1)-\Sigma\nu$; and by definition $\Sigma\nu = \Sigma(2\delta+3\kappa)$. The equation (ii.) is the correlative of (i.) and needs no separate proof. In the equations (iii.) and (iv.) it is important to take a definition of H which does not involve any special supposition as to the nature of the singularities appertaining to the curve. The simplest, though not the most direct, course is to adopt the method of Riemann, and to define $2H+1$ as the index of multiplicity of connexion of the m-leaved spirally connected surface [Q], which is such that if the complex values of q be represented upon it in the usual manner, p may be regarded as a one-valued function of q. In any such surface the index of multiplicity of connexion $2H+1$, the number of leaves m, and the number of *spires* (spiral points, *windungs-punkte*) N, are connected by the equation $N = 2H+2m-2$. This equation Riemann himself demonstrates by comparing the values of certain contour-integrals (*Theorie der Abelschen Functionen*, Art. 7). But he observes that it is entirely independent of considerations of magnitude, and that it belongs properly to the geometry of situation. The demonstration of it from this point of view, which has been given by M. Neumann (*Vorlesungen* p. 309, § 99), is also independent of any supposition as to the special nature of the singularities of the curve C; and is therefore available for our present purpose. But we may observe that the algebraical demonstration of the same equation, which is given by MM. Clebsch and Gordan (in their *Theorie der Abelschen Functionen*, p. 54, § 16), would here be inadmissible, because in that demonstration it is expressly supposed that the singular points of C are only common nodes and cusps. (See the note at p. 11, *loc. sit.*)

It is not difficult to find the number of spires N on the surface [Q]. There is a one-fold spire for every tangent from P to C; for, if (p_0, q_0) be the point of contact of any such tangent, we have for values of q in the vicinity of q_0 two conjugate developments of the type

$$(p-p_0) = B_1(q-q_0)^{\frac{1}{2}} + B_2(q-q_0)^{\frac{2}{2}} + \ldots\ldots,$$

in which B_1 is different from zero; all the other developments (Art. 3) being of the type (A), because the point P has no speciality of position. Again, there is a $(\Delta-1)$-fold spire for any singular branch which is superlinear and of order Δ; this is apparent from the form of the Δ developments appertaining to the branch (see Riemann, *loc. cit.*, Art. 6; M. Puiseux, *Liouville*, 1st series, Vol. XV. pp. 384–404).

We have therefore $N = n + \Sigma(\Delta-1) = n + \Sigma\kappa$, and Riemann's equation becomes

$$n + \Sigma\kappa = 2H + 2(m-1);$$

or, since

$$n + \Sigma\kappa = m(m-1) - 2\Sigma\delta - 2\Sigma\kappa,$$

$$H = \tfrac{1}{2}(m-1)(m-2) - \Sigma\delta - \Sigma\kappa,$$

which is the equation (iii.) Again, it is an immediate consequence of Riemann's definition of the number H (see his *Abelsche Functionen*, Art. 11) that this number remains unchanged by any unicursal transformation of the equation $F(p, q) = 0$. But (as has been already observed by MM. Clebsch and Gordan) any tangential equation of the curve C may be regarded as an unicursal transformation of the equation $F(p, q) = 0$, because the points and tangents of a curve correspond to one another one to one. The equation (iii.), therefore, involves the equation (iv.); a result which, as we have seen, implies that the six equations of Plücker are satisfied by the numbers $\Sigma\delta$, $\Sigma\kappa$, $\Sigma\tau$, $\Sigma\iota$.

11. The indices τ and ι appertaining to any superlinear branch at a singular point, and the number of tangents common to two osculating superlinear branches, may be ascertained directly from the point-equations $\overline{B}$, without actually forming the corresponding line-equations.

To prove this, we shall establish a relation which subsists between certain terms in the two sets of equations.

If Q and R are given constants, $p = Qq + R$ is the equation of a straight line in the system of parametric point-coordinates which we have been employing. In passing to line-coordinates, we may take Q and R as the coordinates of this straight line; and we may regard Q and R as the parameters of two ranges of points, lying on the lines PQ, PR, respectively, and represented by equations of the form

$$Q(P) + (Q) = 0, \quad R(P) + (R) = 0;$$

the line $p = Q_0 q + R_0$ or (Q_0, R_0) being the line joining the points determined in the two ranges by the values Q_0, R_0 of the parameters. If to the hypotheses of Art. 1 we add the supposition that PR is not a tangent to C, and does not pass through any singular point of C, the line-equation of C, which we may represent by $\Phi(Q, R) = 0$, will have the same sort of freedom from speciality which has been already attributed to the point-equation $F(p, q) = 0$. The parameters of the tangent to C at the point (p, q) are

$$Q = -\left(\frac{dF}{dq}\right) \div \left(\frac{dF}{dp}\right), \quad R = p - qQ.$$

Let (p, q) be a point lying on the branch $\overline{B}$, of which the point-equation is $\quad p - p_0 = B_0(q - q_0) + B_1\omega^{\beta_1}(q - q_0)^{\frac{\beta_1}{\Delta}} + \dots\dots;$

and suppose (p, q) different from (p_0, q_0), but sufficiently near to it (Art. 3) to ensure the convergence of the m series $\overline{A}$ and $\overline{B}$. Writing

$$F(p, q) = M \times (p - p_0 - \overline{B}),$$

where M is a product of factors, none of which can vanish at the point (p, q), because no singular point other than (p_0, q_0) exists within the range of values attributed to q, we find

$$Q = \left(\frac{d\overline{B}}{dq}\right), \quad R = p - q\left(\frac{d\overline{B}}{dq}\right).$$

Putting $\quad B_0 = Q_0, \quad p_0 - q_0 B_0 = R_0,$

so that (Q_0, R_0) is the tangent at (p_0, q_0) to $\overline{B}$, we obtain the equations

$$Q - Q_0 = \frac{\beta_1}{\Delta} B_1\omega^{\beta_1}(q - q_0)^{\frac{\beta_1}{\Delta} - 1} + \frac{\beta_2}{\Delta} B_2\omega^{\beta_2}(q - q_0)^{\frac{\beta_2}{\Delta} - 1} + \dots\dots,$$

$$R - R_0 = -q_0(Q - Q_0) + \left(1 - \frac{\beta_1}{\Delta}\right) B_1\omega^{\beta_1}(q - q_0)^{\frac{\beta_1}{\Delta}} + \left(1 - \frac{\beta_2}{\Delta}\right) B_2\omega^{\beta_2}(q - q_0)^{\frac{\beta_2}{\Delta}} + \dots\dots,$$

which determine the parameters Q, R of the tangent at any point of $\overrightarrow{\overline{B}}$. If we further write

$$\omega (q-q_0)^{\frac{1}{\Delta}} = \xi,\quad Q-Q_0 = \frac{\beta_1 B_1}{\Delta} \times Y^{\beta_1-\Delta},\quad R-R_0+q_0(Q-Q_0) = Z,$$

these equations become

$$Y^{\beta_1-\Delta} = \xi^{\beta_1-\Delta}\left[1+\sigma_2\xi^{\beta_2-\beta_1}+\sigma_3\xi^{\beta_3-\beta_1}+\dots\dots\right] \quad \dots\dots\dots\dots (\alpha),$$

$$Z = \rho_1\xi^{\beta_1}+\rho_2\xi^{\beta_2}+\dots\dots \quad \dots\dots\dots\dots\dots\dots (\beta),$$

where we have written σ_i for $\frac{\beta_i B_i}{\beta_1 B_1}$, and ρ_i for $\left(1-\frac{\beta_i}{\Delta}\right)B_i$. It will be observed that ω has disappeared from these equations, which therefore appertain equally to all the Δ branches composing the superlinear branch $\overline{B}$. To obtain the tangential equation of that branch, *i.e.*, the expansion of $R-R_0$ in a series proceeding by powers of $Q-Q_0$, we have three operations to perform. First, we have to raise each side of the equation (α) to the power $\frac{1}{\beta_1-\Delta}$; we thus obtain an expansion of the form

$$\theta Y = \xi(1+A\xi^a+B\xi^b+\dots) \quad \dots\dots\dots\dots\dots\dots (Y),$$

θ denoting any root of the equation $\theta^{\beta_1-\Delta}=1$. Secondly, we have to revert the series (Y), so as to obtain the series

$$\xi = \theta Y\{1+A'(\theta Y)^{a'}+B'(\theta Y)^{b'}+\dots\} \quad \dots\dots\dots\dots (\xi).$$

Lastly, we have to substitute, in the equation (β), for ξ its value given by the series (ξ); the final result being of the form

$$Z = H_1(\theta Y)^{h_1}+H_2(\theta Y)^{h_2}+\dots \quad \dots\dots\dots\dots\dots\dots (Z);$$

or, if

$$\mu = \left(\frac{\Delta}{\beta_1 B_1}\right)^{\frac{1}{\beta_1-\Delta}},$$

$$R-R_0 = -q_0(Q-Q_0)+H_1\mu^{h_1}\theta^{h_1}(Q-Q_0)^{\frac{h_1}{\beta_1-\Delta}}$$
$$+H_2\mu^{h_2}\theta^{h_2}(Q-Q_0)^{\frac{h_2}{\beta_1-\Delta}}+\dots \quad \dots\dots\dots (H).$$

12. Certain of the terms of H, and indeed precisely those critical terms upon which the determination of τ and ι depends, can be assigned *à priori* by the help of the following considerations.

