

we obtain

$$(a+4b+14c) \begin{vmatrix} 1 & 3b+3c & . & . \\ 4 & a+8c & 2b+4c & . \\ 9 & 7b-21c & a+18c & b+3c \\ 16 & . & 8b-32c & a+32c \end{vmatrix},$$

of which the determinant factor is reducible to the three-line form

$$\begin{vmatrix} a-12b-4c & 2b+4c & . \\ -20b-48c & a+18c & b+3c \\ -48b-48c & 8b-32c & a+32c \end{vmatrix}.$$

The next operation

$$\text{col}_1 + 6 \text{col}_2 + 20 \text{col}_3$$

gives in like manner

$$(a+20c) \begin{vmatrix} 1 & 2b+4c & . \\ 6 & a+18c & b+3c \\ 20 & 8b-32c & a+32c \end{vmatrix}$$

or

$$(a+20c) \begin{vmatrix} a-12b-6c & b+3c \\ -32b-112c & a+32c \end{vmatrix};$$

and finally the operation

$$\text{col}_1 + 8 \text{col}_2$$

enables us to change this two-line determinant into

$$(a-4b+18c) \begin{vmatrix} 1 & b+3c \\ 8 & a+32c \end{vmatrix},$$

or

$$(a-4b+18c)(a-8b+8c).$$

The desired result thus is

$$(a+8b)(a+4b+14c)(a+20c)(a-4b+18c)(a-8b+8c).$$

(3) *The continuant of the n^{th} order whose main diagonal is*

$$a, \quad a+2(1.3)c, \quad a+2(2.4)c, \quad a+2(3.5)c, \dots$$

and whose minor diagonals are

$$\begin{aligned} &(n-1)b, \quad (n-2)(b+c), \quad (n-3)(b+2c), \dots \\ &(n+2)(b-3c), \quad (n+3)(b-4c), \quad (n+4)(b-5c), \dots \end{aligned}$$

is equal to the product of the n factors

$$\begin{aligned} &\{a+2(n-1)b\}, \\ &\{a+2(n-3)b+2(2n-1)c\}, \\ &\{a+2(n-5)b+4(2n-3)c\}, \\ &\dots \\ &\{a-2(n-1)b+6(n-1)c\}. \end{aligned} \quad (\text{II})$$

This is established by proceeding in the same way as in § 2, the sets of column-multipliers now being

$$\left. \begin{array}{l} 1, \quad 2, \quad 3, \quad 4, \quad 5, \dots \\ 1, \quad 4, \quad 10, \quad 20, \dots \\ 1, \quad 6, \quad 21, \dots \\ 1, \quad 8, \dots \\ 1, \dots \end{array} \right\} \text{ instead of } \left\{ \begin{array}{l} 1, \quad 1, \quad 1, \quad 1, \quad 1, \dots \\ 1, \quad 4, \quad 9, \quad 16, \dots \\ 1, \quad 6, \quad 20, \dots \\ 1, \quad 8, \dots \\ 1, \dots \end{array} \right.$$

and the resulting factors greater by

$$0, \quad 4c, \quad 8c, \quad 12c, \quad 16c, \quad \dots$$

respectively.

(4) Changing a into $a + 2c$, n into $n - 1$, b into $b + c$ in § 3 we have

$$\begin{vmatrix} a+2c & (n-2)(b+c) & . & . & . \\ (n+1)(b-2c) & a+8c & (n-3)(b+2c) & . & . \\ . & (n+2)(b-3c) & a+18c & (n-4)(b+3c) & . \\ . & . & (n+3)(b-4c) & a+32c & . \\ . & . & . & . & . \end{vmatrix}$$

$$= \{a+2(n-2)b+2(n-1)c\} \cdot \{a+2(n-4)b+6(n-2)c\} \\ \cdot \{a+2(n-6)b+10(n-3)c\} \dots$$

But the continuant here is the complementary minor of the element in the place (1, 1) of the continuant in § 2. Consequently by division we obtain

$$a - \frac{2(n-1)n(b-c)b}{a+2 \cdot 1^2 \cdot c} - \frac{(n-2)(n+1)(b-2c)(b+c)}{a+2 \cdot 2^2 \cdot c} - \dots - \frac{1 \cdot (2n-2)(b-nc+c)(b+nc-2c)}{a+2 \cdot (n-1)^2 \cdot c}$$

$$= \frac{\{a+2(n-1)b\} \{a+2(n-3)b+2(2n-3)c\} \{a+2(n-5)b+4(2n-5)c\} \dots}{\{a+2(n-2)b+2(n-1)c\} \{a+2(n-4)b+6(n-2)c\} \dots} \quad \text{(III)}$$

(5) If in the results of §§ 2, 3 we annex f as a factor to every term on both sides that is independent of a , the identity is not interfered with. (IV)

