"Indian Engineering," Vol. xxii., Nos. 17-20, Oct. 23-Nov. 13, 1897.

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The Construction of the Straight Line joining Two Given Points. By W. Burnside.

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Euclid's postulate: "Let it be granted that a straight line may be drawn from any one point to any other point" implies the use of a ruler or straight-edge of any required finite length. The postulate is clearly not intended to apply to the case in which the distance between the two points is infinite, *i.e.*, in which no finite number of repetitions of a finite operation will lead from one point to the other. In fact, Prop. xxxi, Book i, gives a compass and ruler construction for the line when one of the points can be reached while the other cannot. The other exceptional case, when neither point can be reached, *i.e.*, when the two given points are the points at infinity on two non-parallel lines, is not dealt with by Euclid. It is, however, not difficult to show by elementary geometrical considerations that in this case no point of the joining line can be reached by a finite number of finite operations.

In elliptic space any one point can be reached from any other by a finite number of finite operations. The line joining two given points can therefore be always constructed with the ruler alone.

In hyperbolic space as in Euclidean, there are, so long as we deal with elementary geometry, three cases of the construction to consider. These are (i) that in which the two points to be joined are points which can be reached from any assigned point by a finite number of finite operations, or say "finite points"; (ii) that in which one of the points is a finite point, and the other a point at infinity on a given straight line; (iii) that in which both the points are points at infinity.

If, however, we deal with projective geometry, we must assume that *every* two straight lines in a plane determine a point. When the two straight lines are non-intersectors the point can neither be a finite point nor a point at infinity. Such a point is termed an "ideal point"; and there can be no projective geometry in hyperbolic space without the consideration of ideal points (lines and planes). The problem of constructing the straight line joining two given points involves therefore three further cases; namely, (iv) that in which one of the points is a finite point and the other an ideal point; (v) that in which one is a point a infinity and the other an ideal point; (vi) that in which both points are ideal points.

For these last three cases, however, the problem may be put in an alternative form. A pencil of lines in a plane which pass through a common ideal point is the same thing as a set of lines which have a common perpendicular. Hence the problem of constructing the line that joins a given point to the point determined by two non-intersectors in a plane is the same as that of finding the common perpendicular of the two non-intersectors and letting fall on it a perpendicular from the given point.

In the first only of the six cases can the construction be carried out by the ruler alone. Some, at least, of the constructions given below for the other five cases are either known or are equivalent to known constructions. The constructions are, however, here verified by purely geometrical and very simple reasoning; whereas, so far as I am aware, those given hitherto depend either directly or indirectly on considerations derived from analysis.

The geometrical results of which use is made were published in a paper "On the Kinematics of non-Euclidean Space," in Vol. XXVI of the Society's *Proceedings*. It will save the reader the trouble of reference if the results required are summarized here.

Let AEB, CFD be any two lines, and EF their common perpendicular. Then successive rotations through two right angles about AB and CD are equivalent to a translation of amplitude $2EF$, of which the line EF is the axis, together with a rotation round EF through twice the angle between the planes AEF and EFC. In particular, successive rotations through two right angles about two intersecting lines are equivalent to a rotation about a line through their point of intersection perpendicular to their plane; successive rotations through two right angles about two non-intersectors in a plane are equivalent to a translation along their common perpendicular, and successive rotations through two right angles about two lines which meet at infinity give a displacement *sui generis* for which no finite point or straight line remains unaltered.

A rotation through two right angles about a line AB will be represented by the symbol πAB, and in dealing with a plane figure a rotation through two right angles about a line through A perpendicular to the plane of the figure will be represented by πA. The successive components of a compound displacement are to be read from left to right.

I.

To join a given finite point to a given point at infinity.

Let A be the given finite point, and I, on the line BI, the point at infinity. Draw AB at right angles to BI, and AE at right angles to AB. Take an arbitrary point C on BI and with A as centre and BC as radius describe a circle. From C draw CDE at right angles to AE, meeting the circle in D. Then AD is the line required.

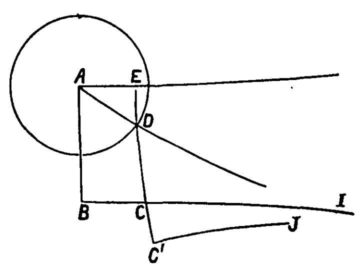

Fig. 1.

Produce EC to C', so that CC' is equal to ED, and through C' draw $C'J$ at right angles to CC'. Then,

$$\begin{aligned}\text{translation } 2DA \,.\, \text{translation } 2BC &= \pi D \,.\, \pi A \,.\, \pi B \,.\, \pi C, \\ &= \pi D \,.\, \pi AE \,.\, \pi BC \,.\, \pi C, \\ &= \pi C'J \,.\, \pi C \,.\, \pi BC \,.\, \pi C, \\ &= \pi C'J \,.\, \pi BC.\end{aligned}$$

Now, if AD and BC intersect in a finite point, a translation $2DA$ followed by a translation $2BC$ is equivalent to a translation along a line which meets BC at an acute angle. On the other hand, $\pi C'J \,.\, \pi BC$ is equivalent to a rotation round a point of BC, a translation along a line at right angles to BC, or a displacement which keeps no finite point or line fixed, according as $C'J$ and BC meet in a finite point, an ideal point, or a point at infinity. Hence AD and BC cannot intersect in a finite point.

