IV. *Notes on the Motion of Viscous Liquids in Channels.* By J. PROUDMAN *.

1. **I**N a recent communication to this Journal † Messrs. Deeley and Parr remark that the conditions of the steady flow of a viscous liquid in a parabolic channel, under a constant force parallel to the length of the channel, have not yet been ascertained. It is implied that the results might be of interest in connexion with the motion of glaciers.

In the present communication the problem is solved for the special case in which the free surface of the liquid passes through the focus of the parabolic section, and also for a particular triangular section. Some remarks are also added in connexion with the mathematical expansions used.

The general problem ‡ for a channel of any section may be reduced to that of finding a function χ which satisfies

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = -2, \quad \ldots \ldots \quad (1)$$

over the section, which vanishes over the sides of the section, and for which $\partial \chi / \partial n = 0$ over the free surface. Here x, y are rectangular Cartesian coordinates in the plane of the section, and $\partial / \partial n$ denotes differentiation along the normal to the free surface.

The velocity of the liquid, which is parallel to the length of the channel, is given by $P\chi / 2\mu$, where P is the pressure gradient along the channel, and μ is the coefficient of viscosity. In applications, the function

$$F = \iint \chi \, dS,$$

where the integral is taken over the area of the section, is required.

Particular Parabolic Section.

2. For convenience, take the length of the latus-rectum of the parabola to be $4\pi^2$. Then if we take polar coordinates r, θ, having for pole the focus S, and for initial line the axis

SA, the equation of the parabola will be

$$r^{\frac{1}{2}} \cos \tfrac{1}{2}\theta = \pi,$$

and that of the latus-rectum $\theta^2 = \tfrac{1}{4}\pi^2.$
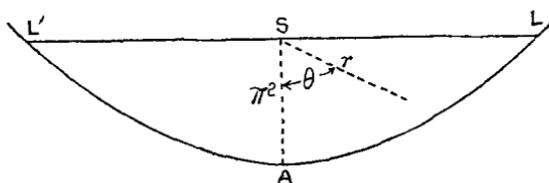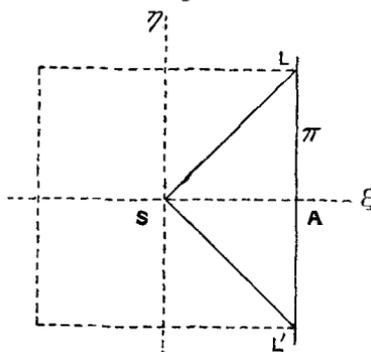
Fig. 1.

Fig. 2.

Let us take

$$\xi = r^{\frac{1}{2}} \cos \tfrac{1}{2}\theta, \qquad \eta = r^{\frac{1}{2}} \sin \tfrac{1}{2}\theta,$$

so that we have a conformal transformation if ξ, η be regarded as Cartesian coordinates in another plane, the parabola transforming into $\xi = \pi$, and the latus-rectum into $\xi^2 = \eta^2$. The correspondence is shown in figs. 1 and 2, where corresponding points are similarly lettered.

Since

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} = 4(\xi^2 + \eta^2), \quad \cdots \cdots \quad (2)$$

the conditions to be satisfied by χ become, with reference to fig. 2,

$$\frac{\partial^2 \chi}{\partial \xi^2} + \frac{\partial^2 \chi}{\partial \eta^2} = -8(\xi^2 + \eta^2), \quad \cdots \cdots \quad (3)$$

over the area L'SL, $\chi = 0$ on $\xi = \pi$, $\partial\chi/\partial\xi = \partial\chi/\partial\eta$ on $\xi = \eta$, and $\partial\chi/\partial\xi = -\partial\chi/\partial\eta$ on $\xi = -\eta$.

Instead of trying to solve this problem directly, let us determine a function which satisfies (3) over the area of the square bounded by $\xi = \pm \pi$, $\eta = \pm \pi$, and which vanishes on the sides of this square. The determination of such a function is known to be unique, and from considerations of symmetry we see that over the triangle L'SL it will be the function we require.

Now

$$4(\pi^2 + \xi^2)(\pi^2 - \eta^2) - \sum_{n=0}^{\infty} A_n \cosh(n+\tfrac{1}{2})\xi \cos(n+\tfrac{1}{2})\eta, \quad (4)$$

where A_n is a constant, satisfies (3) and vanishes over $\eta = \pm \pi$. We shall see that we can choose the constants A_n so that it will vanish on $\xi = \pm \pi$.

From Fourier's theorem, or otherwise, we have

$$\pi^2 - \eta^2 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\tfrac{1}{2})^3} \cos(n+\tfrac{1}{2})\eta,$$

for $-\pi \leq \eta \leq \pi$, by which we see that if we take

$$A_n \cosh(n+\tfrac{1}{2})\pi = 32\pi \frac{(-1)^n}{(n+\tfrac{1}{2})^3},$$

(4) will satisfy all the conditions for χ.

