"Esquisse Historique sur la Marche du Développement de la Géométrie du Triangle," by E. Vigarié; 8vo pamphlet (read before the French Association for the Advancement of Science—Congrès de Paris—1889).

"Sur les Axes de Steiner et l'Hyperbole de Kiepert," et "Sur les Foyers de Steiner d'un Triangle," MM. J. Neuberg and A. Gob (read before the French Association for the Advancement of Science—Congrès de Paris—1889).

"Sur les Projections et Contre-Projections d'un Triangle Fixe, et Sur le Système de Trois Figures directement Semblables," M. J. Neuberg. (Offprint from Tome XLIV. of the "Mémoires Couronnés et autres Mémoires," publiés par l'Académie Royale de Belgique, 1890.)

On a Theorem relating to Bicircular Quartics and Twisted Quartics. *By* R. LACHLAN, M.A.

[*Read May 8th,* 1890.]

The present paper deals with a theorem analogous to the theorem that: Every plane cubic which passes through eight fixed points must pass through a ninth. It will be proved that a very similar theorem exists for plane bicircular quartics, and for twisted quartic curves, and, as might be expected, the theorem leads to as many, if not more, interesting results than the one for the cubic. It has been thought advisable to state for circular curves the corresponding theorems to those which are given in Salmon ("Higher Plane Curves," §§ 30–34) for ordinary plane curves, although of course the former may be easily deduced from these. And it should be stated that a circular curve of the $2n^{\text{th}}$ order is always understood to mean a curve of order $2n$, having each of the circular points at infinity as multiple points of the n^{th} order.

The paper is divided into three sections: in the first are given the general theory of the intersection of circular curves; in the second section a particular theorem relating to plane bicircular quartics is developed; and in the third section the corresponding theorem for twisted quartics is proved independently, and it is explained how the particular theorems in the second section may be translated so as to apply to twisted quartics.

General Theorems on Circular Curves, §§ 1–4.

1. The general equation of a curve of the $2n^{\text{th}}$ order having multiple points of the n^{th} order at each of the circular points, is of

the form

$$u_0(x^2+y^2)^n+u_1(x^2+y^2)^{n-1}+\dots+u_{n-1}(x^2+y^2)$$
$$+v_n+v_{n-1}+\dots v_1+v_0=0,$$

where u_r and v_r are homogeneous expressions of the r^{th} degree in x and y.

Such a curve is determined when we have given

$$(1+2+\dots+n+n+1+n+\dots+1)-1,$$

i.e., $n(n+2)$ points.

2. Two circular curves of the $2m^{\text{th}}$ and $2n^{\text{th}}$ orders intersect in $2mn$ points, excluding the circular points. Hence it follows that every circular curve of the $2n^{\text{th}}$ degree which passes through $n(n+2)-1$ fixed points must also pass through $(n-1)^2$ other fixed points.

We may also deduce the theorem : If of the $2n^2$ points of intersection of two circular curves each of the $2n^{\text{th}}$ order, $2np$ lie on a circular curve of order $2p$, the remaining $2n(n-p)$ will lie on a circular curve of order $2(n-p)$.

3. Again, every circular curve of the $2n^{\text{th}}$ order which passes through $2np-(p-1)^2$ points on a circular curve of order $2p$ (p being less than n) meets this curve in $(p-1)^2$ other fixed points.

For through any other $(n-p)(n-p+2)$ points on the curve of order $2n$ we can draw a circular curve of order $2(n-p)$, and this, with the curve of order $2p$, makes up a curve of order $2n$ passing through

$$2np-(p-1)^2+(n-p)(n-p+2),$$

i.e., $n(n+2)-1$ points; hence, by § 2, it must pass through $(n-1)^2$ other fixed points on the given curve of order $2n$; and these must lie on one or other of the curves of orders $2p$ and $2(n-p)$; but these curves can only meet the given curve in $2np$ and $2n(n-p)$ points, respectively; hence the truth of the theorem is manifest.

