ART. II.—*Description of a method of Reducing Observations of Temperature;* by Professor J. D. EVERETT, of Kings College, Windsor, Nova Scotia.

THE climate of a place, as regards temperature, involves three principal elements—mean temperature—range—and *date of phase*, using this last term to denote the earliness or lateness of the seasons generally, as regards temperature.

The first of these elements is subjected to measurement by nearly all meteorological observers; the other two, and especially the third, have not received equal attention. These three elements appertain alike to daily and to annual variations, but we shall confine our remarks to the latter.

Annual range (i. e. the range that occurs within the year) has been measured in various ways. Sometimes it is assumed as the difference between the two extreme readings which occur within the year—sometimes as the difference between the two extremes of daily mean temperature—sometimes as the difference between the mean temperatures of the warmest and the coldest calendar month—sometimes as the difference between the mean temperature of a certain number of the warmest calendar months, and that of an equal number of the coldest.

The two latter modes of measurement are open to serious objection from the unequal manner in which they apply to different places. It is obvious that the range, if estimated as the difference between the warmest and the coldest calendar month, will (*cœteris paribus*) appear greatest when the maximum and minimum fall precisely in the centres of the two months, and if this condition is more nearly fulfilled at one of two places compared than at the other, the comparison will be unequal. The same remark applies when the mean of three (or any other number of) warm months is compared with that of the same number of cold ones, and the error will (in proportion to the deduced range) be as great as in comparing single months.

The element of "date" which thus interferes with the determination of range from monthly means, is, for its own sake, well worthy of careful investigation; but meteorologists generally content themselves with loose estimates of its amount, and with the exception of the article Meteorology in the new edition of the Enc. Britannica, I am not aware that any work in the English language contains directions for computing it.

We propose to describe a method of deducing both "range" and "date" from the mean temperatures of the twelve calendar months. The method, though it is in fact a modification of that described in the article above mentioned, was not thence derived, but was based on a more elaborate method employed by

Professor W. Thomson of Glasgow, and reduced to its present form by the writer.

It virtually consists in removing the irregularities which characterize the actual curve of temperature at a given place, for any particular year or group of years, so as to obtain in its stead a regular curve which can be expressed by a simple mathematical formula. In the reduced curves thus obtained for various places —or what amounts to the same thing, in the formulæ which express them, we have a definite measure both of the comparative earliness of the phases of temperature and of the amounts of annual range as estimated by a comparison of the warmer half of the year with the colder.

The curve which is thus adopted as the standard of reference is what mathematicians call the "curve of sines," or a "simple harmonic curve," and is expressed by the equation

$$y = A_0 + A_1 \sin(x + E_1)$$

where A_0 denotes the mean temperature of the year, A_1 the amplitude, or greatest departure of the curve from the line of mean temperature, which will be the same above this line as below, and will therefore be equal to half the annual range, and E_1 expresses the "date of phase" being greater in proportion as the phases are earlier. The curve has one maximum and one minimum in the year; which are precisely half a year asunder, and exactly midway between these are the two points where the curve intersects the line of mean annual temperature, corresponding to these two days, one in Spring and the other in Autumn, whose temperatures are on the average the same as the mean of the year.

The curve for a year will consist in fact of four precisely similar portions, the part which is above the line of mean temperature being precisely similar to that which is below, and each of these halves being bisected symmetrically by the points of maximum and minimum temperature respectively.

It is not of course pretended that the actual temperature of any place fulfills these conditions; but merely that when a uniform standard of reduction is to be applied to a number of places (in the temperate or frigid zones) such a curve as we have described is adapted to the purpose. While possessing the necessary amount of uniformity, the curve at the same time admits of infinite variety in respect of its amplitude (i. e. the extent of its departure from a straight line) which may be increased or diminished, without limit, according as we wish to represent a climate where the annual range is great or small.

It is not necessary in practice to draw the curves in question, but merely to calculate the values of the constants A_0 A_1 and E_1, the manner of doing which will be shown further on. We may merely remark in passing, that the labor of deriving these three constants from the monthly means, is less than that of deriving the monthly from the daily means.

The constant A_0 as already stated, is the mean annual temperature.

The constant E_1 represents the interval from that day in Autumn which forms the boundary between the warm and cold halves of the year to the 15th of January, the scale of representation being such that 360° corresponds to an entire year.

The constant A_1 (or the amplitude) is approximately equal to the difference between the mean temperature of the year and that of the warmest or coldest group of 30 days. More accurately it is proportional (but not equal) to the difference between the mean temperatures of the warm and cold halves of the year, bearing to this difference the constant value of 1 : 1·2879. In speaking of the warm and cold halves of the year, I suppose the year divided at two opposite points, that is to say two points which are six months asunder, in such a manner that the greatest possible amount of heat shall be contained in one half, and (consequently) the greatest possible amount of cold in the other.