(i.) If $a, b, c, \dots l, \dots$ are positive and integral numbers, arranged in order of magnitude, of which l is such that it cannot be formed by addition of any multiples of the numbers which precede it, the coefficient of x^l in the expansion of $[\psi(x)]^\sigma$, where σ is any real exponent, and

$$\psi(x) = 1+Ax^a+Bx^b+Cx^c+\dots+Lx^l+\dots,$$

is σL; and, in particular, if all the numbers preceding l are multiples of any number a, of which l is not itself a multiple, a supposition which implies that l cannot be formed by addition of multiples of $a, b, c, \ldots,$ l is the least exponent in the development of $[\psi(x)]^r$, which is not divisible by a.

(ii.) If the series $y = x\psi(x)$ be reverted so as to obtain the equation

$$x = y\psi_1(y) = y(1 + A_1 y^{a_1} + B_1 y^{b_1} + \ldots),$$

the exponents $a_1, b_1, c_1, \ldots$ are all formed by addition of multiples of $a, b, c, \ldots$. For, if this is not so, let h_1 be the least exponent in $\psi_1(y)$, which cannot be formed by adding multiples of $a, b, c, \ldots$; on substituting $x\psi(x)$ for y in $y\psi_1(y)$, a substitution which ought to have x for its result, we find that the coefficient of x^{h_1+1} is H_1; *i. e.*, $H_1 = 0$, or the exponent h_1 does not occur in $\psi_1(y)$. Again, if the exponent l in $\psi(x)$ cannot be formed by adding multiples of the exponents which precede it, the coefficient L_1 of y^l in $\psi_1(y)$ is $-L$; for, on making the same substitution as before, the coefficient of x^{l+1} is found to be $L_1 + L$; *i. e.*, $L_1 = -L$. And, in particular, if l is the lowest exponent in $\psi(x)$ which is not divisible by a, l is also the lowest exponent in $\psi_1(y)$ which is not divisible by a.

Let $\beta_i = \gamma$, be one of the critical exponents $\gamma, \gamma_1, \ldots$ considered in Art. 8; then all the differences $\beta_2 - \beta_1, \beta_3 - \beta_1, \ldots$ up to $\beta_{i-1} - \beta_1$, are divisible by Δ_{s-1}; but $\beta_i - \beta_1$ is not divisible by Δ_{s-1}. Therefore, by (i.), the coefficient of $\xi^{\beta_i - \beta_1}$ in the development of $\frac{\theta Y}{\xi}$ is $\frac{\sigma_i}{\beta_1 - \Delta}$; by (ii.) the coefficient of $(\theta Y)^{\beta_i - \beta_1}$ in the expansion of $\xi \div (\theta Y)$ is $-\frac{\sigma_i}{\beta_1 - \Delta}$, and $\beta_i - \beta_1$ is the least exponent in that expansion which is not divisible by Δ_{i-1}; finally, on substituting in the equation (β), we see that the term $H(\theta Y)^{\beta_i}$ in the development of Z can arise only from the terms $\rho_1 \xi^{\beta_1}$ and $\rho_i \xi^{\beta_i}$ in (β); its coefficient H is therefore $-\rho_1 \frac{\beta_1 \sigma_i}{\beta_1 - \Delta} + \rho_i = B_i$; and the coefficient of $\theta^{\beta_i}(Q - Q_0)^{\frac{\beta_i}{\beta_1 - \Delta}}$, or $\theta^{\gamma_s}(Q - Q_0)^{\frac{\gamma_s}{\gamma - \Delta}}$, in the expansion of $R - R_0$ is $\mu^{\beta_i} B_i$. Nor can any exponent preceding $\frac{\gamma_s}{\gamma - \Delta}$ have a numerator which is not divisible by Δ_{s-1}.

Observing that the greatest common divisor of $\beta_1 - \Delta$, and β_1, is the same as that of Δ and β_1, we infer from this result that the numbers $\Delta_1, \Delta_2, \Delta_3, \ldots; \gamma, \gamma_1, \gamma_2, \ldots$ are the same for the series H as for the series B; and since the numbers $\gamma, \gamma_1, \gamma_2, \ldots$ have no common divisor with Δ, neither have they with $\beta_1 - \Delta = \gamma - \Delta$; *i. e.*, $\gamma - \Delta$ is the least common denominator of the exponents of H. Hence we have, writing $\gamma - \Delta = \Delta_{\prime}$,

$$\iota = \Delta_{\prime} - 1, \quad \gamma = \Delta + \Delta_{\prime} = \iota + \kappa + 2,$$

$$2r + 3\iota = \gamma(\Delta_{\prime} - \Delta_1) + \gamma_1(\Delta_1 - \Delta_2) + \gamma_2(\Delta_2 - \Delta_3 + \ldots;$$

or, subtracting the discriminantal index $2\delta+3\kappa$,

$$(2\tau+3\iota)-(2\delta+3\kappa) = \Delta_{\prime}^{2}-\Delta^{2},$$
$$\tau-\delta = \tfrac{1}{2}(\iota-\kappa)(\iota+\kappa-1),$$

an equation which establishes a relation between the four indices of the superlinear branch.

13. If we consider any term whatever in the series $\overline{\mathrm{B}}$, for example the term $(i) = \beta_i\,\omega^{\beta_i}(q-q_0)^{\frac{\beta_i}{\Delta}}$, we shall in general find a corresponding term (I) in the series H, containing $\mathrm{Q}-\mathrm{Q}_0$ raised to the power $\frac{\beta_i}{\beta_1-\Delta}$: (I) may be considered as the sum of two parts, I_1 and I_2, of which the first, I_1, arises from the term (i) itself, the other, I_2, from the terms preceding (i); (I) being in no way affected by the terms following (i). If β_i is one of the critical exponents, we have just seen that $\mathrm{I}_2=0$, $\mathrm{I}_1 = \mu^{\beta_i}\,\mathrm{B}_i\,(\mathrm{Q}-\mathrm{Q}_0)^{\frac{\beta_i}{\beta_1-\Delta}}$. If β_i is not one of the critical exponents, the first of these equations ceases to subsist, but the second remains true, and its proof requires only a slight modification of the reasoning in Art. 12. Now let two series $\overline{\mathrm{B}}$, appertaining to two different superlinear branches, which have a common tangent, coincide as far as the term (i), but exclusively of it; the two corresponding series $\overline{\mathrm{H}}$ will coincide as far as the term (I), but exclusively of it; we suppose $i>0$. That all terms preceding (I) will coincide in the two developments $\overline{\mathrm{H}}$ is evident, for these terms arise solely from the terms preceding (i), which are identical in the two developments $\overline{\mathrm{B}}$. And the terms (I) themselves are different: for the difference of the two terms (i) is $(\mathrm{B}_i-\mathrm{B}'_i)\,\omega^{\beta_i}(q-q_0)^{\frac{\beta_i}{\Delta}}$, where one of the two B_i, B'_i may be zero, but the difference $\mathrm{B}_i-\mathrm{B}'_i$ is by hypothesis not zero; and the difference of the two terms (I) is $\mu^{\beta_i}(\mathrm{B}_i-\mathrm{B}_i)\,\theta^{\beta_i}\times(\mathrm{Q}-\mathrm{Q}_0)^{\frac{\beta_i}{\beta-\Delta}} = \mathrm{I}_1-\mathrm{I}'_1$, for these two terms have the same part I_2.

