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On the Confluences and Bifurcations of certain Theories. (*Presidential Address.*) By Sir JAMES COCKLE, F.R.S.

[*Read November 8th*, 1888.]

Axioms, says Proclus,[1] are common to all sciences, though each employs them in its peculiar subject-matter. A little further on[2] he cites Aristotle[3] as saying that one science is more certain than another, viz., that which emanates from more simple suppositions than that

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[1] Proclus, "Commentaries on the First Book of Euclid's Elements" (Taylor's Translation, London, 1792), p. 92.
[2] Proclus, *op. cit.*, p. 93.
[3] Taylor (*ib.* p. 93) supplies the reference to the first Analytics, t. 42.

which uses more various principles; and that which tells the why, than that which tells only the simple existence of a thing; and that which is conversant about intelligibles, than that which touches and is employed about sensibles.

Proclus adds that, according to these definitions of certainty, arithmetic is more certain than geometry, since its principles excel by their simplicity. For the conception of unity has no reference to position in space, while that of a point involves such reference. In short, we may say that to count a number of objects is a simpler operation than to measure the distances between them.

All this, and much more, shows how early the notion of what is sometimes called a hierarchy of the sciences arose. Proclus's order of precedence would seem to be this, viz., logic,[1] arithmetic, geometry, mechanics, optics, dioptrics,[2] and so on; the progression being from the more to the less abstract, or from the abstract to the concrete.

Francis Bacon, mindful perhaps of Proclus,[3] and duly appreciating the power of mathematics as an instrument[4] and its value as a discipline,[5] expressly takes the (gree of abstractness of a science as the mark for its classification. He places mathematics, as the most abstract of sciences,[6] at one end of the scale and "policy" at the other. He does not graduate the scale minutely, but it may be that, as in the case of the categories,[7] he attached no great value to such details. Distinguishing philosophy from theology, logic, and mathematics,[8] he assigns to it the axioms which are common to several sciences and the inquiry into essences, as quantity, similitude, diversity, possibility, and the rest. Science he divides between metaphysics, the science of the abstract and permanent, and physics, that of matter and its changes.[9] Bacon, in one place, names the one universal science by the name of philosophy, while in another he treats philosophy and metaphysics as two distinct things.[10] He uses the word metaphysics in a sense different from that in which it was then[11] received. Mathematics he places as a branch of metaphysics,

[1] Proclus, *op. cit.*, p. 79. Hume (Treatise, Vol. I., Lond. 1739, Bk. I., Pt. III., p. 129, *et vid.* p. 128) says that geometry falls short of that perfect precision and certainty which are peculiar to arithmetic and algebra.
[2] Proclus, *op. cit.*, p. 93; *et vid.* pp. 78, 79.
[3] Bacon, "The Proficience and Advancement of Learning" (Oxford, 1633,) pp. 49, 50.
[4] Bacon, *op. cit.*, pp. 151, 152; *et vid.* pp. 119, 120.
[5] Bacon, *op. cit.*, pp. 152, 205, and 231.
[6] Bacon, *op. cit.*, p. 218; *et vid.* pp. 150, 151.
[7] Bacon, *op. cit.*, pp. 130, 131, 140, and 201; *et vid.* p. 161.
[8] Bacon, *op. cit.*, pp. 49, 50; *et vid.* pp. 130, 131, and 140.
[9] Bacon, *op. cit.*, p. 141.
[10] Bacon, *op. cit.*, pp. 130, 140.
[11] Bacon, *op. cit.*, p. 138; *conf.* pp. 146, 147.

and as having determined or determinate quantity for its subject. To the pure mathematics, he says, belong geometry and arithmetic; the one handling continuous, and the other discrete quantity.[1] If he means continuous quantity so far as it is immovable, he agrees with the Pythagoreans.[2]

Quantity, time, and space are placed by Aristotle among his categories, or are implied in them. With regard to space, he does not seem to have reached the Kantian view in any way, nor to be very clear in his meaning, though he apparently feels that to realise space we must have motion. His conception of time as one of the elements required for measuring motion, and his starting the problem as to whether we could have time without a mind to conceive, seem a more distinct approximation, though only an approximation, to Kant's view of time as merely a subjective condition of perception.[3]

