DETERMINANTAL SYSTEMS OF CO-APOLAR TRIADS ON A CUBIC CURVE

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1. *Introduction.*

In the *Proc. London Math. Soc.*, Ser. 2, Vol. 15, pp. 309–313, I studied in some detail the grouping of the triads of points co-apolar with a given triad on a cubic curve. I found that if ABC be an assigned "apolar triangle," and if P be any point chosen arbitrarily on the cubic curve, then *two* complementary dyads of points Q_2R_3 and Q_3R_2 can be found such that $P_1Q_2R_3$ and $P_1Q_3R_2$ are each co-apolar with ABC. If, thereafter, any other point of those obtained above such as Q_2 be taken, the dyad complementary to Q_2, other than P_1R_3 may be denoted by P_3R_1, and so on, with the other points. Proceeding in this way, it was shewn that a *closed* system of nine points was obtained, namely:

$$\begin{matrix} P_1 & P_2 & P_3, \\ Q_1 & Q_2 & Q_3, \\ R_1 & R_2 & R_3, \end{matrix}$$

such that if any one of the six triads defined by the terms of the expansion of the above determinantal form were taken, each triad so obtained would be co-apolar with ABC. Such a system of nine points was called a "determinantal system." The configuration is in its essence a symmetrical one, but the method of obtaining it in the above paper gave no clue as to the symmetry of its form, nor any convenient means of visualizing its structure as a geometrical unit. The purpose of the present communication is to establish the symmetry of the configuration.

2. *The Direct Proposition.*

Let UVW and XYZ be any two coplanar triangles, and let us consider the pencil of cubic curves circumscribing these two triangles and apolar to them. This pencil of cubics will have in common three other points ABC, which we shall refer to as the triangle "complementary" to UVW and XYZ. It was shewn in the *Proceedings of the Edinburgh Mathematical Society*, Vol. 30, Session 1911–12, that every member of the pencil of cubics is apolar to the complementary triangle ABC. It will now be proved that *every member of the pencil of cubic curves circumscribing two triangles and apolar to them cuts the nine lines joining the vertices of these triangles in a determinantal system of points co-apolar with the complementary triangle.*

Let a cubic of the pencil cut the straight lines

$$\begin{array}{ccc} UX, & WY, & VZ, \\ VY, & UZ, & WX, \\ WZ, & VX, & UY, \end{array}$$

in the respective points,

$$\begin{array}{ccc} P_1 & P_2 & P_3, \\ Q_1 & Q_2 & Q_3, \\ R_1 & R_2 & R_3. \end{array}$$

Let us consider first of all the nodal members of the pencil. It was shewn in the *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 235–243, that the necessary and sufficient condition that two apolar triads on a nodal cubic be co-apolar is that the pencils subtended by them at the node shall be apolar. It was also shewn in the above-mentioned paper in the *Proc. Edin. Math. Soc.*, that the canonical form for three complementary apolar triangles on the nodal cubic $x = 6t^2$, $y = 6t$, $z = -(1+t^3)$ can be taken to be

$$ABC \equiv t^3+3pt^2+3qt+pq = 0, \tag{1}$$

$$XYZ \equiv t^3+3rt^2+3st+rs = 0, \tag{2}$$

$$UVW \equiv pqrst^3+3qst^2+3prt+1 = 0. \tag{3}$$

If now we project the triad XYZ through the point t_0 of the nodal cubic on to the nodal cubic again, so as to obtain the triad

$$rst_0^3t^3-3st_0^2t^2+3rt_0t-1 = 0, \tag{4}$$

we see that the apolar triads (1) and (4) subtend pencils at the node which are apolar to one another, if