In the definition here given of E_1 and in the second of the definitions of A_1, not only *annual* harmonic variations, but also *half-yearly*, are taken into account.

As a specimen of the manner in which the proposed method of reduction may be employed for comparing climates, I subjoin a table[1] showing its results as applied to all those stations of the Scottish Meteorological Society which have furnished observations of temperature for the three years 1856–7–8. The data are the mean temperatures of the stations for each calendar month, on the average of the 3 years above named, as given in the Society's Report for the quarter ending June 30th, 1859.

The names of stations are entered in the order in which they appear in the Society's Reports, being nearly that of latitude, proceeding from north to south.

The first column of numbers contains the values of A_0 or the mean annual temperature, obtained in the usual manner.

In the second and third columns are the values of A_1 and E_1 (amplitude and epoch) determined in the manner already explained.

The fourth column shows the number of days and tenths of a day by which each station is earlier or later (as regards the phases of the temperature) than the mean of all; days earlier than the mean being denoted by the sign +, and days later than the mean by the sign −.

The fifth column exhibits the difference between the mean temperatures of the warm and cold halves of the year.

The numbers in the fourth column have been obtained from

[1] Table I.

those in the third by taking the difference between the value of E_1 for each particular station, and its value for the mean of all, and converting it into days at the rate of 1° to $1\frac{1}{72}$ day, (since 360 : 365 : : 72 : 73.)

The numbers in the fifth column are proportional to those in the second, and have been obtained from them by the formula log. A_1 + ·1099 = log. n, (since ·1099 is the logarithm of 1·2879.)

TABLE I.—*Results for three years*, 1856–8.

Stations.	Values of A_0	Values of A_1	Values of E_1	Days earlier + later − than mean.	Difference between warm and cold half.
Stornoway	46·4	8·99	75° 34′	− 3·8	11·6
Culloden	47·5	10·17	80° 57′	+ 1·6	13·1
Elgin	47·2	10·33	79° 24′	+ 0·1	13·3
Castle Newe	44·2	10·83	80° 29′	+ 1·2	13·9
Braemar	44·9	10·89	79° 6′	− 0·2	14·0
Aberdeen	45·9	10·69	78° 50′	− 0·5	13·8
Fettercairn	46·9	11·56	83° 22′	+ 4·1	14·9
Arbroath	46·6	11·09	79° 53′	+ 0·5	14·3
Barry	47·7	10·54	78° 24′	− 1·0	13·6
Kettins	45·8	11·20	80° 39′	+ 1·3	14·4
Callton Mor	47·2	10·37	80° 37′	+ 1·3	13·4
Greenock	48·4	10·97	76° 37′	− 2·7	14·1
Baillieston	46·6	11·59	80° 28′	+ 1·1	14·9
Edinburgh	49·0	10·71	77° 6′	− 2·3	13·8
Dalkeith	48·3	12·02	79° 25′	+ 0·1	15·5
East Linton	47·3	10·66	76° 16′	− 3·1	13·7
Thurston	47·0	10·74	71° 43′	− 7·7	13·8
Yester	46·2	11·55	83° 56′	+ 4·7	14·4
Thirlestane	45·1	11·83	80° 46′	+ 1·4	15·2
Milnegraden	47·0	11·08	78° 43′	− 0·6	14·3
Bowhill	44·2	11·17	82° 9′	+ 2·9	14·4
Makerstoun	46·8	10·65	77° 22′	− 2·0	13·7
Drumlanzig	47·0	11·95	81° 1′	+ 1·7	15·4
Kirkpatrick	46·5	10·85	81° 5′	+ 1·8	14·0
Means	46·7	10·94	79° 20′		14·1

TABLE II.—*Results for single years.*

Stations.	Values of E_1. 1856.	1857.	1858.	Values of A_1. 1856.	1857.	1858.
Bressay (Shetland)	....	55° 41′	73° 48′	...	7·8	8·6
Sandwick (Orkney)	....	62° 2′	75° 49′	...	8·6	8·3
Tongue	....	72° 29′	84° 19′	...	9·7	9·1
Stornoway	81° 50′	72° 3′	73° 27′	8·6	9·4	9·1
Culloden	80° 29′	74° 56′	87° 25′	9·8	10·4	10·4
East Linton	76° 4′	68° 27′	84° 2′	...	...	...
Thurston	70° 56′	62° 10′	82° 40′	...	...	...
Yester	87° 4′	....	....	...	...	...