Thus,

$$\chi = 4(\pi^2 + \xi^2)(\pi^2 - \eta^2)$$
$$- 32\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\tfrac{1}{2})^3} \frac{\cosh(n+\tfrac{1}{2})\xi}{\cosh(n+\tfrac{1}{2})\pi} \cos(n+\tfrac{1}{2})\eta. \quad (5)$$

The value of χ on SL, which gives the velocity on the free surface, is obtained by putting $\eta = \xi$ in (5). Doing this, we obtain

$$4(\pi^4 - \xi^4) - 32\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\tfrac{1}{2})^3} \frac{\cosh(n+\tfrac{1}{2})\xi}{\cosh(n+\tfrac{1}{2})\pi} \cos(n+\tfrac{1}{2})\xi, \quad (6)$$

in which ξ is connected with the distance r from the focus, by $\xi = r^{\frac{1}{2}}/\sqrt{2}$.

For the flux of liquid through the channel we require the function

$$F = 4 \iint \chi(\xi^2 + \eta^2)\, d\xi\, d\eta,$$

taken over the area of the triangle L'SL (fig. 2), or, again from symmetry, taken over the area of the square SL. The integration is straightforward, the series for χ being uniformly convergent over the area, and we obtain

$$\frac{F}{128\pi^2} = \frac{\pi^6}{15} + 2\sum_{n=0}^{\infty} \frac{1}{(n+\tfrac{1}{2})^6} - 2\pi \sum_{n=0}^{\infty} \frac{1}{(n+\tfrac{1}{2})^5} \tanh(n+\tfrac{1}{2})\pi,$$

or, since

$$\sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^6} = \frac{\pi^6}{15},$$
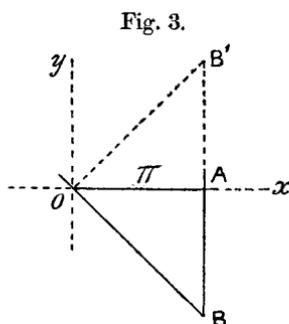
$$\frac{F}{128\pi^2} = \frac{\pi^6}{5} - 2\pi \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^5} \tanh (n+\tfrac{1}{2})\pi.$$

If now we take the latus-rectum to be $4a$ instead of $4\pi^2$, F will be multiplied by $(a/\pi^2)^4$, so that

$$\frac{F}{a^4} = \frac{128}{5} - \frac{256}{\pi^5} \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^5} \tanh (n+\tfrac{1}{2})\pi. \quad . \quad . \quad (7)$$

Particular Triangular Section.

3. The section is that in which one side of the channel is vertical and the other inclined at an angle $\frac{1}{4}\pi$ to it (fig. 3).

Fig. 3.

The solution for this case can be derived from an expression previously given[*], but it is just as easy to verify directly that all the conditions of the problem are satisfied by

$$\chi = (x+y)(\pi - x) - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^3 \sinh (2n+1)\pi}$$

$$\times \{\sinh (n+\tfrac{1}{2})(2\pi - x + y) \sin (n+\tfrac{1}{2})(x+y)$$

$$-\sinh (n+\tfrac{1}{2})(x+y) \sin (n+\tfrac{1}{2})(x-y)\}, \quad . \quad . \quad . \quad (8)$$

the axes being as shown in fig. 3. The boundary conditions are that $\chi = 0$ on $x = \pi$ and on $x = -y$, and that $\partial\chi/\partial y = 0$ on $y = 0$; but instead of the latter we take $\chi = 0$ on $x = y$, again appealing to symmetry. We have taken $OA = \pi$ for convenience.

* Lond. Math. Soc., Records for March 13th, 1913.

The value of χ on OA, which gives the velocity at the free surface, is

$$x(\pi - x) - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\sinh (n+\tfrac{1}{2})(\pi - x) \sin (n+\tfrac{1}{2})x}{(n+\tfrac{1}{2})^3 \sinh (n+\tfrac{1}{2})\pi}, \quad (9)$$

and the function F, when OA is taken to be a instead of π, is given by

$$\frac{F}{a^4} = \frac{1}{12} - \frac{1}{2\pi^5} \sum_{n=0}^{\infty} \frac{1}{(n+\tfrac{1}{2})^5} \coth (n+\tfrac{1}{2})\pi. \quad (10)$$

Remarks on the Expansions.

4. The normal derivatives of (5) and (8) must vanish over SL and OA respectively. For (5) this gives us

$$\xi = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\tfrac{1}{2})^2} \frac{1}{\cosh (n+\tfrac{1}{2})\pi}$$
$$\times \{\cosh (n+\tfrac{1}{2})\xi \sin (n+\tfrac{1}{2})\xi + \sinh (n+\tfrac{1}{2})\xi \cos (n+\tfrac{1}{2})\xi\}, \quad (11)$$

for $-\pi \le \xi \le \pi$, while for (8) it gives us an expansion which is easily transformed into (11).