For, taking (in the fourth order, for shortness' sake) the continuant dealt with in § 2, and putting a/f for a we have

$$\begin{vmatrix} \frac{a}{f} & 3b & . & . \\ 6(b-3c) & \frac{a}{f} + 6c & 2(b+c) & . \\ . & 7(b-4c) & \frac{a}{f} + 16c & b+2c \\ . & . & 8(b-5c) & \frac{a}{f} + 30c \end{vmatrix}$$

$$= \left(\frac{a}{f} + 6b\right) \left(\frac{a}{f} + 2b + 14c\right) \left(\frac{a}{f} - 2b + 20c\right) \left(\frac{a}{f} - 6b + 18c\right),$$

whence on multiplying both sides by f^4 there results

$$\begin{vmatrix} a & 3bf & . & . \\ 6(b-3c)f & a+6cf & 2(b+c)f & . \\ . & 7(b-4c)f & a+16cf & (b+2c)f \\ . & . & 8(b-5c)f & a+30cf \end{vmatrix}$$

$$= (a+6bf)(a+2bf+14cf)(a-2bf+20cf)(a-6bf+18cf),$$

as asserted.

(6) *The sum of the elements of the main diagonal of either of the continuants in §§ 2, 3 is equal to the sum of the factors into which the continuant is resolved.* . . . (V)

This is true of any continuant of the form

$$\begin{vmatrix} a+x_1 & p_1 & . & . & . & . \\ q_1 & a+x_2 & p_2 & . & . & . \\ . & q_2 & a+x_3 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{vmatrix}_n$$

that is resolvable into factors linear in a . By way of proof we have only to note (1) that since the diagonal term is the only term of the continuant that contains either the n^{th} or $(n-1)^{\text{th}}$ powers of a , it follows that the coefficient of a^{n-1} in the continuant is $x_1 + x_2 + x_3 + \dots$, or Σx say: and (2) that if $a + \mu_1, a + \mu_2, a + \mu_3, \dots$ be the factors into which the continuant is resolved, the coefficient of a^{n-1} in their product is $\mu_1 + \mu_2 + \mu_3 + \dots$, or $\Sigma \mu$ say. We thus have

$$\Sigma x = \Sigma \mu,$$

and \therefore

$$na + \Sigma x = na + \Sigma \mu,$$

as was to be proved.

(7) The full table of multipliers used in § 2 is found to be

$$\begin{array}{l} 1, 1, 1, 1, 1, \dots, 1 \\ 1, 4, 9, 16, \dots, \frac{r}{1} C_{r,1} \\ 1, 6, 20, \dots, \frac{r}{2} C_{r+1,3} \\ 1, 8, \dots, \frac{r}{3} C_{r+2,5} \\ 1, \dots, \frac{r}{4} C_{r+3,7} \\ \dots \end{array}$$

—in other words, each multiplier is of the form

$$\frac{r}{s} C_{r+s-1, 2s-1};$$

and the question next arises whether the continuant resolved in § 2 is the only one which this set of multipliers is capable of dealing with. In order to make suitable answer we have to ascertain the relations which must exist between the twelve quantities

$$\begin{array}{cccc} \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ p, & q, & r, & s \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{array}$$

in the continuant

$$\begin{vmatrix} a & 2.4\beta_1 & . & . & . \\ 5\gamma_1 & a+p & 3\beta_2 & . & . \\ . & 6\gamma_2 & a+q & 2\beta_3 & . \\ . & . & 7\gamma_3 & a+r & \beta_4 \\ . & . & . & 8\gamma_4 & a+s \end{vmatrix}$$

in order that it may be resolvable into linear factors by means of the operations of § 2.

The performance of the first operation on the columns evolves the equations

$$\begin{aligned} a + 8\beta_1 &= 5\gamma_1 + a + p + 3\beta_2, \\ &= 6\gamma_2 + a + q + 2\beta_3, \\ &= 7\gamma_3 + a + r + \beta_4, \\ &= 8\gamma_4 + a + s. \end{aligned}$$

Then the factor $a + 8\beta_1$ being removed, the cofactor becomes expressible as a determinant of the next lower order, viz.

$$\begin{vmatrix} a + p - 8\beta_1 & 3\beta_2 & . & . \\ 6\gamma_1 - 8\beta_1 & a + q & 2\beta_3 & . \\ -8\beta_1 & 7\gamma_3 & a + r & \beta_4 \\ -8\beta_1 & . & 8\gamma_4 & a + s \end{vmatrix};$$

and as by hypothesis the performance of the operation

$$\text{col}_1 + 4 \text{col}_2 + 9 \text{col}_3 + 16 \text{col}_4$$

enables us to remove the factor $a + p - 8\beta_1 + 12\beta_2$ we obtain three other conditional equations, to be followed at the next stage by two others, and last of all by one. Of these $4 + 3 + 2 + 1$ equations the four first obtained are obviously best used to determine p, q, r, s in terms of the β 's and γ 's, the results being

$$\begin{aligned} p &= 8\beta_1 - 3\beta_2 - 5\gamma_1, \\ q &= 8\beta_1 - 2\beta_3 - 6\gamma_2, \\ r &= 8\beta_1 - \beta_4 - 7\gamma_3, \\ s &= 8\beta_1 - 8\gamma_4. \end{aligned}$$