If AD and BC intersect in an ideal point, let PQ be their common perpendicular. On ADP, BCQ produced, take p, q such that

$$Pp = Qq = BC,$$

and through p and q draw pp' and qq' at right angles to AD and BC. Then a translation $2DA$ followed by a translation $2BC$ is equivalent to $\pi pp'$ followed by $\pi qq'$. If pp' and qq' intersect, this is equivalent to a rotation round some point lying between AD and BC; and, if they do not, it is equivalent to a translation along a line which is not at right angles to BC. In neither case can it be equivalent to $\pi C'J.\pi BC$. Hence AD and BC do not meet in an ideal point. Finally, then, AD and BC must meet in a point at infinity, or, in other words, AD passes through I.

II.

To join a given finite point to a given ideal point.

Since the joining line is at right angles to the common perpendicular of the two non-intersectors, which determine the ideal point, this is the same as:—*To find the common perpendicular to two given non-intersectors.*

(The preceding construction gives a test to determine whether or no two given lines are non-intersectors.)

Lemma.—Let AA' and BB' be at right angles to AB, and let CC' bisect AB at right angles. From C' draw $C'A'$ at right angles to AA' and $C'B'$ at right angles to BB'.

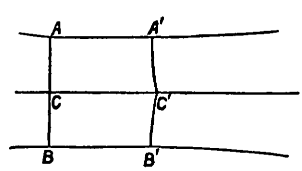

FIG. 2.

Then, translation $2A'A$. translation $2BB'$

$$= \pi A'C' . \pi AB . \pi AB . \pi B'C'$$
$$= \pi A'C' . \pi B'C'.$$

Hence translations of equal amplitude and opposite sense along two non-intersectors can always be taken so small that they are equivalent to a rotation. They are, in fact, equivalent to a rotation unless the translations are so great that $A'C'$, $B'C'$, constructed as in Fig. 2, are either non-intersectors or meet at infinity.

Let AB, CD be the two given non-intersectors whose common perpendicular is required. On AB take two points A and B sufficiently near together. Draw AC and BD at right angles to CD, and produce them to A' and B', so that

$$CA' = AC \quad \text{and} \quad DB' = BD.$$

Then, since πCD changes AB into $A'B'$, the common perpendicular of AB and CD is also at right angles to $A'B'$. Produce CA and DB to a and b, so that

$$Aa = CA \quad \text{and} \quad Bb = DB;$$

and through a and b draw aa' and bb' at right angles to Aa and Bb.

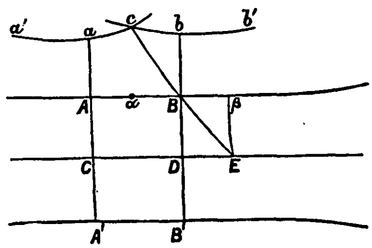

FIG. 3.

If AB is sufficiently small, $\pi B \,.\, \pi A \,.\, \pi A' \,.\, \pi B'$ is a rotation (Lemma). Hence also $\pi A \,.\, \pi A' \,.\, \pi B' \,.\, \pi B$ is a rotation. Now

$$\begin{aligned} \pi A \,.\, \pi A' \,.\, \pi B' \,.\, \pi B &= \text{translation } 2aC \,.\, \text{translation } 2Db \\ &= \pi aa' \,.\, \pi CD \,.\, \pi CD \,.\, \pi bb' \\ &= \pi aa' \,.\, \pi bb'. \end{aligned}$$

Hence aa' and bb' must intersect. Let c be the point of intersection. Join cB and produce it. Then, since Bb is equal to BD, while the angles at D and b are right angles and the vertically opposite angles at B are equal, cB produced must meet CD in a point E, such that BE is equal to Bc.

Now $$\begin{aligned} \pi B \,.\, \pi A \,.\, \pi A' \,.\, \pi B' &= \pi B \,.\, \pi ac \,.\, \pi bc \,.\, \pi B \\ &= \text{rotation round } E. \end{aligned}$$

Hence finally by the lemma, if $E\beta$ is at right angles to AB, and α is taken on AB, so that $\beta\alpha$ is equal to and in the same sense as BA, then the common perpendicular to AB and CD passes through α.

III.

To draw the line joining two ideal points.

If AB, CD are the non-intersectors determining one ideal point, and $A'B'$, $C'D'$ those giving the other; $\alpha\beta$, the common perpendicular to AB, CD, and $\alpha'\beta'$, that to $A'B'$, $C'D'$, can be found by the preceding construction.

If $\alpha\beta$, $\alpha'\beta'$ are non-intersectors, their common perpendicular, which is the required line, can also be found by the preceding construction.

If $\alpha\beta$ and $\alpha'\beta'$ meet at infinity or are intersectors, there are no finite points on the required line.