Newton, in the Scholium to his Definitions, distinguishes between absolute and relative time, the latter being time conceived in its relation to phenomena. Of absolute time (otherwise called duration) which has no relation to a..ything external, he says that it flows equably, and that its rate of flow and the order of its parts are immutable. In his "Fluxions" he uses the word time in a somewhat different sense, viz., as meaning the independent variable, characterised by an equable increase, fluxion, or flow.[4] Sir W. Rowan Hamilton treated algebra as the Science of Pure Time, but his doctrine is not entirely[5] assented to by De Morgan, nor by Professor Cayley, who indeed, in his Southport Address (page 19), intimates dissent from it. Proclus does not connect arithmetic with time, and Professor Cayley suggests (*ib.* p. 18) that, in any case, the notion of number or plurality is not more dependent on time than on space. By the

[1] Bacon, *op. cit.*, pp. 150, 151.

[2] Proclus (Taylor's Translation), p. 74.

[3] For this summary of Aristotle's views I am indebted to Mr. Reginald H. Roe, who referred me to Ueberweg's " Hist. of Phil.," p. 164, for a more general statement, and to p. 165 for a list of the best books for its fuller elucidation, adding that in Ritte and Preller's extracts, pp. 288 and 289, will be found all the important passages from Aristotle bearing on the question. As to the views of Boole, see his " Laws of Thought " (London, 1854), pp. 162 *et seq.*; see also p. 419. Boole treats of space at pp. 163, 175, and 418; and at p. 175 he quotes Aristotle's statements respecting the existence of space in three dimensions.

[4] Newton, " Fluxions," pp. 26 and 38 of the small edition (London, 1737). This is a genuine work of Newton's. As to its bibliography, see *Notes and Queries*, 2nd S., Vol. x., pp. 163, 232, 233 ; 3rd S., Vol. xi., pp. 514, 515 ; 4th S., Vol. ii., p. 316 ; 5th S., Vol. iv., p. 401 ; 6th S., Vol. iv., pp. 129, 130 ; Vol. v., pp. 263, 264, 304, 305, and 426. This octavo edition is very scarce. Indeed, I only know of two copies, viz., my own copy and one in the library of the Royal Astronomical Society.

[5] De Morgan, " On the Foundation of Algebra," *Cambridge Transactions*, Vol. vii., pp. 173—187 ; see pp. 175, 176. The remarks of Prof. Cayley on Whewell, at p. 18 of his Southport Address, are applicable to Rowan Hamilton.

logicians, time seems to be regarded as the more abstract of the pure
intuitions. In fact, time is implied in memory and in thought itself, and
Professor Francis W. Newman observes that no man could get through a
syllogism if he forgot the first premiss while dwelling on the second.[1]
Moreover, he has recourse to the idea of time when he comes to discuss
propositions,[2] and Boole investigates the nature of the connection of
his own Secondary Propositions with the idea of time.[3] The ancient
Indians had their cyclical periods, but not therefore necessarily any
notion of a uniform curvature (so to say) of time.

Absolute space, says Newton, perpetually remains similar to itself
and immovable; and, further on in the Scholium, he adds that the
order of its parts is immutable. In the Preface to the Principia he
had observed that the description of straight lines and circles, on
which geometry is founded, belongs to mechanics, and he follows up
this train of thought. But, whether he means to detach himself from
Plato, I must leave others to say. It is said to be certain that he was
familiar with Bacon's works; that he uses the word axiom, not in
Euclid's sense, but in Bacon', thus giving the name of axioms to the
laws of motion, which, of course, are ascertained by the scrutiny of
nature, and to those general experimental truths which form the
groundwork of optics.[4] Now Bacon says that, in his judgment, the
senses are sufficient to certify and report truth, either immediately or
by way of comparison.[5] Moreover, he suggests that the rule *Quæ in
eodem tertio conveniunt, et inter se conveniunt,* a rule so potent in logic
as that all syllogisms are built upon it, is taken from the mathematics.[6]
In seeking an origin for the more abstract in the less abstract, Bacon
is not solitary. Thomas Stephens Davies suggested[7] that the argu-
ment from superposition had its origin in mechanical considerations,
and from the fitting together of material figures. Moreover, it is
conceivable that some observant person among the ancient Egyptians,
whose custom it was to stamp their bricks, noticing the resemblances
of the marks and the correspondence of the impressions with the

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[1] Newman, "Lectures on Logic, or on the Science of Evidence," etc. (Oxford,
1838), p. 15.