To find the centres of the warm and cold halves of the year, we may proceed as follows: The mean value of E_1 for all the stations is 79° 20′. To reduce to the beginning of the year, subtract 15°, since our reckoning has been taken from the middle of the first month. This leaves 64° 20′, which is the interval from the beginning of the cold half to the end (or beginning) of the year. The complement of this or 25° 40′ is the interval from the beginning of the year to the centre of the cold half, which again is 180° distant from the centre of the warm half.

25° 40′ corresponds to 26 days (nearly)
205° 40′ " " 209 " "

The 26th and 209th days of the year are January 26th and July 28th, which are therefore the centres of the cold and warm halves of the year, for the mean of the stations. The corresponding dates for any particular station, will be later or earlier than these by the amounts shown in the fourth column.

An expeditious method of finding the centre of the cold half is to assume the complement of E_1 as representing the interval from Jan. 15th to the required centre. Thus the complement of 70° 20′ is 10° 40′ corresponding to 11 days nearly, hence the centre of the cold half is 11 days later than January 15th. This determination it will be observed coincides with that above given. In like manner the centre of the warm half will be 11 days later than July 17th.

By taking the sum and the difference of A_0 and A_1 we should obtain approximately the mean temperatures of the warmest and coldest groups of 30 days; or if the difference between the temperatures of these two periods is required, it can be found by simply doubling A_1. These determinations are however only first approximations, and this is my reason for omitting them, all the numbers contained in the Table being second approximations at least.

With the joint purpose of testing the powers of the method, and comparing different years, I have calculated the values of A_1 and E_1 for single years for a few of the Society's stations, including three (the first three) which are not contained in the former table. The results are given without any reservation in Table II.

Bressay (Shetland) appears to be the latest of the Society's stations, being about 13 days behind the mean of the 24 stations included in Table I. Sandwick (Orkney) precedes Bressay by about 2 days, and this interval is preserved nearly constant from 1857 to 1858, although the absolute times differ by nearly a fortnight. The amplitudes are also less for these two stations than for any others, the amplitude (and consequently the range) at Bressay being only about four-fifths of the average derived from the 24 stations. The extreme lateness of Thurston (near Dunbar) seems to be borne out by the results from single years, as appears from a comparison with the neighboring station, East Linton. The extreme earliness of Yester cannot be so satisfactorily tested, as the interpolations (in defect of observations) at this station are numerous during the years 1857–8. In the year 1856, which is entirely free from interpolation, Yester appears to have been 16 days earlier than Thurston, and 11 earlier than East Linton, a remarkable difference, considering that all three places are in the same county (East Lothian). Comparing one year with another, it appears that the seasons were latest in 1857, being fully a week later than in 1856, and at some places about a fortnight later than in 1858. At Thurston the difference between the last two years amounts to nearly 21 days. All the inferences as to dates contained in this paragraph, are derived from mere inspection of the values of E_1 bearing in mind that a degree nearly corresponds to a day, and that the phases are earlier in proportion as E_1 is greater.

As an instance of the convenience afforded by the present method, for comparing the climates of different countries, I subjoin the values of A_0, A_1 and E_1 for Edinburgh, and for Albion Mines, N. S., the former derived from the monthly means of the late Mr. Adie's observations, embracing a period of 40 years, for which I am indebted to a paper by Principal J. D. Forbes, as epitomised in the Ed. New Phil. Journal for July, 1860, the latter from 11 years observations by Mr. Henry Poole, Manager of the mines. The monthly means themselves are—

For Edinburgh.

36·69	37·99	40·61	44·83	50·27	55·66	58·27	57·44	53·73	47·47	41·21	36·60

For Albion Mines.

19·85	19·90	27·41	37·38	48·58	58·14	66·10	65·19	56·05	46·28	35·59	24·47

from which are derived the following values of mean temperature, amplitude, and epoch:

Edinburgh,	$A_0=46·9$	$A_1=10·8$	$E_1=83°\ 27'$
Albion Mines,	$A_0=42·1$	$A_1=23·0$	$E_1=78°\ 13'$

Hence, cleared of technicalities, the relation between the two climates may be expressed by saying that the village of Albion Mines is on the average of the year about 5° colder than Edinburgh, that its range is rather more than double, and that its seasons are on the average 5 days later. No such definite information is obtained by inspecting the monthly means.