IV.

To draw the line joining a given point at infinity to a given ideal point.

This is the same thing as :—

To draw a perpendicular on a given line from a given point at infinity.

Let AB be the given line and I the given point at infinity. Join any two points C and D of AB to I. From A and B draw Aa and Bb at right angles to CI, and produce then to A' and B', so that

$$aA' = Aa, \quad bB' = Bb.$$

From A' and B' draw $A'a'$ and $B'b'$ at right angles to DI and produce them to A_1 and B_1, so that

$$a'A_1 = A'a', \quad b'B_1 = B'b'.$$

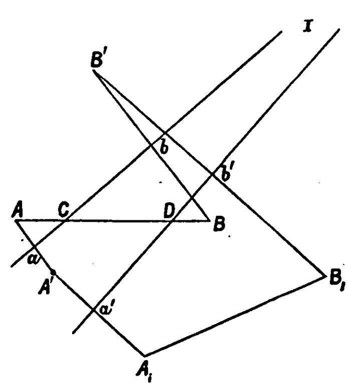

Fig. 4.

Also from A and B construct similarly A_2 and B_2, except that in this case the perpendiculars are first let fall on DI, and afterwards on CI. Then A_1B_1 is the line that results from giving AB the displacement $\pi CI.\pi DI$; and A_2B_2 is the line that results from giving AB the inverse displacement $\pi DI.\pi CI$.

If C and D are taken so that, J being the point at infinity on CD produced, ICJ and IDJ are both acute angles, then AB and A_1B_1 will certainly intersect, as also will AB and A_2B_2.

For, by the displacement $\pi CI.\pi DI$, C is displaced to a point C_1, on the far side of AB from I. Also, if JJ' is the perpendicular on CI from J, and $J'J_1$ the perpendicular from J' on DI, J' and J_1 both being points at infinity, then JJ' is on the same side of AB as I and $J'J_1$ is on the same side of JJ' as I; so that J_1, the displaced position of J, is on the same side of AB as I. Hence A_1B_1 has points on both sides of AB, or, in other words, AB and A_1B_1 intersect.

Let E and F be the points in which A_2B_2 and A_1B_1 intersect AB, and let G be the middle point of EF. Then GI will be the line required.

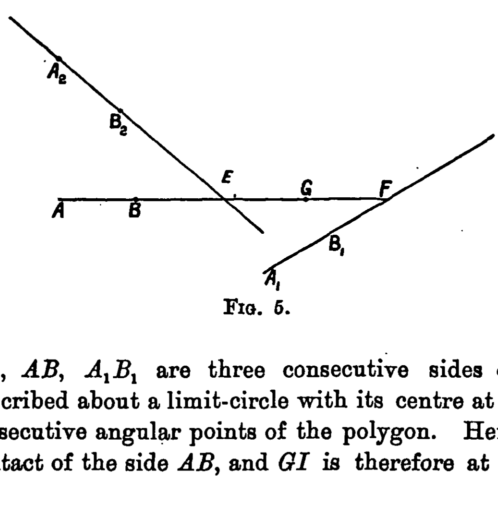

FIG. 5.

For A_2B_2, AB, A_1B_1 are three consecutive sides of a regular polygon described about a limit-circle with its centre at I; and E, F are two consecutive angular points of the polygon. Hence G is the point of contact of the side AB, and GI is therefore at right angles to AB.

V.

To draw the straight line joining two points at infinity.

Let I and J be the two points at infinity. Join I and J to any finite point A, and draw AK bisecting the angle IAJ. Then πAK interchanges I and J, and therefore leaves the line IJ unchanged; so that IJ is at right angles to AK. The line IJ is therefore given on

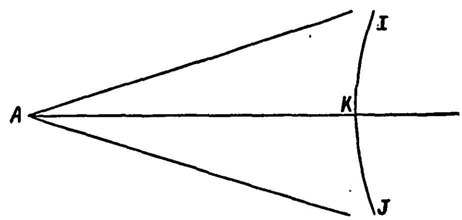

FIG. 6.

drawing the perpendicular from I on AK by the preceding construction.

A Theorem concerning the Special Systems of Point-Groups on a Particular Type of Base-Curve. By Miss FRANCES HARDCASTLE.

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The "special" systems of point-groups on a base-curve of deficiency p are known to be connected in pairs, viz. (employing the usual notation, originally introduced by Brill and Noether*), the special system g_R^r always exists simultaneously with a g_Q^q when the relations

$$Q+R = 2p-2, \qquad \text{(i.)}$$

$$Q-R = 2(q-r) \qquad \text{(ii.)}$$

hold among the positive integers Q, R, q, r.

Given p, these equations, as they stand, are capable of being satisfied by an infinite number of sets of values of Q, R, q, r,† but by means of a further relation, an inequality, namely, $Q \geqslant 2q$ (which also implies that $R \geqslant 2r$), the number of these sets, or, in

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* *Math. Ann.*, Vol. VII., 1873, "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie."

† This is, of course, only true if we assume no knowledge of the application of the equations, and hence no connexion at all among the unknown integers.