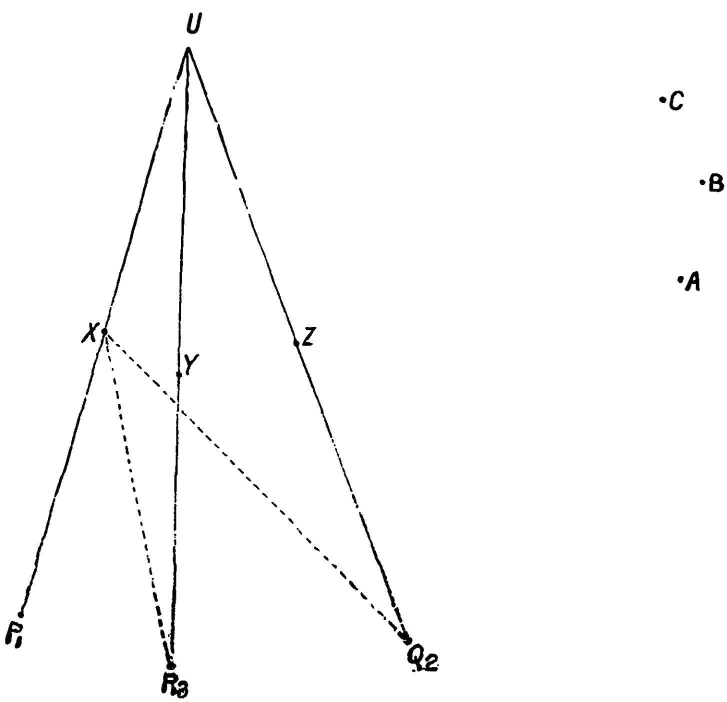
$$pqrst_0^3+3qst_0^2+3prt_0+1 = 0,$$

i.e., if t_0 belong to the triad (3).

Hence we obtain the following result:—

If a nodal cubic circumscribe the two triangles ABC, XYZ and be apolar to them, and if XYZ be projected through any one of the vertices of the complementary triangle UVW on to the cubic again, the apolar triangle thus obtained is co-apolar with ABC.

If U be the vertex chosen through which the projection is to take place, we have thus shewn that the triangle $P_1 Q_2 R_3$ is co-apolar with ABC. Similarly with the other members of the determinantal configuration defined by the intersections of the nodal cubic with the lines joining the vertices of the triangles XYZ, UVW. The required theorem has therefore been proved for the nodal members of the pencil of cubics. We proceed to establish the general result for those members of the pencil which do not possess singular points.



In the above figure, let U be one of the vertices of the triangle complementary to ABC, XYZ. Let R_3 be any point on UY, and let the cubic circumscribing ABC, XYZ and apolar to each of them which passes through R_3, cut UZ in Q_2. Then plainly a one-to-one algebraic correspondence exists between R_3 and Q_2. Again, let Q_2' be taken on UZ, so that $X[U, R_3, Q_2']$ is apolar to $X[ABC]$. It is plain, once more, that a one-to-one algebraic correspondence exists between R_3 and Q_2'. But it has been shewn above that in the case of the *twelve* nodal cubics of the pencil, Q_2 and Q_2' coincide. Hence Q_2 and Q_2' must coincide for *every* member of the pencil.

We thus have the following result:—

If U be one of the vertices of the triangle complementary to ABC, XYZ, and if a cubic circumscribing ABC, XYZ, and apolar to each of them, cut UX, UY, UZ in $P_1R_3Q_2$ respectively, then $P_1R_3Q_2$ subtends at X a pencil of lines which is apolar to $X[ABC]$. Similarly for Y and Z.

But since U lies on the "apolar locus" of ABC, XYZ, we see that $U[ABC]$ is apolar to $U[XYZ]$, *i.e.* to $U[P_1R_3Q_2]$. Hence the "apolar locus" of ABC, $P_1Q_2R_3$ must pass through the points X, Y, Z, U, and therefore have *ten* points in common with the above cubic circumscribing ABC, XYZ, and apolar to each of them, and passing through $P_1Q_2R_3$. The two cubics must therefore coincide. We see therefore with respect to the cubic of the above pencil under consideration that $P_1Q_2R_3$ is a triangle co-apolar with ABC. The final result is now established:

The lines joining the vertices of any two apolar triangles on a cubic curve cut the curve again in nine points forming a determinantal system, co-apolar with the complementary triangle of the given apolar triangles.

3. *The Converse Proposition.*

We next proceed to establish the converse:

If ABC be a given apolar triad and $|PQR|$ a definite determinantal system of co-apolar triads, then $|PQR|$ can in a singly-infinite number of ways be regarded as the further intersections of the cubic curve with the lines joining the vertices of two apolar triangles XYZ, UVW complementary with ABC.