With the view of ascertaining the amount of error entailed by assuming (as our method does) that the calendar months are all of equal length, I have calculated the values of A_0 A_1 and E_1 for Edinburgh in four different ways, my data being the mean temperature of Edinburgh for every day in the year, as contained in the number of the Phil. Journal above referred to, viz:

1st. When the last 2 days of January and first 2 days of

March are reckoned part of February, giving February 33 days, and leaving January and March only 29 days each.

2d. When the last 3 days of February are reckoned part of March, so that January will have 31 days, February 26, and March 34.

3d. When the last day of January and first of March are reckoned part of February, so that January will have 30 days, February 31, and March 30.

4th. When calendar months are adopted, giving January 31 days, February 29, and March 31.

The resulting values of A_0 A_1 and E_1 are as under.

Days.	Days.	Days.		A_0	A_1	E_1
Jan. 29,	Feb. 33,	March 29,	gives	46·91	10·87	83° 37′
" 31,	" 26,	" 34,	"	46·88	10·81	83° 19′
" 30,	" 31,	" 30,	"	46·90	10·78	83° 33′
" 31,	" 29,	" 31,	"	46·90	10·78	83° 27′

Here a difference of 7 days in the length of February causes a difference of ·03 in the mean temperature, ·06 in amplitude, and 18′, or about $\frac{1}{3}$ of a day, in date. From the last two lines it appears that the difference between giving February 29, and 31, days does not affect either mean temperature or amplitude, to two places of decimals, and only affects date by about $\frac{1}{10}$ of a day.

Apart from the small error arising from treating calendar months as twelfth parts of a year, conclusions deduced from monthly means are as accurate as those from daily means, the correction necessary for reducing monthly to daily results being extremely simple and easy of application, the value of E_1 being the same for both, and the values of A_1 differing in the constant ratio of 1 : 1·0115.

I shall not attempt to show in detail the advantages which meteorology may be expected to derive from the extensive application of the method of reduction here proposed. The superiority of definite measures to mere general estimates, is recognized in every branch of statistical enquiry, yet no such measure is usually applied to "date of phase," and the measures commonly used in determining range are subject to an error which affects different places very unequally.

The determination of the "date of phase" will furnish a precise measure of the retarding effect of the sea, and also of the different varieties of soil. The general effect of the interchange of heat between the soil and the air must obviously be to retard the air temperature, but I am not aware that different soils have ever been compared in this respect.

The laws which connect date of phase with extent of range also offer an interesting field of investigation. Generally speaking, the causes which retard the former diminish the latter.

In the application of meteorology to agriculture, date of phase cannot, without serious error, be overlooked. The earliness of crops at one place as compared with another, must necessarily depend upon this element as well as upon mean temperature and range, and it will be interesting to ascertain how much of the effect is due to each of these causes.

Thus far we have endeavored to describe in general terms the objects and principles of the proposed method of reduction. The remainder of this paper will be devoted to the mathematical investigation on which the method rests.

By taking observations of temperature at any place for a sufficiently long series of years, it would be possible to ascertain the average temperature of each day in the year, and if the mean daily temperatures thus found were projected into a curve, its course would be free from those sudden and irregular deviations which characterize the curve of temperature for any particular year.

Such a curve would admit of being expressed, to any required degree of accuracy, by an equation of the form

$$y = A_0 + A_1 \sin(x+E_1) + A_2 \sin(2x+E_2) + A_3 \sin(3x+E_3) + \&c.$$

x and y being the coördinates of any one point in the curve, and A_0 A_1 E_1, &c. being constants. The coefficients A_1 A_2 A_3, &c., are the *amplitudes* of the terms in which they occur, and E_1 E_2 E_3, &c. are *epochs*. The term which involves A_1 and E_1 attains one maximum and one minimum in the space of a year, it is therefore called the annual term. The term involving A_2 and E_2 attains one maximum and one minimum in half a year, it is therefore called the half yearly term; and in general the term $A_n \sin(nx+E_n)$ goes through its entire cycle of values in the $\frac{1}{n}$th part of a year. We assume of course that a year is represented in arc by 2π, or the entire circumference.

For places in the temperate zones the amplitudes of successive terms in the above series diminish so rapidly, that for ordinary purposes all terms involving A_3 and higher coefficients may be neglected.

The mean daily temperatures for any single year or for the average of a few years are too irregular to admit of being expressed with accuracy by any simple formula, but it is possible to represent by a few terms of the above series the probable curve of annual temperature as deduced from the actual daily temperatures even of a single year. It is one object of the present communication to show how this may conveniently be done.

We shall now proceed to the solution of the following problem.

Given the temperatures at twelve equidistant points in the year, it is required to deduce the values of the constants in an expression of the above form which shall be applicable to them.