Let $\overline{\mathrm{D}}$ be the number of points, $\overline{\mathrm{T}}$ the number of tangents common to the two branches $\overline{\mathrm{B}}$ at the point (p_0, q_0); $\overline{\mathrm{T}}$ is given by the formula

$$\overline{\mathrm{T}} = \beta_i\frac{\gamma'-\Delta'}{\sigma-d} + \frac{(\gamma-\Delta)(\gamma'-\Delta')}{(\sigma-d)^2}\{\sigma(\sigma-d-d_1)+\sigma_1(d_1-d_2)+\ldots\ldots\},$$

which is derived from the expression for N′ in Art. 8, by writing $\gamma-\Delta$ for Δ, $\gamma'-\Delta'$ for Δ', and $\sigma-d$ for d. Observing that $\frac{\gamma}{\Delta}=\frac{\gamma'}{\Delta'}=\frac{\sigma}{d}$,

whence $\quad \frac{\gamma'-\Delta'}{\sigma-d}=\frac{\Delta'}{d},\ \frac{\gamma-\Delta}{\sigma-d}=\frac{\Delta}{d},\ \frac{\Delta}{d}\sigma=\gamma=\iota+\kappa+2,$

$$\frac{\Delta'}{d}\sigma = \gamma' = \iota'+\kappa'+2, \quad \frac{\iota+1}{\kappa+1} = \frac{\iota'+1}{\kappa'+1},$$

we find
$$\overline{\mathrm{T}}-\overline{\mathrm{D}} = \frac{\Delta\Delta'}{d^2}\sigma(\sigma-2d)$$
$$= (\iota-\kappa)(\iota'+\kappa'+2) = (\iota'-\kappa')(\iota+\kappa+2)$$
$$= (\iota+1)(\iota'+1)-(\kappa+1)(\kappa'+1) = \Delta_{,}\Delta_{,}'-\Delta\Delta'.$$

We have supposed in the demonstration that $i > 1$, or that the two developments of $p-p_0-\mathrm{B}_0(q-q_0)$ coincide for at least one term. But, for the validity of the formulæ, it is only necessary that the first exponent should be the same in the two developments; and indeed the last two expressions for $\overline{\mathrm{T}}-\overline{\mathrm{D}}$ hold universally for any two superlinear branches having a common tangent.

14. The *species* of a superlinear singularity may be regarded as defined by the series of numbers Δ and $\Delta_{,}$; $\Delta_1, \Delta_2, \ldots\ldots, \gamma_1, \gamma_2, \ldots\ldots$, so that two superlinear singularities, for which these indices have the same values, may be considered as belonging to the same species. A rougher classification, however, which is sometimes useful, may be obtained in the following way. Leaving out of sight the case in which two superlinear singularities present themselves as conjugate imaginaries, and attending only to the case of a real superlinearity, we may distinguish four varieties differing from one another in the appearance which they present to the eye. (See a Memoir by M. Stolz, *Mathematische Annalen*, Vol. VIII., p. 440.)

(i.) Δ uneven, $\Delta_{,}$ uneven; no apparent cusp or inflexion.
(ii.) Δ even, $\Delta_{,}$ uneven; an apparent cusp, no apparent inflexion.
(iii.) Δ uneven, $\Delta_{,}$ even; an apparent inflexion, no apparent cusp.
(iv.) Δ even, $\Delta_{,}$ even; an apparent cusp, and an apparent inflexion.

The form of (ii.) is that of the common or keratoid cusp; (iv.) has the form of the cusp of the second species, or rhamphoid cusp. There is an apparent inflexion at the rhamphoid cusp, because, if a person describing the curve continuously passes through the cusp, the concavity of the curve is to his right after he has passed through the cusp, if it was to his left hand before, and *vice versâ.* We may further observe that, in case (iv.), Δ and $\Delta_{,}$, being both even, have a common measure; thus $\Delta_2 > 1$, and the superlinearity is *composite.* The cases (ii.) and (iii.) are correlative; the cases (i.) and (iv.) are their own correlatives.

15. The curvature of a curve at two points infinitely near to a given superlinear point, and at equal distances from it on either side, is always the same; and is infinite, finite, or zero, according as $\Delta > \Delta_{,}$, $\Delta = \Delta_{,}$, or $\Delta < \Delta_{,}$. Thus, in each of the cases (i.) and (iv.), there are

three sub-varieties of form; and two in each of the cases (iii.) and (ii.). The following are the simplest examples of each of these sub-varieties: for the sake of completeness, the cases in which either of the two numbers Δ or $\Delta_{,}$ is unity, are included.

(i.) Δ and $\Delta_{,}$ uneven.

$$\Delta > \Delta_{,}: \quad y = x^{\frac{3}{2}}; \quad y = x^{\frac{5}{2}}.$$
$$\Delta = \Delta_{,}: \quad y = x^{2}; \quad y = x^{2} + x^{\frac{5}{2}}.$$
$$\Delta < \Delta_{,}: \quad y = x^{4}; \quad y = x^{\frac{5}{3}}.$$

(ii.) Δ even, $\Delta_{,}$ uneven.

$$\Delta > \Delta_{,}: \quad y = x^{\frac{5}{2}}; \quad y = x^{\frac{7}{4}}.$$
$$\Delta < \Delta_{,}: \quad y = x^{\frac{5}{4}}.$$

(iii.) Δ uneven, $\Delta_{,}$ even.

$$\Delta > \Delta_{,}: \quad y = x^{\frac{3}{2}}.$$
$$\Delta < \Delta_{,}: \quad y = x^{3}; \quad y = x^{\frac{7}{3}}.$$

(iv.) Δ and $\Delta_{,}$ even.

$$\Delta > \Delta_{,}: \quad y = x^{\frac{3}{2}} + x^{\frac{7}{4}}.$$
$$\Delta = \Delta_{,}: \quad y = x^{2} + x^{\frac{9}{4}}.$$
$$\Delta < \Delta_{,}: \quad y = x^{3} + x^{\frac{7}{2}}.$$

It should be noticed that in the equation $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$, the only independent radical is $x^{\frac{3}{4}}$, and that $x^{\frac{3}{2}}$ is to be interpreted as $(x^{\frac{3}{4}})^2$. Thus, supposing x positive, and understanding by $\sqrt{x^3}$ and $\sqrt[4]{x^7}$ the real and positive values of the radicals, we have for the four partial branches the equations

$$y = \sqrt{x^3} + \sqrt[4]{x^7}, \qquad y = \sqrt{x^3} - \sqrt[4]{x^7},$$
$$y = -\sqrt{x^3} - i\sqrt[4]{x^7}, \qquad y = -\sqrt{x^3} + i\sqrt[4]{x^7},$$

of which the first two appertain to a real rhamphoid cusp. If we were to change the sign of $\sqrt{x^3}$, we should pass from the equation

$$U = (y^2 + x^3)^2 - x^3(x^2 + 2y)^2 = 0,$$

which is the rationalized equivalent of $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$, to the equation

$$V = (y^2 + x^3)^2 - x^3(x^2 - 2y)^2 = 0,$$

which is the rationalized equivalent of $y = -x^{\frac{3}{2}} + x^{\frac{7}{4}}$. It is, of course, quite possible that two developments, such as $y = \pm x^{\frac{3}{2}} + x^{\frac{7}{4}} + \ldots$, may both belong to the same curve (as indeed they do both belong to the curve $UV + x^{15}\phi(x, y) = 0$), but such a curve would have two distinct superlinear branches touching one another at the point $x=0$, $y=0$.

16. Let O be any point whatever on a curve line; let the arc $OP = \sigma$, P being a point on the curve infinitely near to O; let M be the orthogonal projection of P on the tangent at O; and let the tangents at O

and P intersect at T, making the infinitesimal angle ω. Then it will be found that $\frac{\Delta_{\prime}}{\Delta} = \frac{\log \omega}{\log \sigma} = \frac{OT}{TP} = \frac{\angle TPO}{\angle TOP} = \frac{\log MP}{\log OM} - 1.$

The fraction $\frac{\Delta_{\prime}}{\Delta}$ which admits of these various geometrical interpretations may perhaps be called the *logarithmic curvature* of the curve at the point O. At any ordinary point it is unity; and in a geometrical curve it is always rational, but in a transcendental curve it may have any value rational or irrational.