[2] Newman, *op. cit.*, pp. 32—34.

[3] Boole, "An Investigation of the Laws of Thought" (London, 1854), pp. 162,
et seq.

[4] See the Account of the "Novum Organon" in the Library of Useful Knowledge,
p. 10.

[5] Bacon, "Advancement of Learning" (cited *supra*), p. 193.

[6] Bacon, *op. cit.*, p. 132.

[7] T. S. Davies, Geometrical Notes, "Mechanics' Magazine," Vol. LIII. (1850),
pp. 150, 169, 262, 291, 442. Davies points out "the connection between parallels
and similar triangles." He thinks that Aristotle's secession from the school of
Plato arose from his enforcement of his own logical doctrines. Davies rejects the
notion of a geometry built upon definitions alone without the assistance of axioms.

impressing tool, may have been led to a recognition of the rule
quoted by Bacon. The doctrine that there enters into geometry an
element derived from the senses has, indeed, appeared in books
designed for ordinary readers. Thus, Prof. Newman, writing
in 1836–38, although in one part of his " Logic " (p. 25), he
says that in geometry no results are admitted by help of observation
and testimony, but only by reasoning from the definition, yet he
afterwards (p. 55) states that, as space and its properties appear un-
deniably to be learned by sense, the argument seems to him to pre-
ponderate for naming geometry a Mixed Science, and believing that
its propositions are real and not verbal truths. And Potts[1] says that
geometry seems to rest on the simplest inductions from experience
and observation, and that its principles are founded on facts cog-
nisable by the senses.

But it is to Reid[2] that the idea of a more precise mathematical
treatment of the subject is due, and his name ought to head the roll
on which will be inscribed the names of Lobatschewsky, Riemann,
and other investigators. Kant, indeed, disposes of such questions
summarily, by saying that it follows from his premisses that the
propositions of geometry are not the determinations of a mere crea-
ture of our feigning fancy, but that they necessarily hold of space,
and consequently of all that may be met within it, because space is
nothing else than the form of all the external phenomena, in which
alone objects of sense can be given (" Prolegomena,"[3] p. 51). He
adds (pp. 51 and 53) that external phenomena must necessarily and
precisely agree with the propositions of the geometer. Whether
Kant's allusion to " superficial metaphysicians " points to the
Pyrrhonists and Epicureans[4] or to others, and, possibly, even to Reid,
whom he had mentioned before (Pref., p. viii.), does not appear.
Whatever opinions be formed of Kant's theory, or of the nature of
space, his view is impressive. Confine that view to two dimensions,
and suppose the surface of a sphere to be inhabited by a being desti-
tute of any conception of a third dimension, and whose senses are un-
affected by any point not situated or any motion not taking place on that

[1] Potts (Robert), " Euclid's Elements of Geometry," etc. (Cambridge and London,
1845); Notes to Book I., p. 41.
[2] Thomas Reid, " An Inquiry into the Human Mind on the Principles of Common
Sense " (1764). My pagings refer to the Calcutta Reprint of 1869. Chapter VI.
treats (pp. 94—277) of Seeing; its Sect. VII. (pp. 120—124), of Visible Figure and
Extension; and its Section IX. (pp. 132—145), of the Geometry of Visibles. In
Sect. VIII. (pp. 125—132), we have Some Queries concerning Visible Figure
answered.
[3] I cite from Richardson's Translation (London, 1819); and cannot now give the
corresponding paging in that of Prof. Mahaffy.
[4] Montucla, " Histoire " (2nde édition, An. VII.), p. 21.

surface. He could only estimate direction and position by the tangent to the path of the visual ray at the point where that path meets his visual organ, and would think that all objects were situate in one plane. His geometry would be Euclidian; for, if he could form a notion of the actual paths of rays, he would have a conception of the third dimension in space.[1] Here Kant and Riemann would apparently be at issue; for, if a more general conception of space is to be rendered special by actual measurements on the sphere, then, after an enlarged experience, the Euclidian conception would have to be expelled and replaced by some other. And all this would have to be done without praying in aid the excluded third dimension.