Let U be any point on the cubic curve and let us join U to the three points P_1, Q_2, R_3, so that UP_1, UR_3, UQ_2 cut the cubic again in X, Y, Z respectively. Then, since $P_1Q_2R_3$ is an apolar triangle, we know that XYZ is an apolar triangle. Further, since $U[XYZ]$ is apolar to $U[ABC]$, we see that U is one of the vertices of the triangle complementary to ABC, XYZ. Let V, W be the other two vertices of the triangle complementary to ABC, XYZ. Let XV, XW meet the curve again in R_2', Q_3'. Then, since $X[UVW]$ is apolar to $X[ABC]$, we see from § 2 that P_1, R_2', Q_3' are co-apolar with ABC. Thus the complementary dyads of P_1 are Q_2, R_3, and Q_3', R_2' (according to the notation of the *Proc. London Math.*

Soc., Ser. 2, Vol. 15, pp. 309–313). Hence Q_3', R_2' must coincide with Q_3, R_2 of the determinantal system $|PQR|$. Proceeding in this way, we see that all the lines joining pairs of vertices of XYZ and UVW must pass each through a member of the determinantal system $|PQR|$. Moreover, we chose U arbitrarily on the cubic curve, and thence obtained the other five points XYZ, VW uniquely. Hence, since U can be chosen in a singly-infinite number of ways, we see that the given determinantal system can be regarded as being deduced from a singly-infinite system of pairs of apolar triangles.

4. *Apolar Triangles on a Conic and Tangents to the Cayleyan.*

The case when the two fundamental apolar triangles XYZ, UVW lie on a conic is of considerable geometrical interest. In this case the complementary triad ABC consists of three collinear points lying on the line joining the poles of the conic (regarded as a polar conic) with respect to the triangles XYZ, UVW (regarded as cubic curves), while the tangents at A, B, C to any cubic of the pencil circumscribing XYZ, UVW and apolar to them are concurrent. Moreover, the triads $P_1P_2P_3$, $Q_1Q_2Q_3$, $R_1R_2R_3$ consist each of three collinear points, as do also the triads $P_1Q_1R_1$, $P_2Q_2R_2$, $P_3Q_3R_3$. In fact, we obtain the determinantal system dealt with in the *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 235–243, in which the given cubic is shewn to pass through the nine intersections of three concurrent lines with three other concurrent lines, it being further shewn that the points of concurrency are "corresponding points" on the Hessian curve. Hence in the above configuration, $P_1P_2P_3$, $Q_1Q_2Q_3$, $R_1R_2R_3$ are straight lines passing through one Hessian point of the collinear triad ABC, while $P_1Q_1R_1$, $P_2Q_2R_2$, $P_3Q_3R_3$ pass through the other.

We may proceed further in the search for geometrical properties. It is known that an apolar triad becomes a tangent to the Cayleyan if the three points constituting the triad are collinear. Hence we may take XYZ, UVW to be any two tangents m, n to the Cayleyan. The complementary triad ABC will lie on a line l which is also a tangent to the Cayleyan. Given m and n, it is easy to construct l. For, it was shewn in the *Proc. Edin. Math. Soc.*, in the paper quoted above, that the conic $XYZUVW$ of last paragraph harmonically separates the Hessian points of the triad ABC. Hence, the conic composed of l, m must harmonically separate the Hessian points of UVW, *i.e.* the line l is the join of the harmonic conjugates of the point mn with respect to the Hessian points of XYZ and UVW. We thus see that the lines joining the points of the collinear triads lying on the lines m and n cut the cubic again in nine points which lie three by three on six straight lines that pass through one or other of two "corresponding points" on the Hessian of the given cubic curve, whose position is defined on the line l in the way stated above.

If m and n together constitute a polar-conic of the given cubic, and m', n' be the polar-conic of the point mn, it is easy to shew that the line l is in this case that line which with the join of mn and $m'n'$ constitutes a polar-conic of the original cubic, since every polar-conic cuts the join of a pair of "corresponding points" harmonically. We thus obtain the following theorem:—

If a degenerate polar-conic of the original cubic curve consist of the lines m and n, and if m', n' be the polar conic of the point mn, the nine lines joining the points of section of m and n with the cubic cut the curve again in nine points which lie three by three on six lines passing through one or other of the pair of "corresponding points" in which the line l cuts the Hessian curve, where l is the line which together with the join of mn and $m'n'$ constitutes a degenerate polar-conic of the original cubic.

Symmetry shews *that the joins of the points of section of m', n' with the cubic curve pass through one or other of the same pair of "corresponding points" on the Hessian curve.*

Many other theorems of a like nature might be stated by taking particular positions for two tangents to the Cayleyan.

I take the opportunity of expressing my indebtedness to Dr. William L. Marr, of Aberdeen, a communication from whom first suggested the mode of treatment in the present paper.