Since Δ or $\kappa+1$ is the number of points in which the superlinear branch is cut by any line passing through O, other than its tangent at the point O, we infer that, correlatively, $\iota+1$ or $\Delta_{\prime}$ is the number of tangents drawn to the superlinear branch from any point on the tangent at O, other than O itself. Thus, if d be the discriminantal index of O, or the number of points in which the curve is cut at O by the polar of any arbitrary point, $d+\Delta_{\prime}$ is the number of points in which the curve is cut at O by the polar of any point on the tangent at O, other than O itself; there is, of course, a correlative definition of $d+\Delta$. Lastly, since $\Delta+\Delta_{\prime}$ is the number of points common at O to the tangent and the curve, it is also, correlatively, the number of tangents drawn from O to touch the curve at that point. Thus the polar of the point O intersects the curve at O in $d+\Delta+\Delta_{\prime}$ points, and the tangent at O counts as $d+\Delta+\Delta_{\prime}$ tangents common to the curve, and to the tangential polar of OT with regard to the curve. For the numbers $\Delta_1, \Delta_2, \ldots \gamma_1, \gamma_2, \ldots$ no simple geometrical definition has as yet presented itself.

17. The proof of Plücker's formulæ, which is indicated in Art. 10, may appear very indirect. Some further observations on these formulæ, and on the various modes of demonstrating them, may not be out of place.

(1.) If we write $D = \Sigma(2\delta+3\kappa)$, $T = \Sigma(2\tau+3\iota)$, $I = \Sigma\iota$, $K = \Sigma\kappa$, $\frac{1}{3}(I-K) = \Omega$, Plücker's formulæ become

$$n = m(m-1) - D,$$
$$m = n(n-1) - T;$$
$$\Omega = m(m-2) - D,$$
$$-\Omega = n(n-2) - T;$$

giving
$$T - D = n^2 - m^2, \quad \Omega = 3(n-m),$$
$$\Omega^4 - 2\Omega^2(T+D) - 4\Omega(T-D) + (T-D)^2 = 0.$$

It is thus apparent that Plücker's equations do not contain either K or I separately, but only the difference I−K.

(2.) The discriminantal index $d = 2\delta+3\kappa$ of any given point is defined geometrically as the number of intersections of the polar of an arbitrary point with the curve at the given point. But the definition which we have given in Art. 9 of the cuspidal index κ is an analytical one, and does not readily admit of interpretation in coordinate geometry. The Hessian does not serve to define either ι or κ, for in all the cases that have as yet been rigorously investigated, it has been found that the number of intersections of the Hessian with the curve at a point of discriminantal index d is $3d+\iota-\kappa$, so that, even if the number of these intersections at any singular point should be determined by a general method, we should only obtain a definition of the difference $\iota-\kappa$. Again, if several superlinear branches have a common tangent OT at the point O, it will be seen that the geometrical definitions of Art. 16 only give the numbers $\Sigma(\iota+1)$ and $\Sigma(\kappa+1)$; viz., if d is the total discriminantal index of all the branches intersecting at O, the first polar of any point on OT (other than O) intersects the curve at O in $d+\Sigma(\iota+1)$ points; the polar of O intersects the curve at O in $d+\Sigma(\iota+\kappa+2)$ points; and there are correlative definitions of the numbers $d+\Sigma(\kappa+1)$, and $d+\Sigma(\iota+\kappa+2)$. By combining these definitions, we obtain a geometrical definition of the difference $\Sigma(\iota-\kappa)$, the summation extending to all the branches which touch one another at O. But here it is to be observed (1) that to deduce the values of $\Sigma\iota$ and $\Sigma\kappa$ from those of $\Sigma(\iota+1)$ and $\Sigma(\kappa+1)$, we should require to determine the number λ of distinct superlinear branches which touch OT at O; and (2) that, even if $\Sigma\iota$ and $\Sigma\kappa$ were known, it would still remain to determine the decomposition of these sums, and to assign the partial indices appertaining to each of the λ branches; whereas no determination of the number λ, or of the indices ι and κ of each separate superlinear branch, has as yet been obtained by considering the intersections of the given curve with any concomitant or system of concomitants.

(3.) The difficulty, which thus presents itself in obtaining a definition of the indices ι and κ, ceases to exist when we leave the domain of coordinate geometry, and consider either the analytical expansions, or the geometrical representations (depending on principles foreign to coordinate geometry) which correspond to those expansions. If several superlinear branches touch one another at a given point, the analytical expansions separate them, and assign the cuspidal and inflexional indices proper to each of them. If we apply to the equation $F(p, q) = 0$ the geometrical methods of double algebra, the cuspidal indices appear in the cycles of values of p, which present themselves at the points answering to the discriminantal values of q. (See the memoir of M. Puiseux, *Liouville*, Vol. XV., p. 384.) If, instead of the simple plane of double algebra, we use the multiple plane of Riemann, the cuspidal indices are represented by the spires which connect the leaves of the multiple plane. But it is important to remember that, in employing the methods of double algebra, and *à fortiori* in employing the surfaces of Riemann, we are entirely abandoning the methods of coordinate and projective geometry. The present question is perhaps not directly affected by the fundamental distinction between the "infinite" of double algebra, which is a point, and the infinite of projective geometry, which is a straight line. But the duality, characteristic of projective geometry, is lost in double algebra; so that, when the complex values of p and q which satisfy the equation $F(p, q) = 0$ are regarded as developed on a plane, or on one of Riemann's surfaces, we do indeed obtain a direct representation of the cuspidal index κ, but no corresponding representation (unless we first transform the equation into its reciprocal) of the correlative index ι. Indeed, it may be asserted that, whereas the character of any given superlinearity mainly depends on a series of indices $\Delta=\kappa+1$, $\Delta_{\prime}=\iota+1$, $\Delta_1, \Delta_2, \ldots, \gamma_1, \gamma_2, \ldots$, the modes of geometrical representation, to which we are here referring, offer a sensible image of the first of these indices only. If we employ a simple plane, any one of the Δ values of p, which come to coincide with one another at the discriminantal point, must describe Δ elementary contours around that point before it acquires again its original value. If for simplicity we suppose that $\Delta_2=1$, the Δ values of p, which form the cycle, will divide themselves into Δ_1 sub-cycles, each containing $\frac{\Delta}{\Delta_1}$ values; and any value, belonging to one of these sub-cycles, will acquire approximately its original value, after describing $\frac{\Delta}{\Delta_1}$ elementary contours around the discriminantal point, the order of the error being $\frac{\gamma_1}{\Delta}$ if the order of the infinitesimal radius be taken as unity. And upon this approximate return to the original value depends the only indication which the method affords of the existence of sub-cycles, and of the values of the numbers Δ_1 and γ_1. If we employ the multiple plane of Riemann, we may perhaps represent the relations of the Δ expansions to one another by taking a Δ_1-leaved plane, repeated $\frac{\Delta}{\Delta_1}$ times, and having a spire of order $\Delta-1$, so arranged that after $\frac{\Delta}{\Delta_1}$ revolutions we return to the same Δ_1-leaved plane upon which we were when we set out, but not to the same leaf of that plane. And we can give to this image a certain amount of clearness by supposing that the Δ_1 leaves of any Δ_1-leaved plane are infinitely nearer to one another than are any two of the $\frac{\Delta}{\Delta_1}$ repetitions of the Δ_1-leaved plane.