Aristotle[2] notices that the nature of everything is best seen in its smallest portions, and Kant[3] remarks that there was a time when mathematicians, who were philosophers too, began to doubt, not the truth of their geometrical propositions as far as they regard space, but the objective validity and applicability of the conception itself, and of all its determinations, to nature; as they were apprehensive that a line in nature might consist of physical points, and that consequently true space in the object might consist of simple parts, though space as conceived by the geometer cannot so consist. Clifford[4] would have given due weight to the doubts of the philosophical mathematicians. He even suggests that the properties of space may change with time. Now, a number may be a function of an angle; the very angle itself determines those numbers (ratios of lines) which we call sines and cosines. But, says De Morgan,[5] in every case but this it is impossible to conceive number a function of magnitude. It seems almost equally difficult to entertain Clifford's conjecture, which, nevertheless, measurements might verify. The sentence, *Nam tempora et spatia sunt sui ipsorum et rerum omnium quasi Loca*, in Newton's Scholium, though it may suggest that omnipresence does not involve extension in space, implies no functional relation between space and time. The words "then and there," accompanying every material allegation in indictments, would suffice to show that the opinions of the world at large on certain characteristics

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[1] See Cayley, Southport Address, pp. 11, 12.
[2] See Bacon, " Advancement of Learning," p. 108.
[3] Kant, " Prolegomena," p. 52.
[4] William Kingdon Clifford, "Mathematical Papers" (London, 1882). See pp. xl. and xlii. of the Introduction, by H. J. S. Smith.
[5] De Morgan, "On the General Principles of which the Composition or Aggregation of Forces is a Consequence " (*Camb. Trans.*, Vol. x., Part ii., pp. 291, 295, foot-note).

of time and space[1] were in accord with that of the philosophers. Indeed, their isolation, as forms of intuition, may no more be a peculiarity of Kant's system than is his distinction between analytical and synthetical judgments. This distinction was present to the mind of Bacon,[2] as well as to that of Locke, whom Kant cites (" Prolegomena," p. 25), and who, elsewhere than in the place cited, adverts to the distinction. That which Locke had styled a trifling proposition, Kant called an analytical judgment; and that which Locke (Essay concerning Human Understanding, Book IV., Chap. VIII., Sect. 8) styled a real truth, Kant would have called a synthetical judgment. With Hume, too, Kant is in some respects in close relation. Hume (Treatise, Vol. I., Bk. I, Pt. II., pp. 53—124) treats specially of the ideas of space and time. Hume, again (Inquiry, p. 17; Essay iv., p. 50), distinguishes between results attained by reasonings *à priori* and results arising entirely from experience (Inq., p. 17; Ess., p. 49). He seems to allow conception a sufficiently wide range, for he urges (Inq., p. 13; Ess. II., pp. 26, 27) that, in one exceptional instance, there may be an idea not arising from a corresponding impression; viz., in the case when from the impressions of two distinct shades of a particular colour, a conception is formed of an intermediate shade of the same colour. He asserts (*ib.*, p. 118) that the only objects of the abstract sciences or of demonstration are quantity and number.

If, as Clifford[3] seems to think, there are no sufficient grounds for maintaining that, if our space has curvature, it must be contained in a space of more dimensions and no curvature, one difficulty is apparently removed. The one-dimensioned time is something very different to space, from which the higher-dimensioned entity might differ still more; and if a solid be treated as the shadow or projection in Euclid's space of, say, a four-dimensioned body, that part of the body which lies outside the shadow seems to have no quality analogous to impenetrability or inertia, nor indeed any quality which affects the senses or deranges the results of calculation. Prof. Cayley says (Southport Address, p. 11) that Riemann's idea seems to be that of modifying the notion of distance, not that of treating it as a locus in four-dimensional space. The suggestion (Cayley, *ib.* p. 10) of a rule

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[1] I should have been glad to have given Locke's and Kant's descriptions of space and time, and to have compared them with Newton's. But I cannot omit to refer to a Smith's Prize paper, by Mr. Robert Franklin Muirhead, printed in the " Philosophical Magazine " for June, 1887 ; S. 5, Vol. XXIII., pp. 473—489.