The general term in the expression is $A_n \sin(nx+E_n)$. Let this be assumed equal to $P_n \cos nx + Q_n \sin nx$. This assumption gives

$$A_n \sin E_n = P_n, \quad A_n \cos E_n = Q_n$$

whence
$$\frac{P_n}{Q_n} = \tan E_n, \qquad A_n^2 = P_n^2 + Q_n^2.$$

The transformed series is

$$y = A_0 + P_1 \cos x + Q_1 \sin x + P_2 \cos 2x + Q_2 \sin 2x + \text{\&c.}$$

Let the given temperatures be denoted by

$$y_0 \; y_1 \; y_2 \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; y_{11}$$

Then if the time 0 correspond to the temperature y_0, the times, or values of x, corresponding to the 12 given values of y are respectively 0°, 30°, 60°, ... 330°.

Let the sines of 0°, 30°, 60° and 90°, be denoted by the abbreviations S_0, S_1, S_2 and S_3. Then we have

$$\begin{aligned}
y_0 &= A_0+P_1S_3+Q_1S_0+P_2S_3+Q_2S_0+P_3S_3+Q_3S_0+P_4S_3-Q_4S_0\\
y_1 &= A_0+P_1S_2+Q_1S_1+P_2S_1+Q_2S_2+P_3S_0+Q_3S_3-P_4S_1+Q_4S_2\\
y_2 &= A_0+P_1S_1+Q_1S_2-P_2S_1+Q_2S_2-P_3S_3-Q_3S_0-P_4S_1-Q_4S_2\\
y_3 &= A_0+P_1S_0+Q_1S_3-P_2S_3-Q_2S_0-P_3S_0-Q_3S_3+P_4S_3-Q_4S_0\\
y_4 &= A_0-P_1S_1+Q_1S_2-P_2S_1-Q_2S_2+P_3S_3+Q_3S_0-P_4S_1+Q_4S_2\\
y_5 &= A_0-P_1S_2+Q_1S_1+P_2S_1-Q_2S_2+P_3S_0+Q_3S_3-P_4S_1-Q_4S_2\\
y_6 &= A_0-P_1S_3-Q_1S_0+P_2S_3-Q_2S_0-P_3S_3-Q_3S_0+P_4S_3-Q_4S_0\\
y_7 &= A_0-P_1S_2-Q_1S_1+P_2S_1+Q_2S_2-P_3S_0-Q_3S_3-P_4S_1+Q_4S_2\\
y_8 &= A_0-P_1S_1-Q_1S_2-P_2S_1+Q_2S_2+P_3S_3+Q_3S_0-P_4S_1-Q_4S_2\\
y_9 &= A_0-P_1S_0-Q_1S_3-P_2S_3+Q_2S_0+P_3S_0+Q_3S_3+P_4S_3-Q_4S_0\\
y_{10} &= A_0+P_1S_1-Q_1S_2-P_2S_1-Q_2S_2-P_3S_3-Q_3S_0-P_4S_1+Q_4S_2\\
y_{11} &= A_0+P_1S_2-Q_1S_1+P_2S_1-Q_2S_2-P_3S_0-Q_3S_3-P_4S_1-Q_4S_2
\end{aligned}$$

Subtracting y_6 from y_0, y_7 from y_1, y_8 from y_2, &c., all the terms which contain P_2 Q_2 P_4 and Q_4 will disappear. Similarly, adding y_6 to y_0, y_7 to y_1, y_8 to y_2, &c., all the terms which contain P_1 Q_1, P_3 and Q_3 will disappear.

Let
$$\begin{aligned}
y_0-y_6 &= k_0\\
y_1-y_7 &= k_1\\
y_2-y_8 &= k_2\\
y_3-y_9 &= k_3\\
y_4-y_{10} &= k_4\\
y_5-y_{11} &= k_5
\end{aligned}$$

Also let
$$\begin{aligned}
k_0 \times S_3 &= l_0\\
(k_1-k_5) \times S_2 &= l_1\\
(k_2-k_4) \times S_1 &= l_2\\
k_3 \times S_0 &= l_3
\end{aligned}$$

And let
$$\begin{aligned}
k_0 \times S_0 &= m_0\\
(k_1+k_5) \times S_1 &= m_1\\
(k_2+k_4) \times S_2 &= m_2\\
k_3 \times S_3 &= m_3
\end{aligned}$$

It will be found that the sum of l_0, l_1, l_2 and l_3 is $6P_1$, and the sum of m_0, m_1, m_2 and m_3 is $6Q_1$. Hence P_1 and Q_1 are found as in the arithmetical example below; and E_1 and A_1 are obtained by the equations,

$\tan E_1 = \frac{P_1}{Q_1}$, $A_1^2 = P_1^2 + Q_1^2$, or using E_1 as a subsidiary angle, $A_1 = Q_1, \sec E_1$.