(4.) The demonstrations of Plücker's formulæ, which are usually given, apply only to the case in which the singularities are simple; the cases of multiple points, or multiple tangents, or of branches having contact with one another of any order, being made to depend, by the method of limits, on the simple cases of double points, or double tangents (see Dr. Salmon's *Higher Plane Curves*, p. 53). But these demonstrations do not admit of immediate extension to the case of the higher singularities properly so called, because it has not as yet been established, in any general manner, that a higher singularity may be regarded as the limit of an equivalent number of lower singularities situated infinitely near to one another. It would seem that Plücker himself was well aware of the incompleteness (in this respect) of the demonstration of his equations; for he supplements that demonstration by separately considering the case of a common cusp of the second species. Assuming the equation $n = m(m-1) - D$, and its reciprocal, (about the rigorous proof of which there is no doubt,) we have only to establish one other equation of the system. Two different methods are given by Plücker (*Theorie der Algebraischen Curven*, Part ii., Arts. 77—81): (i.) He establishes directly the theorem that, at a cusp of the second species, the curve

$$\frac{d^2F}{dp^2}\left(\frac{dF}{dq}\right)^2 - 2\frac{d^2F}{dp\,dq}\frac{dF}{dp}\frac{dF}{dq} + \frac{d^2F}{dq^2}\left(\frac{dF}{dp}\right)^2 = 0$$

(which may be used for our present purpose instead of the Hessian) intersects the given curve in $3d+\iota-\kappa = 15$ points. We have already stated that, in all the cases which have been examined hitherto, the number of intersections of the Hessian with the curve at any point has been found to be $3d+\iota-\kappa$; but no general demonstration of this theorem has as yet been given. The only method at present known for determining the number of intersections of two curves at a point which is singular on each of them, consists in obtaining the developments of the various branches of the two curves at the point, and in comparing these developments with one another. The discussion in Art. 18 of the development of the polar curve in the vicinity of a superlinear branch, may serve to show that the corresponding enquiry in the case of the Hessian is one of considerable intricacy. (ii.) The other method employed by Plücker depends on a determination of the number of double tangents lost by a curve of the fourth order in consequence of the presence of a cusp of the second species. In the absence of any demonstration that a higher singularity can be regarded as the limit of simple singularities existing infinitely near to one another, it is difficult to see how this mode of proof can be rendered universally applicable.

(5.) We have seen (Art. 10) that the theorem of the invariance of the number $\frac{1}{2}(m-1)(m-2) - \Sigma\delta - \Sigma\kappa$ in any unicursal transformation of the curve suffices to establish the equation

$$\text{(A)} \ldots\ldots \tfrac{1}{2}(m-1)(m-2)-\delta-\kappa = \tfrac{1}{2}(n-1)(n-2)-\tau-\iota,$$

and thus to complete the proof of the formulæ of Plücker. Among the demonstrations of this theorem which have been given in recent times that of MM. Bertini and Zeuthen (*Giornale di Mathematica*, Vol. VII., p. 105; *Mathematische Annalen*, Vol. III., p. 150; Dr. Salmon's *Higher Plane Curves*, p. 314) is remarkable for its simplicity; and appears, as we shall now attempt to show, to admit of extension to the case in which the curves have any singularities whatever. We begin by assuming that when a curve is subjected to an unicursal, or one-to-one transformation, the continuity of its branches is invariably preserved, even when the position of these branches with regard to one another has undergone great distortion. For example, if a curve have two branches intersecting at the point O, these two branches will certainly be represented by two corresponding branches in the transformed curve; but these two branches may have no point of intersection, and the point O may be represented by two different points one on each of the two branches. Again, two branches which osculate one another with any degree of approximation may be transformed into branches having no contact and no point in common. But a superlinear branch behaves as one branch, and always is transformed into one branch and one only. Consider, for example, a real branch which is superlinear at O, and suppose for simplicity that no other branch passes through O; whatever be the nature of the superlinearity, we have one continuous branch passing through O, and if a point describe this branch, the track of the image-point in the transformed figure cannot be anything but one continuous branch.

Let C_1, C_2 be two curves of the orders m_1, m_2, and of the classes n_1, n_2, lying in the same plane and corresponding to one another unicursally; and let P_1, P_2 be points upon them corresponding unicursally. Taking two arbitrary points S_1, S_2, we consider, with M. Zeuthen, the locus Γ of the intersection of the rays S_1P_1, S_2P_2; and we propose to determine the number of tangents that can be drawn to Γ from each of the two points S_1 and S_2. We may suppose that S_1S_2 cuts each of the two curves in points which do not have singular points of the other curve for their corresponding points; then it is evident that Γ will have m_2 ordinary branches passing through S_1, and m_1 ordinary branches passing through S_2. We may further suppose that the n_1 tangents drawn from S_1 to C_1 are none of them singular tangents, and that to the points of contact of these n_1 tangents there answer on C_2 points having no singularity: each of these tangents will then be a tangent of Γ, but not a singular tangent of that curve. Beside the $2m_2+n_1$ tangents, which we have now drawn from S_1 to Γ, there may be others, coinciding in direction with the rays running from S_1 to the singular points of C_1. Let X_1 be a superlinear point on C_1, having the cuspidal index κ_1; and to X_1 let X_2 answer on C_2, the cuspidal index of X_2 being κ_2, where $\kappa_2 \gtreqless 0$. We may suppose at first that only one branch passes through X_1 and only one through X_2. The ray S_1X_1 meets C_1 at X_1 in precisely κ_1+1 coincident points, because S_1X_1 is not a tangent at X_1; similarly S_2X_2 is not a tangent at X_2, but meets C_2 in precisely κ_2+1 points at X_2, since we may attribute to S_2 the requisite generality of position with regard to C_2. Thus, if Q is the intersection of S_1X_1, S_2X_2, the locus Γ is intersected at Q κ_1+1 times by S_1X_1, and κ_2+1 times by S_2X_2. The points of the curve Γ answer, one to one, to the points of C_1 or C_2; thus at Q there is but one branch answering to the one branch at X_1, or to the one branch at X_2. If $\kappa_1 = \kappa_2$, the cuspidal index of this branch is $\kappa_1 = \kappa_2$, while its inflexional index remains unknown. If $\kappa_1 > \kappa_2$, its cuspidal index is κ_2, its inflexional index is $\kappa_1-\kappa_2-1$; similarly, if $\kappa_2>\kappa_1$, these indices are κ_1 and $\kappa_2-\kappa_1-1$; *i.e.*, in the first case, S_1X_1 counts $\kappa_1-\kappa_2$ times as a tangent to Γ at Q, and S_2X_2 is not a tangent at all; in the second case, S_2X_2 counts $\kappa_2-\kappa_1$ times as a tangent, and S_1X_1 is not a tangent at all. When $\kappa_1 = \kappa_2$, neither S_1X_1 nor S_2X_2 are tangents. The preceding reasoning will not be affected, if we now introduce the supposition that several linear or superlinear branches intersect or osculate at X_1, and that branches corresponding to some or all of them pass through X_2. Several branches will now pass through Q, but each of them may be considered separately, and the number of times that it is touched by S_1Q or S_2Q may be ascertained as above. Equating the results appertaining to the points S_1 and S_2, we now obtain

$$2m_2+n_1+\Sigma'(\kappa_1-\kappa_2) = 2m_1+n_2+\Sigma'(\kappa_2-\kappa_1);$$

where Σ' extends only to those differences which are positive. Written in the form

$$n_1+\Sigma\kappa_1-2m_1 = n_2+\Sigma\kappa_2-2m_2,$$

this equation coincides with the formula

$$\tfrac{1}{2}(m_1-1)(m_1-2)-\Sigma(\delta_1+\kappa_1) = \tfrac{1}{2}(m_2-1)(m_2-2)-\Sigma(\delta_2+\kappa_2),$$

which it was required to prove.

The assumption, which we have explicitly made, that a linear or superlinear branch is always transformed by a one-to-one transformation into one branch, and one only, is indispensable in the preceding proof; as upon it depends the determination of the number of times that Γ is touched by S_1X_1 or S_2X_2. In the case of a real branch transformed by a real transformation, the assumption may be regarded as evident; in the general case, we should have to consider, instead of two plane curves, the two corresponding surfaces of Riemann. For our immediate purpose, however, we do not need to establish the assumption as universally true in all cases; because the only one-to-one transformation (beside that of C_1 or C_2 into Γ) which is here employed is the transformation by polar reciprocation; and the investigation of Art. 11 affords a direct proof that in this transformation any one linear or superlinear branch is always transformed into one branch (linear or superlinear).