[2] Bacon, " Advancement of Learning," p. 47.

[3] Clifford, " The Universal Statements of Arithmetic " *Nineteenth Century* (1879, Vol. V., pp. 513—522 ; *vide* p. 522).

changing its length by an alteration of temperature facilitates appre-
hension. Prof. von Helmholtz has considered the effect of the
changes in sensible phenomena which a transition to a spherical or
pseudo-spherical world, if such things be, would produce; and he has
taken an independent view of the subject in other respects.[1]

De Morgan[2] professed to have been puzzled to know on which side
the meeting of parallels took place, or whether on both. He con-
cludes that they never meet. This, however, does not shake, nor is
it to be supposed that he wished[3] it to shake, the belief in modern
methods, for he apparently admits that interpretation of forms may
demand conclusions which can be reached by reasoning on infinity, if
increase without limit show approach. He observes that it is clearly
conceived by the logicians that all division is reducible to simple
dichotomy and its repetitions, and that when the logician has once
shown division, difference, he does not trouble himself with the diffi-
culty of repetitions. De Morgan's remark is easily verified by turn-
ing to Potts' Note on Euc. I. 10 (p. 49). Turning again to Boole,
(L. of T., p. 91), it would se_n that the logician does not completely
detach himself from the notion of infinity; he has to interpret $1 : 0$
as well as $0 : 0$.[4]

Bacon differs from Plato, who considered forms as absolutely
abstracted from matter, and not as confined and determined by it,
and agrees with Aristotle in saying that words are the images of
thoughts;[5] so that the agreement of the views of Bacon with those
of Prof. Max Müller, would seem to be tolerably close. It is easy to
find cases in which a doubtful meaning of a word may give rise to
disagreement on matters of substance. Boole (L. of T., pp. 407, 408)

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[1] A paper in *Mind*, by Prof. von Helmholtz, elicited a criticism from Prof.
Land, which produced a reply, and with a brief note appended to a paper on
another subject, by Prof. Land, the discussion closed. See *Mind*, Vol. I., pp. 301—
321; Vol. II., pp. 38—46; Vol. III., pp. 212—225; and Vol. IV.

[2] De Morgan, "On Infinity," etc. (*Camb. Trans.*, Vol. XI., Part I., 1865,
pp. 145—189; *vid.* pp. 173, 176, 180, 147). In connection with this paper, the
comments of Mr. W. S. B. Woolhouse in the *Educational Times* (*Reprint*, Vol. VI.,
pp. 49—52) should be considered. And in connection with a paragraph at pp. 161,
162 of De Morgan's paper, the leading paragraph of p. 424 of a previous paper of
his, "On the Theory of Errors of Observation" (C. T., Vol. X., Pt. II., 1862),
should be read. In the last-mentioned passage he distinguishes between the zero
and the indivisible of probability. Hamilton, of Edinburgh, following earlier
authorities, expressly restricts the application of logic to finite things. But it does
not therefore follow that logicians in general turn a deaf ear to all reasoning upon
infinites and infinitesimals, and that they reject results stamped with authority and
universally accepted.

[3] This sufficiently appears from a statement at p. 15 of his paper, "On the Root,"
&c. (*Camb. Trans.*, Vol. XI., Pt. II.).

[4] See the last footnote but one.