The remaining six form a very interesting set: after simplification they are

$$\begin{aligned} 12\beta_1 - 18\beta_2 + 5\beta_3 &= -10\gamma_1 + 9\gamma_2 \\ 64\beta_1 - 81\beta_2 + 7\beta_4 &= -45\gamma_1 + 35\gamma_3 \\ 15\beta_1 - 18\beta_2 &= -10\gamma_1 + 7\gamma_4 \\ 45\beta_2 - 60\beta_3 + 14\beta_4 &= -36\gamma_2 + 35\gamma_3 \\ 24\beta_2 - 25\beta_3 &= -15\gamma_2 + 14\gamma_4 \\ \beta_3 - \beta_4 &= -\gamma_3 + \gamma_4. \end{aligned}$$

Taking the first three and using with them the multipliers 7, -1 , 1 respectively, we find, on adding, that

$$5(\beta_1 + \gamma_1) - 9(\beta_2 + \gamma_2) + 5(\beta_3 + \gamma_3) - (\beta_4 + \gamma_4) = 0;$$

similarly from the subset of two there results by subtraction

$$3(\beta_2 + \gamma_2) - 5(\beta_3 + \gamma_3) + 2(\beta_4 + \gamma_4) = 0;$$

and the final set, of course, is

$$(\beta_3 + \gamma_3) - (\beta_4 + \gamma_4) = 0.$$

By means, therefore, of these three derived equations we arrive at the proposition that *in the determinant under discussion the sum of any β and the corresponding γ is constant.*

This being equivalent to only three equations, and other three being still unaccounted for, we put

$$\gamma_1, \gamma_2, \gamma_3, \gamma_4 = \sigma - \beta_1, \sigma - \beta_2, \sigma - \beta_3, \sigma - \beta_4.$$

and learn (1) that one of the equations is not independent of the others, (2) that the β 's are connected by the equation

$$\beta_1 - 2\beta_2 + \beta_3 = 7(\beta_2 - 2\beta_3 + \beta_4),$$

and (3) that σ is expressible in terms of any three of the β 's, for example,

$$\sigma = -2\beta_1 + 9\beta_2 - 5\beta_3.$$

The conclusion thus is that in the continuant with which we started we can retain any three of the β 's, and express in terms of these the fourth β , all the γ 's, and p, q, r, s ,—thus obtaining a function of *four* variables which is resolvable into linear factors.

(8) Had the determinant operated upon been of the sixth order, we should still have found $\sigma = -2\beta_1 + 9\beta_2 - 5\beta_3$ and the first four β 's connected by the same equation as in the preceding case, but there would have been a fresh equation of condition connecting the second set of four consecutive β 's, viz.

$$\beta_2 - 2\beta_3 + \beta_4 = 3(\beta_3 - 2\beta_4 + \beta_5).$$

Similarly the case of the seventh order would be found to differ from that of the sixth merely in having the additional equation

$$5(\beta_3 - 2\beta_4 + \beta_5) = 11(\beta_4 - 2\beta_5 + \beta_6);$$

and so on.

As the result of all this we therefore affirm that — *If the continuant*

$$\begin{vmatrix} a & 2(n-1)\beta_1 & . & . & . & . \\ n\gamma_1 & a+p & (n-2)\beta_2 & . & . & . \\ . & (n+1)\gamma_2 & a+q & (n-3)\beta_3 & . & . \\ . & . & (n+2)\gamma_3 & a+r & . & . \\ . & . & . & . & . & . \end{vmatrix}_n$$

be resolvable into linear factors by means of the set of multipliers

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & \\ & 1 & 4 & 9 & 16 & \\ & & 1 & 6 & 20 & \\ & & & 1 & 8 & \\ & & & & 1 & \\ & & & & & \end{array}$$

then (1) every four consecutive β 's are connected by a linear relation, viz.

$$\begin{array}{l} 1 \cdot (\beta_1 - 2\beta_2 + \beta_3) = 7 \cdot (\beta_2 - 2\beta_3 + \beta_4), \\ 3 \cdot (\beta_2 - 2\beta_3 + \beta_4) = 9 \cdot (\beta_3 - 2\beta_4 + \beta_5), \\ 5 \cdot (\beta_3 - 2\beta_4 + \beta_5) = 11 \cdot (\beta_4 - 2\beta_5 + \beta_6), \\ \end{array}$$

thus making all the β 's expressible in terms of any three; (2) all the γ 's are expressible in terms of the same three β 's because of the fact that $\beta_m + \gamma_m = -2\beta_1 + 9\beta_2 - 5\beta_3$ for all values of m ; and (3) $p, q, r, . . .$ are also so expressible because the sum of the elements of any row of the continuant is constant. (VI)