(6.) Abandoning for a time the hypotheses of Art. 1, let us suppose that P is a singular point on the curve C, Q retaining its generality of position. And first let P be a point through which only one superlinear branch passes, having the indices $\kappa = \Delta - 1$, $\iota = \Delta_{\prime} - 1$; let us also suppose that no singular tangent of C (other than the tangent at P) passes through P. The order of p in the equation $F(p, q) = 0$ is now $m - \Delta$, instead of m; and the number of tangents that can be drawn from P to the curve C (other than the coincident tangents at P itself) is $n - \Delta - \Delta_{\prime}$ (see Art. 16), instead of n. To all the singular points of C, other than P, there will appertain developments of precisely the same form as in the case in which P has no speciality of position. Let q_0 be the value of q corresponding to the tangent at P; the parameters of the point P are $p = \infty$, $q = q_0$. We cannot, therefore, in examining the superlinear branch at P, develope p in a series proceeding by powers of $q - q_0$; but we may so develope $\frac{1}{p}$, or any linear function of p, such as $\frac{c+dp}{a+bp}$, which assumes a finite value $p_0 = \frac{d}{b}$, when $p = \infty$. The exponents in any such development will have $\Delta_{\prime}$ instead of Δ, for their least common denominator, because the tangent to C at P meets the curve (Art. 16) in $\Delta + \Delta_{\prime}$ points, so that, if $q = q_0$, $\Delta_{\prime}$ of the $m - \Delta$ values of $\frac{c+dp}{a+bp}$ become equal to p_0. Setting out from the given equation $F(p, q) = 0$, let us form the developments appertaining to all the *singular* discriminantal values of q; and in each group of conjugate developments let us consider the greatest common divisor θ of its exponents. The sum $\Sigma(\theta - 1)$ will be equal to $\Sigma\kappa + \Delta_{\iota} - \Delta_{\prime\prime}$, instead of $\Sigma\kappa$; and the three numbers, by which we have now replaced m, n, and $\Sigma\kappa$, will satisfy the equation

$$(n - \Delta - \Delta_{\prime}) + (\Sigma\kappa + \Delta_{\prime} - \Delta) - 2(m - \Delta) = n + \Sigma\kappa - 2m.$$

The cases in which (α) more than one branch passes through P, (β) one or more singular tangents pass through P, (γ) Q as well as P has some speciality of position with regard to C, may all be treated by the same method. In any of these cases, let $E(p)$ be the highest exponent of p in the equation $F(p, q) = 0$; and let $\omega(p) = \Sigma(\theta - 1)$, the sign of summation now extending to *all* the discriminantal values of q, so that $\Sigma(\theta - 1)$ contains an unit for every ordinary tangent that can be drawn from P to touch the curve elsewhere. If any of the discriminantal values of q, or any of the corresponding equal values of p, are infinite, we are to employ linear functions of p and q, instead of p and q themselves, in forming the developments from which we are to infer the numbers θ. We shall thus obtain the equation

$$\text{(B)} \quad \ldots\ldots\ldots \quad \omega(p) - 2\text{E}(p) = \omega(q) - 2\text{E}(q) = n + \Sigma\kappa - 2m,$$

from which, as Clebsch has shown, the general theorem of the invariance of the deficiency may be immediately deduced. (See a Memoir by M. Nöther, *Mathematische Annalen*, Vol. VIII., p. 497.)

In the memoir to which we have just referred, M. Nöther offers a demonstration of the equation (B). But this demonstration is perhaps not wholly free from obscurity. (See the words, p. 499, *loc. cit.*, "Dieses findet . . . ergiebt," with the accompanying reference to the Göttingen *Nachrichten*.) A similar remark applies to a second demonstration, in the same memoir, of the invariance of the deficiency. [See p. 501, "Man hat aber dann . . . das Glied $\Sigma i_1 (i_1 - 1)$."]

M. Nöther has returned to the same subject, in a recent memoir of great interest (*Mathematische Annalen*, Vol. IX., p. 166), in which he considers the resolution of a higher singularity by successive applications of a simple quadratic transformation, and infers (though by a method which can hardly be accepted as rigorous) that any higher singularity may be regarded as the limit of a certain number of lower singularities situated infinitely near to one another. We may observe (α) that the use of a quadratic transformation for the resolution of complicated singularities is due to Cramer (*Analyse des Lignes Courbes*); (β) that to establish the complete system of the formulæ of Plücker, M. Nöther selects the same three equations, which we have been led to employ in the present paper [viz., the equations (i.), (ii.), and (iii.) $\doteq$ (iv.), of Art. 10].

18. The expansions of Arts. 3 and 4 enable us to examine the relation of a curve at a singular point to its polar curves. Putting for brevity $p - p_0 = \eta$, $q - q_0 = \xi$, $\text{F}(p, q) = \text{F}_1(\eta, \xi)$, we have $\text{F}_1(\eta, \xi) = \Pi(\eta - \bar{\text{A}}) \times \Pi(\eta - \bar{\text{B}})$, $\dfrac{d\text{F}}{dp} = \dfrac{d\text{F}_1}{d\eta}$. From the expression of $\text{F}_1(\eta, \xi)$ as a product of m factor-series, we infer that if, on writing $\text{K}_1\xi$ for η in $\text{F}_1(\eta, \xi)$, we obtain a result of which the order of evanescence with ξ is higher than μ, $\eta = \text{K}_1\xi + \ldots\ldots$ is the beginning of one at least of the expansions $\bar{\text{B}}$. Again, let us substitute for η in $\text{F}_1(\eta, \xi)$ an expression of the form

$$\text{K} = \text{K}_1\xi + \text{K}_2\xi^{\alpha_2} + \text{K}_3\xi^{\alpha_3} + \ldots\ldots + \text{K}_\nu\xi^{\alpha_\nu},$$

in which $1 < \alpha_2 < \alpha_3 \ldots\ldots < \alpha_\nu$. If the order of evanescence of $\text{F}_1(\text{K}, \xi)$ with ξ experiences an abrupt diminution when either α_ν or K_ν (the exponent and coefficient of the last term of K) is affected by any small variation, the terms K are the initial terms of one at least of the expansions $\bar{\text{B}}$. This observation (which admits of some useful applications) enables us to deduce the developments appertaining to the polar curve $\frac{dF}{dp}$, in the vicinity of the point (p_0, q_0), from the developments appertaining to C.

Let k of the developments $\overline{B}$ coincide with one another and with K, as far as the term $K_\nu \xi^{a_\nu}$ inclusively, so that for any one of these k developments we have

$$\overline{B}_i = K + L_i \xi^{l_i}, \quad l_i > a_\nu \ldots\ldots\ldots\ldots (K),$$

$$L_i = \lambda_i + \lambda_i' \xi^{l_i'} + \ldots\ldots,$$

the terms $\lambda_i \xi^{l_i}$ not being all identical.

Put $$V = \eta - K, \qquad \prod_1^k (V - L_i \xi^{l_i}) = \phi(V);$$

then $$F_1(\eta, \xi) = M \times \phi(V), \quad \text{and} \quad \frac{dF_1}{d\eta} = \frac{dM}{d\eta}\phi(V) + M\phi'(V),$$

M being a product of $m-k$ factors, viz., of the $m-\mu$ factors $\eta - \overline{A}$, and of those $\mu - k$ factors $\eta - \overline{B}$ which do not coincide with $\eta - K$ as far as the term $K_\nu \xi^{a_\nu}$ inclusively. Suppose, at first, that $l_1, l_2, \ldots\ldots l_k$ are all unequal, and arranged in order of magnitude; it is easily ascertained that the first terms in the expansions of the roots of $\phi'(V) = 0$ are

$$V_1 = \frac{k-1}{k}\lambda_1 \xi^{l_1}, \quad V_2 = \frac{k-2}{k-1}\lambda_2 \xi^{l_2}, \quad V_3 = \frac{k-3}{k-2}\lambda_3 \xi^{l_3}, \quad \ldots\ldots$$

$$V_{k-1} = \tfrac{1}{2}\lambda_{k-1}\xi^{l_{k-1}}.$$

Substitute for η in $\frac{dF_1}{d\eta}$ an expression of the form $\eta = K + H\xi^h$, where $h > a_\nu$, and H is independent of ξ. If $H\xi^h$ is not the same as any one of the quantities $V_1, V_2, \ldots V_{k-1}$, the order of evanescence of $\phi(V)\frac{dM}{d\eta}$ surpasses that of $M\phi'(V)$; for the order of evanescence of M cannot surpass that of $\frac{dM}{d\eta}$ by a number greater than a_ν, whereas the order of $\phi(V)$, on the supposition that none of the equations $H\xi^h = V_i$ is satisfied, surpasses the order of $\phi'(V)$, at least by one of the numbers $l_1, l_2, \ldots l_k$. If we now suppose H and h to vary continuously, the order of evanescence of $\phi'(V)$ is abruptly increased when $H\xi^h$ comes to coincide with any one of the roots $V_0, V_1 \ldots\ldots V_{k-1}$; and, since the order of evanescence of M remains unchanged, that of $\frac{dF}{d\eta}$ is also increased abruptly. Hence $k-1$ of the developments appertaining to $\frac{dF}{d\eta}$ are of the type