[5] Bacon, "Advancement of Learning," p. 143; *Conf.*, pp. 130, 140. See also
ɔ. 192, 209.

observes that the term "necessary" may be applied either to the
observed constancy of nature or to the logical connection of proposi-
tions.   He expresses no decided preference for either meaning.   The
meanings should be kept carefully apart.   If an axiom be a necessary
truth, in the strictest sense, then Newton's laws of motion are laws
à priori, viz., giving Kant's meaning to the term (Prol., p. 103), they
are known independently of all experience.   But Laplace (Méc. Cél.,
pp. 14—18[1]) treats them as results of experience.   Moreover, he
treats (pp. 65—69) the laws of motion under all the relations mathe-
matically possible between force and velocity.   Newton, in fact,
usually speaks of "law," and gives the term "axiom," Bacon's
meaning.

   Boole's Chapter xx. (L. of T., pp. 320—375) relates to problems
on causes, but his use of the word "cause" has given rise to much
discussion.   He proposed a question on causes in 1851, which was
answered by Prof. Cayley in 1853.   The solution was criticised by
Boole in 1854, who arrived at a different result, and, in 1854, Mr. H.
Wilbraham examined both solutions.   Prof. Cayley returned to the
subject in 1862, and Boole thereupon admitted that it would have
been better, in stating his problem, not to have employed the word
"cause" at all.[2]   One mode of stating the nature of the relation be-
tween "cause" and "effect," may be this, viz., when a certain (ante-
cedent) change is immediately and invariably followed by a certain
other (subsequent) change, then the relation in which the antecedent
stands to the subsequent (which may now be called the consequent)
change is that of cause and effect.   This is, in substance, if not in form,
a view common to Algazel, Glanvil,[3] Hume,[4] Brown,[5] Kant, and, as I

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   [1] My pagings refer to the 2nd ed. of the "Mécanique Céleste," Vol. I. (Paris,
1829).
   [2] Boole, "C. and D. M. J.," Vol. VI., p. 286; "L. of T.," pp. 321—326; "Phil.
Mag.," S. 4, Vol. VII., pp. 29—32; Vol. XXIII., pp. 361—363; Wilbraham, "Phil.
Mag.," S. 4, Vol. VII., pp. 465—476; Cayley, "Phil. Mag.," S. 4, Vol. VI., p. 259;
S. 4, Vol. XXIII., pp. 352—365, and 470.   A short letter by Boole ("Phil. Mag.,"
S. 4, Vol. XXIV. (1862), p. 80) concludes the discussion.
   [3] Glanvill (Joseph) "Scepsis Scientifica," etc. (Lond. 1665, 4°); Lond. 1885,
8°.   On Causation I have only mentioned comparatively recent authors.
But, going further back, we find Thales (with his elemental analysis),
Xenophanes (with his one cosmic substance), and Pythagoras (with his arithmetical
and geometrical combinations), all recognizing invariable sequences in nature; and
Socrates admitted a class of phenomena wherein the connection of antecedent and
consequent was invariable and ascertainable by human study (Grote, "History of
Greece," Vol. I., 1846, pp. 495—498).   Socrates applied similar scientific reason-
ings to moral and social phenomena (*ib.* p. 504).
   [4] David Hume, "A Treatise of Human Nature," etc. (Lond., Vols. I. and II.,
1739; Vol. III., 1740.   His name does not appear on the title-pages).   Philoso-
phical Essays concerning Human Understanding" (2nd ed., Lond. 1750).   "An
Inquiry concerning Human Understanding" (Lond. 1861 marks the issue to which
I refer).
   [5] Thomas Brown, "Inquiry into the Relation of Cause and Effect" (Third Edi-

believe, Reid; for the question seems to be one about words. It differs but slightly from the view (C. T., Vol. x., Pt. ii., p. 300) of De Morgan. Perhaps "unvarying" might be a better word than "invariable," for one instant of time is the immediate and invariable antecedent of its consecutive instant; but the idea of "cause" does not seem to arise. When "cause" is used in the above sense, the solutions of Boole and Prof. Cayley agree. Boole's question has been dealt with in our *Proceedings* (Vol. xi., p. 118) by Mr. McColl.