(9) Instead, however, of taking a and three of the β 's as variables it is better to take a and $\beta_1, \beta_1 - \beta_2, \beta_1 - 2\beta_2 + \beta_3$. Doing this and calling the last three quantities b, c, d ,—a change implying the substitution

$$\begin{aligned}\beta_1 &= b, \\ \beta_2 &= b - c, \\ \beta_3 &= b - 2c + d,\end{aligned}$$

—we can by using the equations of condition obtain the requisite expressions for $\sigma, \beta_4, \beta_5, \dots, \gamma_1, \gamma_2, \dots$ in terms of b, c, d . The theorem to which this course ultimately leads is—*The continuant*

$$\begin{vmatrix} A_1 & 2(n-1)\beta_1 & . & . & . & . \\ n\gamma_1 & A_2 & (n-2)\beta_2 & . & . & . \\ . & (n+1)\gamma_2 & A_3 & (n-3)\beta_3 & . & . \\ . & . & (n+2)\gamma_3 & A_4 & . & . \\ . & . & . & . & . & . \end{vmatrix}_n$$

is resolvable into linear factors if

$$\begin{aligned}\beta_m &= b - (m-1)c + \frac{(m-1)(m-2)}{2(2m-1)} \cdot 5d, \\ \gamma_m &= b + mc - \frac{(m+1)m}{2(2m-1)} \cdot 5d, \\ A_m &= a - 2(m-1)^2c + \frac{(m-1)^2}{(2m-3)(2m-1)} \{2m(m-2) + 3n\} \cdot 5d,\end{aligned}$$

the s^{th} factor being

$$a + 2(n-2s+1)b - 2(s-1)(2n-2s+1)c + (n-s+2)(s-1) \cdot 5d \quad \text{. (VII)}$$

For example, when $n=4$ we have

$$\begin{vmatrix} a & 2 \cdot 3b & . & . \\ 4(b+c-5d) & a-2c+20d & 2(b-c) & . \\ . & 5(b+2c-5d) & a-8c+24d & b-2c+d \\ . & . & 6(b+3c-6d) & a-18c+36d \end{vmatrix} \\ = (a+6b)(a+2b-10c+20d)(a-2b-12c+30d)(a-6b-6c+30d).$$

On putting $d=0$ the β 's and γ 's form equidifferent progressions, and the theorem degenerates into that of § 2.

(10) Out of this effort to obtain greater generality an unexpected and curious result arises; for, whereas at first sight both members of the identity are functions of the four variables a, b, c, d , it is found on careful examination that the right-hand side is expressible as a function of one less. In fact it is readily verified that the s^{th} factor given above can also be written in the form

$$\{a + 2(n-1)b\} - 2(s-1)\{2b-3c\} - (n-s+2)(s-1)\{4c-5d\}$$

so that the factorial expression for the continuant of the n^{th} order, besides being

$$\begin{aligned}\{a + 2(n-1)b\} &\cdot \{a + 2(n-3)b - 2(2n-3)c + 1 \cdot n \cdot 5d\} \\ &\cdot \{a + 2(n-5)b - 4(2n-5)c + 2 \cdot (n-1) \cdot 5d\} \\ &\cdot \{a + 2(n-7)b - 6(2n-7)c + 3 \cdot (n-2) \cdot 5d\} \\ &\cdot \{a - 2(n-1)b - (2n-2) \cdot 1 \cdot c + (n-1) \cdot 2 \cdot 5d\},\end{aligned}$$

(12) There is a special case of the theorem (VII) in § 9 which deserves particular attention, viz. the case where σ vanishes, i.e. where $5d = 2b + c$. In this case the s^{th} factor

$$\begin{aligned} &= \{a + 2(n-1)b\} - 2(s-1)\{2b-3c\} + (n-s+2)(s-1)\{2b-3c\} \\ &= \{a + 2(n-1)b\} + (n-s)(s-1)(2b-3c); \end{aligned}$$

and the $(n-s+1)^{\text{th}}$ factor

$$\begin{aligned} &= \{a + 2(n-1)b\} - 2(n-s)\{2b-3c\} + (s+1)(n-s)\{2b-3c\} \\ &= \{a + 2(n-1)b\} + (n-s)(s-1)(2b-3c). \end{aligned}$$

This means that when $5d = 2b + c$ in the continuant of § 9 the s^{th} factor from the beginning is the same as the s^{th} factor from the end, and consequently that an even-ordered continuant of this kind is a square. (IX)

(13) The question of the generalisation of the theorem of § 3 may be investigated in a manner perfectly similar to that followed in the preceding paragraphs with regard to the theorem of § 2. The essential point of difference is to be found in the new set of column-multipliers, which are now all of the form $C_{r+2s, r-1}$ instead of ${}^rC_{r+s-1, 2s-1}$. It will suffice merely to enunciate the results. The first is—