$$\eta = K + \frac{k-1}{k}\lambda_1 \xi^{l_1} + \ldots, \quad \eta = K + \frac{k-2}{k-1}\lambda_2 \xi^{l_2} + \ldots,$$

$$\eta = K + \tfrac{1}{2}\lambda_{k-1}\xi^{l_{k-1}} + \ldots.$$

Again, suppose that s of the indices l are equal; let for example the s lowest indices be equal; then s roots of the equation $\phi'(V) = 0$ are of the form $H_i \xi^l + \ldots$, where $l = l_1 = l_2 \ldots = l_s$; and if

$$\psi(\theta) = (\theta - \lambda_1)(\theta - \lambda_2) \ldots\ldots (\theta - \lambda_s),$$

the s coefficients H_i are the roots of the equation

$$(k-s)\,\psi(\theta) + \theta\psi'(\theta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\theta).$$

If the s equal indices $l_1 \ldots l_s$ are followed by another set of s' indices equal to one another and to l', l' being $> l$, put

$$(\theta - \lambda_{s+1})(\theta - \lambda_{s+2}) \ldots\ldots (\theta - \lambda_{s+s'}) = \psi_1(\theta);$$

then the equation $\phi'(V) = 0$ has s' roots of the form $H'_i \xi^{l'} + \ldots$, the coefficients H'_i being the roots of the equation

$$(k-s-s')\,\psi_1(\theta) + \theta\,\psi'_1(\theta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\theta'),$$

and so on continually. Lastly, considering any group of equal indices l, for example the group $l_{s+1}, l_{s+2} \ldots\ldots l_{s+s'}$, let σ of the corresponding coefficients λ be supposed equal (in which case σ of the developments K coincide with one another for one term at least after $K_\nu \xi^{\alpha_\nu}$); the corresponding equation (θ') will have $\sigma - 1$ roots (and no more) equal to one another and to the equal coefficients λ; so that $\sigma - 1$ of the developments appertaining to the polar will coincide, as far as the term next after $K_\nu \xi^{\alpha_\nu}$, with the σ developments appertaining to C. To carry on these $\sigma - 1$ developments until their complete separation from one another, we must repeat the preceding process as often as may be necessary, using in the first instance $K + \lambda\xi^l$ instead of K, and confining our attention to the σ developments, appertaining to C, in which $K + \lambda\xi^l$ are the initial terms.

As the roots of the equations $\psi(\theta) = 0$, $\psi_1(\theta) = 0$, ... are all different from zero, so also are the roots of the equations (θ), (θ'), ..., except when the highest index l is one of a group of equal indices. In this case, if $\psi(\theta) = \Pi(\theta - \lambda)$, the sign of multiplication extending only to those coefficients λ_i which occur in terms having the greatest exponent l, the last of the equations (θ) is of the form $\psi'(\theta) = 0$, and r of its roots may be equal to zero. When this happens, in the r polar developments corresponding to the zero roots, the terms K are not followed by a term of the form $H\xi^l$, but by a term of higher exponent. To determine this term in each of the r developments, we must use, in forming $\psi(\theta)$, not simply the quantities λ_i, but as many terms of the series $\lambda_i + \lambda'_i \xi^{l'_i} + \ldots$ as may be necessary. The zero roots of $\psi'(\theta) = 0$ are then replaced by roots of the form $H\xi^a$, a being positive, and the initial terms of the r polar developments are given by the formula $K + H\xi^{l+a}$.

We shall employ the preceding method to examine the nature of the polar branches in the vicinity of a superlinear branch. We suppose the superlinear branch to be of the type

$$[\Delta, \Delta_1, \Delta_2, \ldots \Delta_s, \Delta_{s+1} = 1; \ \gamma_1, \gamma_2, \ldots \gamma_s];$$

and we consider only the case in which this superlinear branch (Δ) is not touched by any other branch. The polar has $\Delta - 1$ branches (Δ') touching the superlinear branch. Their developments coincide with one another, and with those of (Δ), as far as the term $[x^{\frac{\gamma}{\Delta}}]$ exclusively. But at this term $\frac{\Delta}{\Delta_1} - 1$ of them cease to osculate any branch of (Δ); they do not contain the term $[x^{\frac{\gamma}{\Delta}}]$, which is replaced in each of them by a term of higher exponent, yet so that the aggregate of the $\frac{\Delta}{\Delta_1} - 1$ exponents cannot exceed $\frac{\gamma}{\Delta_1} - 1$. The remaining $\frac{\Delta}{\Delta_1}(\Delta_1 - 1)$ branches divide themselves into $\frac{\Delta}{\Delta_1}$ groups of $\Delta_1 - 1$ each. The $\Delta_1 - 1$ branches of each group are identical with one another, and with Δ_1 of the branches (Δ), as far as the term $[x^{\frac{\gamma_1}{\Delta}}]$ exclusively. At this term $\frac{\Delta_1}{\Delta_2} - 1$ branches out of each group cease to osculate any branch of (Δ), and the remaining $\frac{\Delta_1}{\Delta_2}(\Delta_2 - 1)$ divide themselves, in the same way as before, into $\frac{\Delta_1}{\Delta_2}$ groups of $\Delta_2 - 1$ each; the branches of each group being identical with one another, and with Δ_2 of the branches of (Δ), as far as the term $(x^{\frac{\gamma_2}{\Delta}})$ exclusively. In this way we obtain the following theorem in which i is to have every value from 0 to s, both inclusively.

"The polar curve of an arbitrary point has $\frac{\Delta}{\Delta_{i+1}} - \frac{\Delta}{\Delta_i}$ branches which form $\frac{\Delta_i}{\Delta_{i+1}} - 1$ superlinear branches of the type.

$$\left[\Delta' = \frac{\Delta}{\Delta_i}, \quad \Delta'_1 = \frac{\Delta_1}{\Delta_i}, \ \ldots\ldots \ \Delta'_{i-1} = \frac{\Delta_{i-1}}{\Delta_i}, \quad \Delta'_i - 1; \right.$$

$$\left. \gamma' = \frac{\gamma}{\Delta_i}, \quad \gamma'_1 = \frac{\gamma_1}{\Delta_i}, \ \ldots\ldots \ \gamma'_{i-1} = \frac{\gamma_{i-1}}{\Delta_i} \right].$$

These superlinear branches coincide with one another, and with the branches of (Δ) as far as the term $[x^{\frac{\gamma_i}{\Delta}}]$ exclusively; instead of the term $[x^{\frac{\gamma_i}{\Delta}}]$ each of them contains a term of higher exponent; the $\frac{\Delta_i}{\Delta_{i+1}} - 1$ superlinear branches may, but do not necessarily, group themselves into higher superlinear branches."