The import of the word "principle," is not the same when we speak of the Principle of Contradiction or of Excluded Middle, as when we speak of the Principle of the Permanence of Equivalent Forms, or of the Sufficient Reason, or of Continuity. That of sufficient reason has been assailed by Brown (C. and E., Sect. iv., pp. 222, misnumbered 322, to 306), and by De Morgan (C. T., x., Pt. ii., pp. 290—304). Clifford (*Op. cit.*, p. xli.) was prepared to sacrifice the principle of continuity, even in the case of space, and the author of anonymous "Strictures" on Peacock's Algebra (Camb., 1837), who was (so at least I was told many years ago by Davies) Hind, concludes (p. 21) that number is perfectly abstract, that it is the only thing which is so, and that it is not rightly denominated a species of quantity, being equally connected with every species. An instance of a striking failure of the principle of the permanence of equivalent forms is given by Dr. J. W. L. Glaisher in the *Messenger of Mathematics*, N. S., Vol ii. (1872), p. 95. Again, take another word, viz., "disparity." Supposing it to be said that there are two persons in a room, whose united ages are twenty-one years, and between whose ages there is the greatest disparity possible. This is intelligible if one be a newborn or nascent infant, and the other a person aged twenty-one. But suppose the same statement made of three persons; the proficient in language might have to inquire of the mathematician what meaning, if any, the statement bears. Or, again, the mathematician might be asked what, or whether any, numerically definite meaning can be attached to the words, "triangle of maximum scalenity."

Prof. Newman ("Logic," 1838, p. 52) says that the truths of arithmetic are verbal. Perhaps this, and the corresponding statements of Dugald Stewart, would not now be insisted on. They are opposed to the views of Kant, Clifford, and De Morgan (C. T., xi., Pt. i., p. 160). The identities $3^2 + 4^2 = 5^2$ and $3^3 + 4^3 + 5^3 = 6^3$

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tion, Edinburgh, 1818). Draper does not admit the construction put upon Algazel's words by Whewell ("Hist. Ind. Sc.," Lond., 1837, i., p. 251). A facsimile reprint of Glanvil has been published within the last few years. Buckle pronounced Brown's to be one of the best books ever written.

seem to be something very different from definitions of words. Kant considers $7 + 5 = 12$ to be a synthetical judgment (Proleg., pp. 22, 23).

Metaphysics and mathematics are consorts in the East as well as the West. Bhascara says that the analytical art is merely sagacity exercised, and is independent of symbols, which do not constitute the art.[1] If De Morgan[2] be right in placing Diophantus as late as the beginning of the seventh century, Aryabhatta was earlier, by two centuries, than Diophantus. The name, certainly, seems to have been a very common one. Josephus[3] relates that Alexander (a son of Herod the Great) said that Diophantus the scribe had imitated his hand. But Mr. Heath's work[4] renders it scarcely possible to sustain De Morgan's contention.

On Raabe's Bernoullians. By J. D. HAMILTON DICKSON, M.A.

[*Read Nov. 8th, 1888.*]

Raabe has given the name Bernoullian to the function

$$\varphi\,(z,\,m) = z^m + m_1 B_0 z^{m-1} + m_2 B_1 z^{m-2} + m_3 B_2 z^{m-3} + \ldots,$$

where there is no term without z; $m_1, m_2, \ldots$ being the successive Binomial coefficients after the first in the expansion of $(1+t)^m$, and $B_0, B_1, \ldots$ being Bernoulli's numbers, viz.,

$$B_0 = -\tfrac{1}{2}, \quad B_2 = B_4 = B_6 = \ldots = B_{2n} = \ldots = 0,$$

$$B_1 = \tfrac{1}{6}, \quad B_3 = -\tfrac{1}{30}, \quad B_5 = \tfrac{1}{42}, \quad B_7 = -\tfrac{1}{30}, \quad B_9 = \tfrac{5}{66}, \ \&c.$$

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[1] Colebroke, "Algebra," etc. (London, 1817), p. xix.
[2] De Morgan, "Arithmetical Books" (London, 1847), p. 47.
[3] Josephus, "Antiquities of the Jews" (Burder's Translation, Vol. i., pp. 616, 617). Burder's Preface is dated London, October 1, 1811.
[4] T. L. Heath, "Diophantos of Alexandria: a Study in the History of Greek Algebra" (Cambridge University Press, 1885).