If the continuant

$$\begin{vmatrix} a & (n-1)\beta_1 & . & . & . & . \\ (n+2)\gamma_1 & a+p & (n-2)\beta_2 & . & . & . \\ . & (n+3)\gamma_2 & a+q & (n-3)\beta_3 & . & . \\ . & . & (n+4)\gamma_4 & a+r & . & . \\ . & . & . & . & . & . \end{vmatrix}$$

be resolvable into linear factors by means of the set of column-multipliers

$$\begin{array}{ll} 1, 2, 3, 4, 5, & \dots\dots C_{r,1} \\ 1, 4, 10, 20, & \dots\dots C_{r+2,3} \\ 1, 6, 21, & \dots\dots C_{r+4,5} \\ 1, 8, & \dots\dots C_{r+6,7} \\ 1, & \dots\dots C_{r+8,9} \end{array}$$

then (1) every four consecutive β 's are connected by a linear relation, viz.

$$\begin{aligned} 3(\beta_1 - 2\beta_2 + \beta_3) &= 9(\beta_2 - 2\beta_3 + \beta_4), \\ 5(\beta_2 - 2\beta_3 + \beta_4) &= 11(\beta_3 - 2\beta_4 + \beta_5), \\ &\dots\dots\dots \end{aligned}$$

thus making all the β 's expressible in terms of any three; (2) all the γ 's are expressible in terms of the same three β 's because of the fact that for all values of m $\beta_m + \gamma_m = -9\beta_1 + 25\beta_2 - 14\beta_3$; and (3) p, q, r, \dots are also so expressible because of the mode of removing the first factor from the continuant. (X)

It should be noted that the linear relation connecting the first four consecutive β 's is that which in § 8 connects the second four,—that, in fact, the relation here is

$$(2r+1)\{\beta_r - 2\beta_{r+1} + \beta_{r+2}\} = (2r+7)\{\beta_{r+1} - 2\beta_{r+2} + \beta_{r+3}\}$$

whereas in § 8 it is

$$(2r-1)\{\beta_r - 2\beta_{r+1} + \beta_{r+2}\} = (2r+5)\{\beta_{r+1} - 2\beta_{r+2} + \beta_{r+3}\}.$$

Further, there is a similar difference in the expressions for σ : here the expression is

$$-9\beta_1 + 25\beta_2 - 14\beta_3,$$

whereas in § 8 when given in terms of $\beta_2, \beta_3, \beta_4$ it is

$$-9\beta_2 + 25\beta_3 - 14\beta_4.$$

The second theorem is—*The continuant*

$$\begin{vmatrix} A_1 & (n-1)\beta_1 & . & . & . & . & . \\ (n+2)\gamma_1 & A_2 & (n-2)\beta_2 & . & . & . & . \\ . & (n+3)\gamma_2 & A_3 & (n-3)\beta_3 & . & . & . \\ . & . & (n+4)\gamma_3 & A_4 & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix}_n$$

is resolvable into linear factors if

$$\beta_m = b - (m-1)c + \frac{(m-1)(m-2)}{2(2m+1)} \cdot 7d,$$

$$\gamma_m = b + (m+2)c - \frac{(m+3)(m+2)}{2(2m+1)} \cdot 7d,$$

$$A_m = a - 2(m^2-1)c + \frac{m^2-1}{4m^2-1}(2m^2+2+5n) \cdot 7d,$$

the s^{th} factor being

$$a + 2(n-2s+1)b - 2(s-1)(2n-2s+3)c \\ + (n-s+4)(s-1) \cdot 7d,$$

or

$$\{a + 2(n-1)b\} - 2(s-1)\{2b-5c\} - (n-s+4)(s-1)\{4c-7d\}. \quad (\text{XI})$$

An immediate deduction from this is that when $14d=2b+3c$ the s^{th} factor is

$$a + 2(n-1)b + \frac{1}{2}(n-s)(s-1)(2b-5c)$$

and is the same as the s^{th} factor from the end, so that when in addition n is even the continuant is a square. (XII)

The third theorem is—*The value of any continuant of the form referred to in § 3 is not altered by adding to its matrix the matrix of the continuant*

$$\begin{vmatrix} -\frac{2}{1.3}(n-1)e & \frac{1}{3}(n-1)e & . & . & . & . \\ -\frac{1}{3}(n+2)e & -\frac{2}{3.5}(n-7)e & \frac{1}{5}(n-2)e & . & . & . \\ . & -\frac{1}{5}(n+3)e & -\frac{2}{5.7}(n-17)e & \frac{1}{7}(n-3)e & . & . \\ . & . & -\frac{1}{7}(n+4)e & -\frac{2}{7.9}(n-31)e & . & . \\ . & . & . & . & . & . \end{vmatrix} \quad (\text{XIII})$$

(14) The corresponding theorems for the set of column-multipliers

$$\begin{array}{l} 1, 1, 1, 1, 1, \dots, C_{r-1,0} \\ 1, 2, 2, 4, \dots, C_{r,1} \\ 1, 3, 6, \dots, C_{r+1,2} \\ 1, 4, \dots, C_{r+2,3} \\ 1, \dots, C_{r+3,4} \end{array}$$

will be found in the *Trans. S. Afr. Philos. Soc.* referred to above.