19. The development appertaining to a superlinear branch can always be obtained from the equation of the curve by successive applications of the "analytical triangle." The process has been described by M. Puiseux in his important memoir "Recherches sur les fonctions algébriques." (*Liouville*, Vol. XV., p. 384; see also a paper by M. de la Gournerie, *ibid.*, 2nd series, Vol. XIV., p. 425, Vol. XV., p. 1.) We propose to conclude the present paper by showing how the numbers γ, γ_1, ... Δ, Δ_1, ... present themselves in the course of the operation. Putting, as in Art. 18, η for $p-p_0$, ξ for $q-q_0$, we first of all write the equation $F_1(\eta, \xi) = 0$ in the form $u_\mu + u_{\mu+1} + \ldots$, where u_μ is a homogeneous function of ξ and η of the order μ, which is that of the singular point. If $(\eta - B_0\xi)^a$ is a multiple factor of u_μ the line $\eta - B_0\xi$ is touched by branches (linear or superlinear) of which the aggregate order is a. Put $\eta - B_0\xi = v$; the resulting equation between v and ξ will give precisely a values of v in which the order of v surpasses that of ξ. Form, by the analytical triangle, the equations (of the aggregate order a in v) which give the initial terms of the expansions of these a values. These equations are of the type

$$(v^\nu - K\xi^\lambda)^{a_1} = 0,$$

where λ and ν are relatively prime, $\lambda > \nu$, and $\Sigma a_1\nu = a$; they are always obtained linearly, except when there are s of them in which the numbers a_1, λ, ν are all the same; in which case the analytical triangle determines an equation, of order s, having constant coefficients, of which the roots are the s quantities K. There are four cases to be considered: (i.) $a_1=1$, $\nu=1$; (ii.) $a_1=1$, $\nu>1$; (iii.) $a_1>1$, $\nu=1$; (iv.) $a_1>1$, $\nu>1$. (i.) To the equation $v-K\xi^\lambda=0$, $\lambda>1$, answers a linear branch which, considered by itself, has no point-singularity (if λ is >2, it is an inflexion). (ii.) To the equation $v^\nu - K\xi^\lambda = 0$ answers a superlinear branch of which the character is defined by the equations $\Delta=\nu$, $\Delta_1=1$, $\gamma=\lambda$; its development proceeds by integral powers of $\xi^{\frac{1}{\nu}}$, and the successive terms are obtained linearly by the analytical triangle. (iii.) To the equation $(v-K\xi^\lambda)^{a_1}=0$ answer a_1 branches, which may be all linear, but which also may group themselves in whole or in part into superlinear branches; if Δ, Δ', Δ'' ... are the orders of these separate linear or superlinear branches, we have $\Sigma\Delta = a_1$. (iv.) To the equation $(v^\nu - K\xi^\lambda)^{a_1} = 0$ answer $a_1\nu$ branches, which may belong to a_1 distinct superlinear branches of the type ($\Delta=\nu$, $\Delta_1 = 1$, $\gamma=\lambda$); these superlinear branches may however themselves be grouped, wholly or in part, into branches of higher superlinearity; if Δ, Δ', Δ'' ... are the orders of the distinct superlinear branches, these numbers are all divisible by ν, and $\Sigma\frac{\Delta}{\nu} = a_1$; we have also for every one of them

$\frac{\Delta}{\Delta_1}=\nu$, $\frac{\gamma}{\Delta_1}=\lambda$, Δ_1 having to be determined subsequently for each of them separately. With the cases (i.) and (ii.) we have nothing further to do; the case (iii.) may be regarded as included under (iv.); we therefore continue the process in this last case only. Put $v-K^{\frac{1}{\nu}}\xi^{\frac{\lambda}{\nu}}=v_1$, $\xi^{\frac{1}{\nu}}$ representing any one determinate value of the radical; and form by the analytical triangle equations of the type

$$(v_1^{\nu_1}-K_1\xi^{\frac{\lambda_1}{\nu}})^{a_1}=0,$$

of which the aggregate order in v_1 is a_1, and which give the initial terms of those a_1 values of v_1, of which the order surpasses that of $\xi^{\frac{\lambda}{\nu}}$; we have of course $\frac{\lambda_1}{\nu\nu_1}>\frac{\lambda}{\nu}$, or $\frac{\lambda_1}{\nu_1}>\lambda$; λ_1 and ν_1 are relatively prime, but we observe that λ_1 is not necessarily prime to ν. We consider the same four cases as before. (i.) To the equation $v_1-K_1\xi^{\frac{\lambda_1}{\nu}}=0$, or more properly to the ν equations comprehended in it, answers a superlinear branch of the type $(\Delta=\nu,\ \Delta_1=1,\ \gamma=\nu)$. (ii.) To the ν equations $v_1^{\nu_1}-K_1\xi^{\frac{\lambda_1}{\nu}}=0$ there also answers a single superlinear branch for which $\Delta=\nu\nu_1$, $\Delta_1=\nu_1$, $\Delta_2=1$; $\gamma=\lambda\nu_1$, $\gamma_1=\lambda_1$; *i.e.*, a superlinear branch of the type $\left(\frac{\Delta}{\Delta_1}=\nu,\ \frac{\Delta_1}{\Delta_2}=\nu_1,\ \Delta_2=1;\ \gamma=\lambda\Delta_1,\ \gamma_1=\lambda_1\Delta_2\right)$. In this case, as well as in (i.), the discussion of the superlinearity is complete. (iii.) To the ν equations $(v_1-K_1\xi^{\frac{\lambda}{\nu}})^{a_2}=0$ there may answer a_2 superlinear branches of the type $\left(\frac{\Delta}{\Delta_1}=\nu,\ \Delta_1=1;\ \gamma=\lambda\Delta_1\right)$; or these may group themselves in any manner into higher superlinear branches for each of which $\frac{\Delta}{\Delta_1}=\nu$, $\gamma=\lambda\Delta_1$; the numbers Δ_1 (which have to be determined for each branch separately), satisfying the condition $\Sigma\Delta_1=a_2$. (iv.) To the equations $(v_1^{\nu_1}-K_1\xi^{\frac{\lambda_1}{\nu}})^{a_2}$ answer a certain number of superlinear branches, for each of which $\frac{\Delta}{\Delta_1}=\nu$, $\frac{\Delta_1}{\Delta_2}=\nu_1$; $\gamma=\lambda\Delta_1$, $\gamma_1=\lambda_1\Delta_2$; while Δ_2 and the subsequent numbers of the series have still to be determined, and may be different for each of them; we have however the equation $\Sigma\Delta_2=\Sigma\frac{\Delta_1}{\nu_1}=a_2$. The process, which we need not follow further, may be considered to terminate for any particular development, when that development is separated from every other, and can be continued linearly. This will happen when, in the series

$a\, a_1\, a_2 \ldots$, we arrive at a term equal to unity. And we shall eventually arrive at such a term; for, though the second of two consecutive indices a may be as great as the first (the equation $(v - \mathrm{K}\xi^{\frac{\lambda}{\nu}})^{a_1}$ may, for example, at the next step in the process, lead to only one equation; and this may be of the type $(v_1 - \mathrm{K}_1 \xi^{\frac{\lambda_1}{\nu}})^{a_1} = 0$, so that we should have $a_2 = a_1$), yet it is impossible for two branches to osculate one another indefinitely, because the discriminantal index is necessarily finite. If a_s be the first of the indices a which is equal to unity, we have

$$\frac{\Delta}{\Delta_1} = \nu, \quad \frac{\Delta_1}{\Delta_2} = \nu_1, \ldots \quad \frac{\Delta_s}{\Delta_{s+1}} = \nu_s, \quad \Delta_{s+1} = 1\,;$$

and the development appertaining to the superlinear branch is of the type

$$(\Delta = \nu\, \nu_1\, \nu_2 \ldots \nu_s, \quad \Delta_1 = \nu_1\, \nu_2 \ldots \nu_s, \quad \ldots\ldots, \quad \Delta_s = \nu_s, \quad \Delta_{s+1} = 1,$$
$$\gamma = \lambda \Delta_1, \quad \gamma_1 = \lambda_1 \Delta_2, \quad \ldots\ldots, \quad \gamma_{s-1} = \lambda_{s-1} \Delta_s, \quad \gamma_s = \lambda_s).$$

On Hamilton's Characteristic Function for a Narrow Beam of Light.

By J. Clerk-Maxwell, M.A., F.R.S.

[*Read January* 8*th*, 1874.]

Hamilton's characteristic function V is an expression for the time of propagation of light from the point whose coordinates are x_1, y_1, z_1 to the point whose coordinates are x_2, y_2, z_2. It is a function of these six coordinates of the two points. The axes to which the coordinates are referred may be different for the two points.

In isotropic media the differential equation of V may be written

$$\left(\frac{d\mathrm{V}}{dx}\right)^2 + \left(\frac{d\mathrm{V}}{dy}\right)^2 + \left(\frac{d\mathrm{V}}{dz}\right)^2 = \mu^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$

where μ is the slowness of propagation at a point in the medium whose coordinates are x, y, z, and is a function of these coordinates. If the time of propagation through the unit of length in vacuum be taken as the unit of time, then μ is the index of refraction of the medium.

The form of the equation in doubly refracting media, as given by Hamilton, is not required for our present purpose.

Let OPQR be the path of a ray of light. Let the part OP be in a homogeneous medium whose index of