To find P_2 and Q_2, proceed as under.

$$\begin{array}{llll}
\text{Let } & y_0 + y_6 = K_0 & \quad & (K_0 - K_3) \times S_3 = L_0 \\
& y_1 + y_7 = K_1 & & (K_1 - K_4) \times S_1 = L_1 \\
& y_2 + y_8 = K_2 & & (K_2 - K_5) \times (-S_1) = L_2 \\
& y_3 + y_9 = K_3 & & (K_0 + K_3) \times S_0 = M_0 \\
& y_4 + y_{10} = K_4 & & (K_1 + K_4) \times S_2 = M_1 \\
& y_5 + y_{11} = K_5 & & (K_2 + K_5) \times S_2 = M_2
\end{array}$$

Then will $L_0 + L_1 + L_2 = 6P_2$

$M_0 + M_1 + M_2 = 6Q_2$

whence E_2 and A_2 can be obtained by the equations

$$\tan E_2 = \frac{P_2}{Q_2} \qquad A_2 = Q_2 \sec E_2.$$

To find P_3 Q_3, P_4 and Q_4 we have

$$k_0 + k_4 - k_2 = 6P_3 \qquad K_0 + K_3 - \tfrac{1}{2}(K_1 + K_2 + K_4 + K_5) = 6P_4$$
$$k_1 + k_5 - k_3 = 6Q_3 \qquad (K_1 + K_4 - K_2 - K_5) \times S_2 = 6Q_4$$

whence E_3 A_3, E_4 and A_4 can be obtained as above.

In the following example, the values of P_1, Q_1, E_1 and A_1, are found for Halifax, N. S., on the assumption that the mean temperatures of the calendar months, may be regarded as identical with the temperatures of 12 equidistant points in the year. The numbers in the first column are the mean temperatures of the months January to June, those in the second column are the mean temperatures of the months July to December.

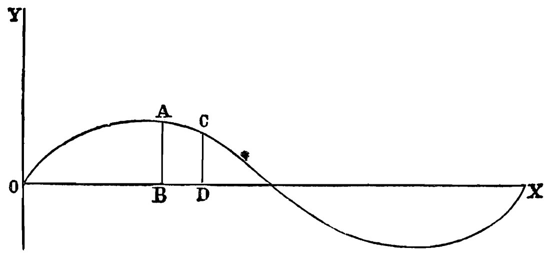
I.	II.	III. = (I.—II.)	IV. (two last in III.)	V. = (III—IV.)	VI.	VII. = (V×VI.)	VIII. = (III+IV.)	IX.	X. = (VIII×IX.)
23·9	64·9	−41·0		−41·0	S_3	−41·00	−41·0	S_0	·00
23·2	65·1	−41·9	+30·6	−72·5	S_2	−62·79	−11·3	S_1	− 5·65
30·1	58·3	−28·2	+10·5	−38·7	S_1	−19·35	−17·7	S_2	−15·33
38·9	48·2	− 9·3		− 9·3	S_0	·00	− 9·3	S_3	− 9·30
48·3	37·8	+10·5			6	−123·14	6		−30·28
58·4	27·8	+30·6			$P_1=$	− 20·52	$Q_1=$		− 5·05

$$E_1 = \tan^{-1} \frac{P_1}{Q_1} = 76° \, 10, \; A_1 = Q_1 \sec E_1 = -21·14.$$

The coefficients A_2, E_2 and those belonging to higher terms are of comparatively little practical use, and it will not be necessary to append examples of the process for obtaining them, as there is no difficulty in the application of the formulæ.

The last edition of the Encyclopaedia Britannica has an article on "Meteorology" by Sir John Herschel, in which the attention of meteorologists is called to the great practical utility of the mode of reduction above described, which has been for some time known but has been little used. The formulæ which Sir John Herschel there gives for deriving the values of the constants from monthly means, are in reality identical with those above given, though the identity is not at first sight obvious. He asserts that the values thus obtained are the most probable values, as derived from the application of the method of least squares. Also that "it is a peculiarly valuable property of these expressions, that if the approximation be stopped at any one term, ... then should it be considered afterwards desirable to carry it a term further, ... it is not necessary to recompute the former coefficients, their values remaining unaltered."[1]

Instead of using the temperatures of 12 equidistant days, as the basis of calculation, there are obvious advantages in employing the mean temperatures of the 12 months which compose the year; but it will be necessary to apply a correction to the results thus obtained; for it is not true, even on the average of a long series of years, that the mean temperature of a month is the same as that of its middle day. We shall proceed to investigate the nature and amount of the correction which must be applied, deducing by the way some interesting relations between the mean and instantaneous values of variable elements.