When the first theorem there given is attempted to be generalised in the manner employed in the present paper the following is the result:—

The continuant of the n^{th} order whose main diagonal is

$$a, \quad a + (b - c) - (n - 2)\gamma, \quad a + 2(b - c) - 2(n - 4)\gamma, \quad a + 3(b - c) - 3(n - 6)\gamma, \dots$$

and whose minor diagonals are

$$\begin{array}{cccc} (n-1)b, & (n-2)(b+\gamma), & (n-3)(b+2\gamma), & \\ c, & 2(c-\gamma), & 3(c-2\gamma), & \dots \end{array}$$

is equal to the product of the n factors

$$\begin{aligned} & \{a + (n-1)b\} \\ & \cdot \{a + (n-2)b - c + 1 \cdot (n-2)\gamma\} \\ & \cdot \{a + (n-3)b - 2c + 2 \cdot (n-3)\gamma\} \\ & \cdot \dots \cdot \{a - (n-1)c\} \end{aligned} \quad \text{(XIV)}$$

It is seen to degenerate into the original theorem when γ is put equal to 0. If, however, we write X in it for the half-sum of the first and last factors, and Y for $b + c$, the factors may be written

$$\begin{aligned} & X + \frac{1}{2}(n-1)Y, \\ & X + \frac{1}{2}(n-3)Y + 1 \cdot (n-2)\gamma, \\ & X + \frac{1}{2}(n-5)Y + 2 \cdot (n-3)\gamma, \\ & \cdot \dots \cdot \\ & X - \frac{1}{2}(n-1)Y; \end{aligned}$$

thus showing that four variables are not necessary for the expression of the identity. An easier way of reaching the same result is to put

$$\begin{aligned} a &= \alpha - (n-1)\xi, \\ b &= \beta + \xi, \\ c &= \beta - \xi, \end{aligned}$$

when it will be found that ξ appears in the continuant but not in its factors; and when there are consequently obtained at one and the same time the case of theorem (XIV) where $b = c$, and an expression for the corresponding nil-factor continuant.

(15) We have thus in all at present three sets of column-multipliers, each of which has associated with it a linearly resolvable continuant of the form

$$\begin{vmatrix} a & \beta_1 & \cdot & \cdot \\ \gamma_1 & a+p & \beta_2 & \cdot \\ \cdot & \gamma_2 & a+q & \beta_3 \\ \cdot & \cdot & \gamma_3 & a+r \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

Other sets will doubtless be discovered, as the only difficulty is the devising of a set which will not lead to unreasonably complicated expressions for the elements of the continuant. *In all cases if*

$$\xi_1, \xi_2, \xi_3, \dots, \xi_{n-1}$$

and thence in the form

$$\begin{vmatrix} a & 2\cdot 4b & . & . \\ 5(b+c) & a-2c & 3(b-c) & . \\ . & 6(b+2c) & a-8c & 2(b-2c) \\ -35(b-3c) & 56(b-3c) & -21(b-5c) & a+8b-42c \end{vmatrix}.$$

Similarly the operation

$$\text{row}_4 - 6 \text{row}_3 + 15 \text{row}_2 - 10 \text{row}_1$$

now enables us to remove the factor $a-4b-18c$; and the operations

$$\begin{aligned} \text{row}_3 &= 4 \text{row}_2 + 3 \text{row}_1, \\ \text{row}_2 &= 1 \text{row}_1 \end{aligned}$$

the remaining factors. The set of column-multipliers

$$\begin{array}{cccc} 1, & 1, & 1, & 1, & 1 \\ & 1, & 4, & 9, & 16 \\ & & 1, & 6, & 20 \\ & & & 1, & 8 \\ & & & & 1 \end{array}$$

is thus equivalent to the set of row-multipliers

$$\begin{array}{cccc} 1, & -8, & 28, & -56, & 35 \\ & 1, & -6, & 15, & -10 \\ & & 1, & -4, & 3 \\ & & & 1, & -1 \\ & & & & 1. \end{array}$$

(17) The general result is that *the table of row-multipliers suitable for the resolution of the continuant of § 2 is*

$$\left. \begin{array}{ccccccc} 1, & -C_{2n-2,1}, & C_{2n-2,2}, & -C_{2n-2,3}, & \dots & (-)^{n+1} \frac{1}{2} C_{2n-2,n-1} \\ & 1, & -C_{2n-4,1}, & C_{2n-4,2}, & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \end{array} \right\} \quad (\text{XVI})$$

Similarly it is found after a little investigation that *the table of row-multipliers suitable for the resolution of the continuant of § 3 is*

$$\left. \begin{array}{ccccccc} 1, & -\frac{2}{1}(n-1)C_{2n-1,0}, & \frac{2}{2}(n-2)C_{2n-1,1}, & -\frac{2}{3}(n-3)C_{2n-1,2}, & \dots & \dots & \dots \\ & 1, & -\frac{2}{1}(n-2)C_{2n-3,0}, & \frac{2}{2}(n-3)C_{2n-3,1}, & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \end{array} \right\} \quad (\text{XVII})$$

the general form of the multiplier being $2_{s+1}^{r-s} C_{2r+1,s}$.