As a particular case, we see that every circular curve of the eighth order which passes through 15 points on a bicircular quartic passes through one other fixed point; and every circular curve of the order $2n$ which passes through $4n-1$ fixed points on a bicircular quartic must pass through one other fixed point.

4. An extension of the theorem in § 3 may be thus stated :—

Any circular curve of the $2r^{\text{th}}$ order (r being greater than m and n, but less than $m+n-1$), which passes through all but $(m+n-r-1)^2$ of the $2mn$ points of intersection of two circular curves of orders $2m$ and $2n$, will pass also through the remaining intersections.

For, if we draw a circular curve of order $2(r-m)$ through

$$(r-m)(r-m+2)$$

arbitrary points on the curve of order $2n$; and a circular curve of order $2(r-n)$ through $(r-n)(r-n+2)$ arbitrary points on the curve of order $2m$; these curves make up, with the curves of orders $2m$ and $2n$, two curves each of order $2r$, and each passing through such a number of the $2mn$ points of intersection of the given curves of orders $2m$, $2n$ as make up $r(r+2)-1$ points in each case; and hence these two curves pass through $(r-1)^2$ other fixed points.

But

$$r(r+2)-1-(r-m)(r-m+2)-(r-n)(r-n-2)$$
$$=2mn-(m+n-r-1)^2;$$

hence the theorem is seen to be true. For the theorem to be applicable, it is easily seen that r must be at least equal to the greater of the integers m and n, also $r-m$ must be $< n$, otherwise it would not be possible to draw the auxiliary curve of order $2(r-m)$; and, as the theorem is nugatory when $r = m+n-1$, we see that we must have

$$r < m+n-1.$$

Application to Bicircular Quartics, §§ 5–12.

5. The general theorem in § 3 states that every circular curve of the eighth degree which passes through 15 fixed points on a bicircular quartic must pass through a sixteenth fixed point; in other words, if two circular systems each of the eighth degree intersect in 15 points which lie on a bicircular quartic, their sixteenth point of intersection also lies on the quartic.

Suppose we have three circles A, B, C cutting a bicircular quartic respectively in the points a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; and let the circles (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), (a_4, b_4, c_4) meet the quartic in the points d_1, d_2, d_3, d_4, respectively; then these points must be concyclic. For let the circles (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), (a_4, b_4, c_4) be denoted by L, M, N, P; and let D denote the circle (d_1, d_2, d_3); then each of the systems A, B, C, D and L, M, N, P are of the eighth degree, and fifteen of their points of intersection lie on the quartic, therefore the remaining point in which the system A, B,

C, D meets the quartic must be the point d_4, which is the sixteenth point in which the system L, M, N, P meets the quartic. Hence the circle D must pass through the point d_4.

6. Suppose now that the circles A, B, C coincide; then the circles L, M, N, P become the osculating circles at the points a_1, a_2, a_3, a_4; hence we have the theorem that: The osculating circles at four concyclic points on a bicircular quartic, meet the quartic in four points which are also concyclic.

The circle D passing through the points d_1, d_2, d_3, d_4, in which the osculating circles at a_1, a_2, a_3, a_4 meet the quartic, might be called the *satellite* circle of the circle A.

7. Let us now suppose the points a_1, a_2, a_3 are cyclic points; then the circle D must coincide with the circle A; and hence the point a_4 must either be another cyclic point, or must coincide with one of the points a_1, a_2, a_3; in other words, the circle which can be drawn passing through any three cyclic points, meets the quartic in a fourth cyclic point or touches it at one of the three points.

Now we know that there are sixteen cyclic points on a bicircular quartic, viz., four cyclic points on each of the principal circles of the quartic.