Let OACX be the curve which represents the variations of temperature through the year. Let the ordinates AB and CD represent the temperatures at the beginning and end of an interval of time represented by BD. It is obvious that the mean temperature of this interval will be obtained by dividing the area ABCD by the distance BD.

[1] Of the theorems to which the remainder of this article is devoted, I believe I have the honor to be the first discoverer. They were first published by me in the Edin. New Phil. Journal for July, 1861. A correction for the difference between the mean temperature of a month and the temperature of the middle of the month had been applied (unknown to me) by Professor (now Principal) J. D. Forbes, in a paper read March 25th, 1860, (Trans. R. S. E., vol. xxii, Part II), accompanied by the remark that the correction has not usually been made. But the method there employed was only approximate and was based on different principles from that here described.

First let us suppose the equation of the curve (or the expression for the temperature in terms of the time) to be

$$y = a \sin x.$$

Let $2c$ denote the length of the interval BD, and let x' be the value of x for its middle point. Then the values of x for points B and D will be $x'-c$ and $x'+c$, and the area ABDC will be the integral of ydx taken between these limits,

$$= a(\cos \overline{x'-c} - \cos \overline{x'+c})$$
$$= 2a \sin x'. \sin c$$
$$= 2 \sin c . y'$$

if y' denote the value of y for the middle point of BD.

Hence the area bounded by two ordinates whose mutual distance is given, varies directly as the ordinate drawn midway between them. The areas of portions of the curve below the line OX must of course be reckoned as negative.

Dividing the expressions for the area by $2c$ we obtain

$$\frac{\sin c}{c} y',$$

which is therefore the mean value of y for the given interval.

Let $c = \frac{\pi}{m}$, then 2π denoting a year, the given interval $2c$ will be the $\frac{1}{m}$th part of a year. Hence the mean temperature of any $\frac{1}{m}$th part of a year is to the temperature of its middle point as $\sin\frac{\pi}{m} : \frac{\pi}{m}$. If the given interval is the $\frac{1}{12}$th part of a year, this ratio becomes $\sin 15° ; \text{arc } 15°$ or $1 : 1{\cdot}0115$.

These conclusions have been drawn on the supposition that the expression for the temperature is $y = a \sin x$. They will still be true if the expression be

$$y = a \sin (x + \mathrm{E})$$

for this change only amounts to removing the origin of coördinates along the axis of x and does not alter the values of the ordinates.

If the expression for the temperature be

$$y = \mathrm{A} + a . \sin (x + \mathrm{E})$$

the ordinates will be greater than before by the constant quantity A, which represents the mean temperature of the year; hence the temperatures will require to be diminished by the mean of the year in order that the above conclusions may be applicable. The following theorem will hold in all three cases, viz:—

The difference between the mean temperatures of any two equal portions of the year will be less than the difference between the temperatures of their respective centres, in the constant ratio of $\sin \frac{\pi}{m} : \frac{\pi}{m}$, each of the portions being supposed to be the $\frac{1}{m}$th part of a year, where m may be either a whole number or a fraction.

Hence the annual range as shown by the curve of monthly mean temperatures will be less than that exhibited by the curve of daily mean temperatures in the ratio of sin 15°: arc 15°.

Strictly speaking, instead of "daily mean temperatures," I ought to say "instantaneous temperatures;" but the difference is so small as to be quite inappreciable, since the former are to the latter nearly in the ratio of sine to arc of 30 minutes or of 1 to 1·000013.

Assuming then that the expression for instantaneous temperature is

$$y = A + a \sin(x + E)$$

the mean temperature Y_m of any $\frac{1}{m}$th portion of a year will be given by the equation

$$Y_m = A + a \cdot \frac{\sin \frac{\pi}{m}}{\frac{\pi}{m}} \sin(x + E)$$

x being the time for the centre of the portion. Hence if the instantaneous temperatures follow a simple harmonic law, the mean temperatures of equal intervals of time will also follow a simple harmonic law. For the mean temperature of any period of $30\frac{5}{12}$ days we have

$$Y_{12} = A + a \cdot \frac{\sin 15°}{\text{arc } 15°} \sin(x + E).$$

Secondly, let the expression for instantaneous temperatures be

$$y = A_0 + a_1 \sin(x + E_1) + a_2 \sin(2x + E_2).$$

The expression for the area bounded by two ordinates whose distance is $2c$ will as in the former case be the integral of ydx between the limits $x'-c$ and $x'+c$

$$= 2A_0c + 2a_1 \sin c \sin(x' + E_1) + 2a_2 \frac{\sin 2c}{2} \sin(2x' + E_2)$$

and dividing by $2c$ we obtain for the mean value of y the expression

$$A_0 + a_1 \frac{\sin c}{c} \sin(x' + E_1) + a_2 \frac{\sin 2c}{2c} \sin(2x' + E_2).$$