Lastly, *the table of row-multipliers suitable for the resolution of the continuant of § 13 is*

$$\left. \begin{array}{ccccccc} 1, & -C_{n-1,1}, & C_{n-1,2}, & -C_{n-1,3}, & \dots & \dots & \dots \\ & 1, & -C_{n-2,1}, & C_{n-2,2}, & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \end{array} \right\} \quad (\text{XVIII})$$

—that is to say, may be got by a rearrangement of the column-multipliers: for

example, in the case of the 5th order the equivalent tables of column-multipliers and row-multipliers are

$$\begin{array}{ccccc} 1, & 1, & 1, & 1, & 1 \\ & 1, & 2, & 3, & 4 \\ & & 1, & 3, & 6 \\ & & & 1, & 4 \\ & & & & 1, \\ & & & & & 1. \end{array} \quad \begin{array}{ccccc} 1, & -4, & 6, & -4, & 1 \\ & 1, & -3, & 3, & -1 \\ & & 1, & -2, & 1 \\ & & & 1, & 1 \\ & & & & 1. \end{array}$$

(18) There falls now to be noted a set of theorems regarding resolvable continuants of a totally different form but connected with and derivable from those of §§ 2, 3, 13. If in any one of these latter theorems we put 0 for the element in the place (1, 1), the continuant is expressible as the negative product of the elements in the places (1, 2), (2, 1), and a continuant of the lower order $n-2$: further, one of the said elements is contained in the first factor of the original continuant and the other in the last factor: in this way, therefore, the resolution of the new continuant of order $n-2$ is secured. Thus, taking the five-line continuant dealt with in § 2 and putting $a=0$ we obtain

$$\begin{aligned} & -2 \cdot 4 \cdot 5b(b-c) \left| \begin{array}{ccc} 8c & 2(b+2c) & . \\ 7(b-3c) & 18c & 1 \cdot (b+3c) \\ . & 8(b-4c) & 32c \end{array} \right| \\ & = 8b(4b+14c)(20c)(-4b+18c)(-8b+8c), \end{aligned}$$

and therefore

$$\left| \begin{array}{ccc} 8c & 2(b+2c) & . \\ 7(b-3c) & 18c & 1 \cdot (b+3c) \\ . & 8(b-4c) & 32c \end{array} \right| = \frac{8}{5} (4b+14c) \cdot 20c \cdot (-4b+18c).$$

The general theorems thus obtained are

$$\left\{ \begin{array}{l} \left| \begin{array}{ccccccc} A_1 & (n-1)\beta_1 & . & . & . & . & . \\ (n+4)\gamma_1 & A_2 & (n-2)\beta_2 & . & . & . & . \\ . & (n+5)\gamma_2 & A_3 & . & . & . & . \\ . & . & . & . & . & . & . \end{array} \right|_n \\ \left\{ \begin{array}{l} 2(n-1)b - 2(2n+1)c + 1 \cdot (n+2) \cdot 5d \\ \{ 2(n-3)b - 4(2n-1)c + 2 \cdot (n+1) \cdot 5d \} \\ \{ 2(n-4)b - 6(2n-3)c + 3 \cdot n \cdot 5d \} \end{array} \right\} \end{array} \right\} \quad (\text{XIX})$$

if $\beta_m = b - (m+1)c + \frac{m(m+1)}{2(2m+3)}5d$, $\gamma_m = b + (m+2)c - \frac{(m+2)(m+3)}{2(2m+3)}5d$,

and $A_m = (m+1)^2 \left\{ -2c + \frac{2m(m+2)+6+3n}{(2m+1)(2m+3)}5d \right\}$.

$$\left\{ \begin{array}{l} \left| \begin{array}{ccccccc} A_1 & (n-1)\beta_1 & . & . & . & . & . \\ (n+6)\gamma & A_2 & (n-2)\beta_2 & . & . & . & . \\ . & (n+7)\gamma_2 & A_3 & . & . & . & . \\ . & . & . & . & . & . & . \end{array} \right|_n \\ \left\{ \begin{array}{l} 2(n-1)b - 2(2n+3)c + 1 \cdot (n+4) \cdot 7d \\ \{ 2(n-3)b - 4(2n+1)c + 2 \cdot (n+3) \cdot 7d \} \\ \{ 2(n-5)b - 6(2n-1)c + 3 \cdot (n+2) \cdot 7d \} \end{array} \right\} \end{array} \right\} \quad (\text{XX})$$

if $\beta_m = b - (m+1)c + \frac{m(m+1)}{2(2m+5)}7d$, $\gamma_m = b + (m+4)c - \frac{(m+4)(m+5)}{2(2m+5)}7d$,

and $A_m = (m+1)(m+3) \left\{ -2c + \frac{2m^2+8m+10+5n}{(2m+3)(2m+5)}7d \right\}$.