Let α_1, α_2, α_3, α_4 be the four cyclic points on the principal circle J_1; β_1, β_2, β_3, β_4 the cyclic points on J_2; γ_1, γ_2, γ_3, γ_4 those on J_3; and δ_1, δ_2, δ_3, δ_4 those on J_4. If α_2 be the inverse point to α_1 with respect to J_2, it is evident that the circles $\alpha_1\alpha_2\beta_1$, $\alpha_1\alpha_2\beta_2$, $\alpha_1\alpha_2\beta_3$, $\alpha_1\alpha_2\beta_4$ will touch the quartic at the points β_1, β_2, β_3, β_4, respectively. And it follows from the theorem that two pairs of inverse points are concyclic, that no other circle passing through α_1, α_2 and a third cyclic point, γ_1 say, touch the curve at either of the points α_1, α_2, γ_1. Now sixteen points may be arranged in groups of three in $16 . 15 . 14/6 = 560$ ways; through each pair of points such as α_1, α_2 we can draw four circles touching the curve at another cyclic point, and we can take a pair such as α_1, α_2 in 6×4 ways (viz., 6 pairs on each principal circle); hence, of the 560 circles that are obtained by taking any three of the 16 points, $6 \times 4 \times 4 = 96$ touch the quartic; we have left 464 circles, and, by what was proved above, each of these must meet the curve in a fourth cyclic point. Dividing this number by 4, we see that 116 circles can be drawn each passing through four cyclic points on a bicircular quartic.

It will be interesting to see how these circles may be grouped. There are the four principal circles; and then we know that pairs of inverse points are concyclic, thus we shall have 48 circles of the type

$a_1a_2\gamma_1\gamma_3$, a_1 and a_2 being inverse points with respect to J_2, as also γ_1 and γ_3: we shall also have the circles $a_1a_2\gamma_2\gamma_4$, $a_1a_2\delta_1\delta_3$, $a_1a_2\delta_2\delta_4$; *i.e.*, four circles for each pair of points a_1a_2 on J_1, and so $\frac{1}{2}.4\times 6\times 4 = 48$ in all; the remaining 64 circles will be circles passing through one point on each of the principal circles.

8. We know that nine circles can always be drawn through any point a on a bicircular quartic which will osculate the curve elsewhere; let us suppose, then, that the osculating circles at the points $a_1, a_2, \ldots a_9$ meet the curve again in the point a; and let the osculating circles at $b_1, b_2, \ldots b_9$ meet the curve in b, the osculating circles at $c_1, c_2, \ldots c_9$ meet in c, and the osculating circles at $d_1, d_2, \ldots d_9$ meet the curve in d. Then, if the points a, b, c, d are concyclic, it follows from § 6 that the circle passing through three points such as a_i, b_j, c_k must pass through one of the points $d_1, d_2, \ldots d_9$.

Consequently, though there corresponds but one satellite circle D for every circle A, yet corresponding to a circle D there are $9\times 9\times 9 = 729$ different circles A.

9. Let us suppose that the points a, c, d coincide; then we see that, if the osculating circle at the point a meets the curve in the point b, and if the osculating circles at $a_1, a_2, \ldots a_9$ meet the curve in the point a, and the osculating circles at $b_1, b_2, \ldots b_8$ meet the curve in b, then the circle passing through any three of the points $a_1, a_2, \ldots a_9$ must pass through one of the points $b_1, b_2, \ldots b_8$. In other words, if $a_1, a_2, \ldots a_9$ are the points, the osculating circles at which meet the curve in the point a, then the $9.8.7/6 = 84$ circles which can be drawn through the points $a_1, a_2, \ldots a_9$, taken three at a time, meet again in eight other points which are also on the quartic, and the osculating circles at these eight points meet the curve again in the same point as the osculating circle at the point a.

10. Returning to the general theorem in § 5, let us suppose that the circles A, B, C are bitangent circles to the quartic at the points a, a'; b, b'; c, c', respectively; then the circles L, M coincide, and also the circles N, P. Hence, if the circles abc, $a'b'c'$ cut the curve again in the points d, d', we infer that the circle which touches the curve at d and passes through d' must touch the curve at d'. If the three bitangent circles aa', bb', cc' belong to the same system, *i.e.*, cut orthogonally the same principal circle, then it is obvious that d, d' are inverse points with respect to the same principal circle; but in the above reasoning it would seem that the three bitangent circles A, B, C need not necessarily belong to the same system.