Hence the mean temperature of any $\frac{1}{m}$th portion of a year is given by the equation

$$Y_m = A_0 + a_1 \frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}} \sin(x+E_1) + a_2 \frac{\sin\frac{2\pi}{m}}{\frac{2\pi}{m}} \sin(2x+E_2).$$

Let $m=2$ and we have for the mean temperature of a half year

$$Y_2 = A_0 + a_1 \frac{1}{\frac{\pi}{2}} \sin(x+E_1),$$

the third term vanishing, since $\sin \pi = 0$. Hence the half yearly term produces no effect upon the mean temperature of a half year (as is also obvious from general considerations), and the amplitude of the half yearly means is to that of the annual term for daily means, as the radius of a circle is to a quadrantal arc.

The range of the half yearly means, being double of the amplitude, is $a_1 . \frac{2}{\frac{\pi}{2}}$ which being divided by $a_1 . \frac{\sin\frac{\pi}{12}}{\frac{\pi}{12}}$ the amplitude of the annual term for monthly means, gives as a quotient $\frac{1}{3\sin\frac{\pi}{12}}$ the numerical value of which is 1·2879. Hence if the amplitude A_1 deduced from monthly means be multiplied by this number, the product will be the difference between the temperatures of the warmest and coldest halves into which the year can be divided.

Lastly, let the expression for instantaneous temperatures take the general form

$$y = A_0 + a_1 \sin(x+E_1) + a_2 \sin(2x+E_2) + \quad . \quad . \quad . \quad . \quad + a_n \sin(nx+E_n)$$

It will be found by proceeding as in the previous cases, that the expression for the mean temperature of the $\frac{1}{m}$th of a year is

$$Y_m = A_0 + a_1 . \frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}} \sin(x+E_1) + a_2 \frac{\sin\frac{2\pi}{m}}{\frac{2\pi}{m}} \sin(2x+E_2) +$$
$$+ a_3 . \frac{\sin\frac{3\pi}{m}}{\frac{3\pi}{m}} \sin(3x+E_3) + \, . \, . \, . \, . \, . + a_n . \frac{\sin\frac{n\pi}{m}}{\frac{n\pi}{m}} \sin(nx+E_n).$$

Hence if $A_1, A_2, A_3 \, . \, . \, . \, . \, A_n$ denote the amplitudes deduced from monthly means, we have

$$A_1 = a_1 . \frac{\sin 15^\circ}{\text{arc } 15^\circ}, \; A_2 = a_2 \frac{\sin 30^\circ}{\text{arc } 30^\circ}, \; A_3 = a_3 \frac{\sin 45^\circ}{\text{arc } 45^\circ},$$

and generally $$A_n = a_n . \frac{\sin n \times 15^\circ}{\text{arc } n \times 15^\circ}.$$

Conversely,

$$a_1 = A_1 \frac{\text{arc } 15^\circ}{\sin 15^\circ} = A_1 \times 1{\cdot}0115$$

$$a_2 = A_2 \frac{\text{arc } 30^\circ}{\sin 30^\circ} = A_2 \times 1{\cdot}0472$$

$$a_3 = A_3 \frac{\text{arc } 45^\circ}{\sin 45^\circ} = A_3 \times 1{\cdot}1107$$

$$a_4 = A_4 \frac{\text{arc } 60^\circ}{\sin 60^\circ} = A_4 \times 1{\cdot}2092$$

&c. &c.

Hence, to reduce monthly to daily results it will simply be necessary to multiply the amplitudes A_1, A_2, &c., as above indicated. The logarithms of the multipliers for A_1, A_2, A_3 and A_4 are as under.

$$\log. \frac{\text{arc } 15^\circ}{\sin 15^\circ} = {\cdot}0049725$$

$$\log. \frac{\text{arc } 30^\circ}{\sin 30^\circ} = {\cdot}0200286$$

$$\log. \frac{\text{arc } 45^\circ}{\sin 45^\circ} = {\cdot}0456049$$

$$\log. \frac{\text{arc } 60^\circ}{\sin 60^\circ} = {\cdot}0824980.$$