$$\begin{vmatrix} 2(b-c-n\gamma+2\gamma) & (n-1)(b+2\gamma) & . & . & . & . \\ 3(c-2z) & (3b-c-n\gamma+4\gamma) & (n-2)(b+3\gamma) & . & . & . \\ . & 4(c-3\gamma) & 4(b-c-n\gamma+6\gamma) & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{vmatrix}_n \quad (XXI)$$

$$= (n+1) \cdot \{n(\beta+\gamma)-c\} \{(n-1)(b+2\gamma)-2c\} \{(n-2)(\beta+3\gamma)-3c\} \dots$$

Of these three results it would be interesting to obtain independent proofs.

(19) In a paper of HEINE's on Lamé's functions (*Crelle's Journ.*, lvi. pp. 77-99) there occurs the continued fraction

$$z - c_0 - \frac{c_0'c_1}{z - c_1 - c_2} - \frac{c_2'c_3}{z - c_3 - c_4} - \dots$$

where the c 's are m in number with the values

$$\frac{1}{2}m(m+1), \quad \frac{1}{4}(m-1)(m+2), \quad \frac{1}{4}(m-2)(m+3), \quad \dots, \quad \frac{1}{4} \cdot 1 \cdot 2m,$$

the fractional factor being $\frac{1}{4}$ in every case except the first. From extraneous considerations the value of the continued fraction was given

$$\frac{z(z-2^2)(z-4^2) \dots (z-m^2)}{(z-1^2)(z-3^2) \dots (z-(m-1)^2)} \quad \text{for } m \text{ even,}$$

and

$$\frac{(z-1^2)(z-3^2) \dots (z-m^2)}{(z-2^2) \dots (z-(m-1)^2)} \quad \text{for } m \text{ odd;}$$

but the author added, "Einen directen Beweis für diese Summirung des Kettenbruches habe ich noch nicht aufgefunden." This, however, is readily obtained by writing the value of the continued fraction as the quotient of two determinants, viz.

$$\frac{\begin{vmatrix} z-c_0 & c_0 & . & . & . \\ c_1 & z-c_1-c_2 & c_2 & . & . \\ . & c_3 & z-c_3-c_4 & . & . \\ . & . & . & . & . \end{vmatrix}}{\begin{vmatrix} z-c_1-c_2 & c_2 & . & . \\ c_3 & z-c_3-c_4 & . & . \\ . & . & . & . \end{vmatrix}},$$

and then using two of the foregoing theorems. Thus, taking the case where $m=6$, and where therefore the c 's are

$$21, \quad 10, \quad 9, \quad 7\frac{1}{2}, \quad 5\frac{1}{2}, \quad 3,$$

we have to evaluate the quotient

$$\begin{vmatrix} z-21 & 21 & . & . \\ 10 & z-19 & 9 & . \\ . & 7\frac{1}{2} & z-13 & 5\frac{1}{2} \\ . & . & 3 & z-3 \end{vmatrix} \div \begin{vmatrix} z-19 & 9 & . \\ 7\frac{1}{2} & z-13 & 5\frac{1}{2} \\ . & 3 & z-3 \end{vmatrix},$$

the dividend of which, changed into

$$\begin{vmatrix} z-21 & 6 \cdot 3\frac{1}{2} & . & . \\ 4(3\frac{1}{2}-1) & z-21+2 \cdot 1^2 & 2 \cdot (3\frac{1}{2}+1) & . \\ . & 5(3\frac{1}{2}-2) & z-21+2 \cdot 2^2 & 1 \cdot (3\frac{1}{2}+2) \\ . & . & 6(3\frac{1}{2}-3) & z-21+2 \cdot 3^2 \end{vmatrix},$$

is seen to be one of the simplest cases of the determinant of § 2, and thus to have for its value

$$z(z-4)(z-16)(z-36).$$

In like manner the divisor may be written

$$\begin{vmatrix} z-19 & 2\cdot 4\frac{1}{2} & . \\ 3(4\frac{1}{2}-2) & z-19+6 & 1(4\frac{1}{2}-1) \\ . & 4(4\frac{1}{2}-3) & z-19+16 \end{vmatrix}$$

and is then recognised to be a very special case of the continuant of § 3, and therefore to be equal to

$$(z-1)(z-9)(z-25).$$

The continued fraction in question is consequently equal to

$$\frac{z(z-4)(z-16)(z-36)}{(z-1)(z-9)(z-25)}.$$