11. Further, let us suppose that the points a, b, c coincide; then we see that, if three of the bitangent circles at the point a touch the curve again at the points a_1, a_2, a_3, respectively, and if the osculating circle at a and the circle $a_1a_2a_3$ meet the curve again in d, d', then d, d' are points of contact of a bitangent circle.

Now there are four circles which touch at a and touch the curve elsewhere; let a_1, a_2, a_3, a_4 be the points of contact (*i.e.*, the inverse points of a with respect to the four principal circles), then we infer that the four circles which can be drawn through these four points meet the curve in four points d_1, d_2, d_3, d_4 which are the points of contact of the bitangent circles which touch the curve at d, the point where the osculating circle at a cuts the curve.

12. Again, let the points of contact of the bitangent circles at a, b, c, d be respectively a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; d_1, d_2, d_3, d_4; then, if a, b, c, d are concyclic, we see, from § 10, that any circle such as $a_ib_jc_k$ must pass through one of the four points d_1, d_2, d_3, d_4.

Hence these 16 points lie on 64 circles, each circle passing through one and only one of the points corresponding to a, b, c, d, respectively.

Analogous Theorems in connexion with Twisted Quartics, §§ 13–15.

13. Since the equation of a bicircular quartic may be written in the form

$$ax^2+by^2+cz^2+dw^2=0,$$

where x, y, z, w are the "power-coordinates" of a point referred to four mutually orthogonal circles, so that they are connected by the identical relation

$$x^2+y^2+z^2+w^2=0,$$

it follows that, corresponding to all theorems connected with bicircular quartics, there exist analogous theorems concerning quartic curves in space. In fact, we have merely to substitute the word "plane" for "circle" in the enunciation of theorems concerning bicircular quartics to obtain the corresponding theorem for twisted quartics.

Thus, in place of the theorem in § 5, we shall have the theorem: Every quartic surface which passes through 15 fixed points on the curve of intersection of two quadrics must pass through one other fixed point on the curve.

14. It may be as well to give an independent proof of this theorem.

Thus, let u, v denote two quadrics, and S any quartic surface passing through 15 fixed points on the curve of intersection of u and v; we have to prove that any other quartic surface which passes through these 15 points must pass through the remaining point of intersection of S with u and v.

Now, through any 8 arbitrary points on the curve of intersection of u and S, and through an arbitrary point P on S, we can draw a quadric, v' say; also, through 8 arbitrary points on the curve of intersection of S and v, we can draw a quadric, u' say, also passing through P (P being supposed not to lie on either u or v); then we have three quartic surfaces, S, uu', vv', each passing through $15+8+8+1 = 32$ fixed points, and every quartic which passes through these points must be of the form $S+\lambda uu'+\mu vv'$, and therefore must pass through the remaining 32 points in which S, uu', and vv' intersect.

Hence every quartic surface which passes through 15 of the points of intersection of S, u, v must pass through the remaining point of intersection.

15. We have at once the theorem, that if three planes A, B, C be drawn cutting a twisted quartic in the points a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4, respectively; the planes $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $a_4b_4c_4$ will cut the quartic in four coplanar points d_1, d_2, d_3, d_4.

Consequently, all the theorems stated previously may be at once translated so as to apply to twisted quartics.

On the Arithmetical Theory of the Form $x^3+ny^3+n^2z^3-3nxyz$. *By* Professor MATHEWS, M.A.

[*Read May 8th*, 1890.]

In the last four papers contained in Vol. I. of Dirichlet's collected works will be found some remarkable propositions relating to certain arithmetical forms of higher degrees.*

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* Dirichlet's *Werke*, I., pp. 619, 625, 633, 639. The titles of the papers are—"Sur la Théorie des Nombres" (*Comptes Rendus*, 1840, p. 285, or *Liouville*, Sér. I., t. v., p. 72); "Einige Resultate von Untersuchungen über eine Classe homogener Functionen des dritten und der höheren Grade" (*Berichte über die Verhandlungen d. Königl. Preuss. Akad. d. Wissensch.*, 1841, p. 280); "Verallgemeinerung eines