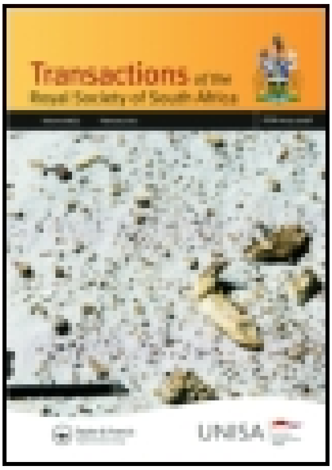

Transactions of the Royal Society of South Africa
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/ttrs20

NOTE ON AN OVERLOOKED THEOREM REGARDING THE PRODUCT OF TWO DETERMINANTS OF DIFFERENT ORDERS

Thomas Muir LL.D.

Published online: 26 Mar 2010.

NOTE ON AN OVERLOOKED THEOREM REGARDING THE PRODUCT OF TWO DETERMINANTS OF DIFFERENT ORDERS.

By Thomas Muir, LL.D.

(Received January 8, 1913.)
(Read April 16, 1913.)

1. Hidden away in an investigation on the common roots of two equations (*Comptes Rendus . . . Acad. des Sci.* (Paris), lxxxviii, pp. 223–224), there occurs the following theorem:—

"*Soient* $\mathrm{A} = \Sigma \pm \mathrm{a}_{11}\mathrm{a}_{22} \dots \mathrm{a}_{\mathrm{mm}}$ *et* $\mathrm{B} = \Sigma \pm \mathrm{b}_{11}\mathrm{b}_{22} \dots \mathrm{b}_{\mathrm{nn}}(\mathrm{m} > \mathrm{n})$ *deux déterminants : si l'on désigne par* B_{ks} *le résultat de la substitution des* n *premiers éléments de la* $\mathrm{k}^{\text{ième}}$ *ligne de* A *à la place de la* $\mathrm{s}^{\text{ième}}$ *ligne de* B, *par* α_{1r} *les mineurs de* A *par rapport au* $\mathrm{r}^{\text{ième}}$ *colonne et par* β_{kr} *le mineur de* B *par rapport au* $\mathrm{r}^{\text{ième}}$ *élément de la* $\mathrm{k}^{\text{ième}}$ *ligne, on a identiquement—*

$$\alpha_{\mathrm{1r}}\mathrm{B}_{\mathrm{1k}} + \alpha_{\mathrm{2r}}\mathrm{B}_{\mathrm{2k}} + \dots + \alpha_{\mathrm{mr}}\mathrm{B}_{\mathrm{mk}} = \mathrm{A}\beta_{\mathrm{kr}},$$

en considérant β_{kr} *comme nul quand* r *est plus grand que* n."

No proof is given, and it is consequently a little difficult to see how the author came to reach a result of such importance without obtaining a much more extensive generalisation.

2. A brief scrutiny suffices to convince one that what the theorem really gives is an expression for the product of any two determinants of the pth and qth orders $(p > q)$ in the form of a sum of products of two determinants of the $(p-1)$th and $(q+1)$th orders. It is at once clear, for example, that the kth row of B in the identity is a fiction, for on the left-hand side it is explicitly supplanted by rows obtained from A, and on the

right-hand side it and the rth column of B are simultaneously excluded. Thus, taking the case

$$m,\ n,\ k,\ r = 4,\ 3,\ 3,\ 3$$

we have

$$|a_{21}\ a_{32}\ a_{44}| \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} - |a_{11}\ a_{12}\ a_{44}| \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$$

$$+ |a_{11}\ a_{22}\ a_{44}| \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - |a_{11}\ a_{22}\ a_{34}| \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

$$= |a_{11}\ a_{22}\ a_{33}\ a_{44}| \cdot |b_{11}\ b_{22}|,$$

where neither on the one side nor on the other do the elements b_{31}, b_{32}, b_{33} (that is to say, the third row of B) occur. As a matter of fact there are only four elements of B found on both sides: and it is the introduction of the two extra elements b_{13}, b_{23} which makes the identity possible.

3. The following mode of establishing the theorem in the foregoing particular case throws additional light on the point raised.

The right-hand member of the identity is clearly equal to

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdot & \cdot \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdot & \cdot \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdot & \cdot \\ b_{11} & b_{12} & b_{13} & \cdot & b_{11} & b_{12} \\ b_{21} & b_{22} & b_{23} & \cdot & b_{21} & b_{22} \end{vmatrix}$$

which, again, by subtraction of the first two columns from the last two columns is equal to

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & -a_{11} & -a_{12} \\ a_{21} & a_{22} & a_{23} & a_{24} & -a_{21} & -a_{22} \\ a_{31} & a_{32} & a_{33} & a_{34} & -a_{31} & -a_{32} \\ a_{41} & a_{42} & a_{43} & a_{44} & -a_{41} & -a_{42} \\ b_{11} & b_{12} & b_{13} & \cdot & \cdot & \cdot \\ b_{21} & b_{22} & b_{23} & \cdot & \cdot & \cdot \end{vmatrix},$$

and the expansion of this by means of Laplace's theorem gives the left-hand member.

Had we omitted the elements b_{13}, b_{23} on starting, the result would have been nugatory, namely—

$$|a_{11}\ a_{22}\ a_{33}\ a_{44}| \cdot |b_{11}\ b_{22}| = |b_{11}\ b_{22}| \cdot |a_{13}\ a_{24}\ a_{31}\ a_{42}|.$$

If on the other hand we had in addition inserted b_{14}, b_{24} we should have obtained for

$$|a_{11}\ a_{22}\ a_{33}\ a_{44}| \cdot |b_{11}\ b_{22}|$$

an expression consisting of six terms of a similar form beginning with

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{vmatrix} \cdot \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}.$$

4. It is thus evident that the product of two determinants of the pth and qth orders is expressible as a sum of products of two determinants: (1) of the $(p-1)$th and $(q+1)$th orders, (2) of the $(p-2)$th and $(q+2)$th orders, (3) of the $(p-3)$th and $(q+3)$th orders, and so on, the number of results being $p-q$.

In Sylvester's similar theorem, which has received every attention from writers of text-books, q is equal to p, and the orders of the determinant factors in the expansion are the same as those in the original product.

Of course any one of the expansions obtained for the given product can be equated to any other, and there is thus originated a series of identities similar to Schweins' of 1825 (see *Philos. Magazine*, xviii (1884), pp. 416–427).