Again, alternative forms can be obtained for (5) and (8), and on equating them respectively to the above forms, identities are obtained. Identities of this nature were noticed by Sir G. Stokes *, and remarked upon by Thomson and Tait †. Those mentioned by these authors were examined by F. Purser ‡, who pointed out their connexion with Elliptic Functions.

Two additional remarks, however, seem worth making.

The first is that the identities can be easily obtained by taking a two-dimensional harmonic in algebraic Cartesian form, and then finding a series of two-dimensional harmonics in normal Cartesian forms (*i. e.* in terms of trigonometric and hyperbolic functions), which satisfies the same conditions at a certain boundary.

The second is that when the expansions of conjugate functions are combined to form a series of functions of a complex variable, the resulting forms appear to be interesting.

For example, we can thus obtain the following expansions, valid over a *square* whose corners are given by

* Math. and Phys. Papers, vol. i. p. 190 (1846).
† Natural Philosophy, part ii. p. 249, 1883 ed.
‡ Messenger of Math. vol. xi. (1882).

$$\frac{z}{\pi} = 1+i, \quad -1+i, \quad -1-i, \quad 1-i:$$

$$z = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})^2 \cosh (n+\frac{1}{2})\pi} \left\{ \sinh (n+\tfrac{1}{2})z + \sin (n+\tfrac{1}{2})z \right\}, \quad (12)$$

$$z^2 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})^3 \cosh (n+\frac{1}{2})\pi} \left\{ \cosh (n+\tfrac{1}{2})z - \cos (n+\tfrac{1}{2})z \right\}, \quad (13)$$

$$z^3 = \frac{12}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})^4 \cosh (n+\frac{1}{2})\pi} \left\{ \sinh (n+\tfrac{1}{2})z - \sin (n+\tfrac{1}{2})z \right\}, \quad (14)$$

$$z^4 + 4\pi^4 = \frac{48}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})^5 \cosh (n+\frac{1}{2})\pi} \left\{ \cosh (n+\tfrac{1}{2})z + \cos (n+\tfrac{1}{2})z \right\},$$

and so on.   . . . (15)

If in (12) we write $z = \xi + i\xi$ and then take the real part, ξ being real, we reproduce the expansion (11).

Numerical Values.

5. The series (6), (7), and (9) have been examined numerically by Mr. J. K. Maddrell, of Liverpool University, who has very kindly supplied the following results :—

$\frac{32}{\pi^2} r.$	$\chi_1.$	$\frac{8}{\pi} x.$	$\chi_2.$
0......	17·41374	0......	0·000000
1......	17·40685	1......	0·208361
4......	17·30824	2......	0·546561
9......	16·89301	3......	0·870728
16......	15·78149	4......	1·096033
25......	13·52752	5......	1·163877
36......	9·73117	6......	1·028783
49......	4·86775	7......	0·650613
64......	0·00000	8......	0·000000

Here r and χ_1 refer to (6), while x and χ_2 refer to (9). The results are shown graphically in figs. 4 and 5.

The *maximum* value of χ_2 is found to be about 1·1656, its position being given by $x/\pi = ·60819$. This maximum has an interest in connexion with several other physical problems which are mentioned in the note referred to in § 3.

The series for F/a^4 in (7) and (10) have the respective values ·9293, ·02610.

In connexion with a mathematically related problem Saint-Venant * pointed out that if we write

$$2F = \kappa A^4/I,$$

where A is the sectional area of the pipe formed by the sides

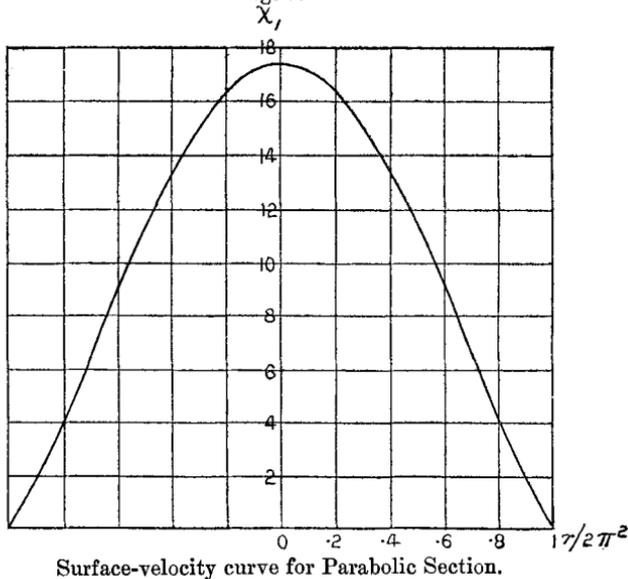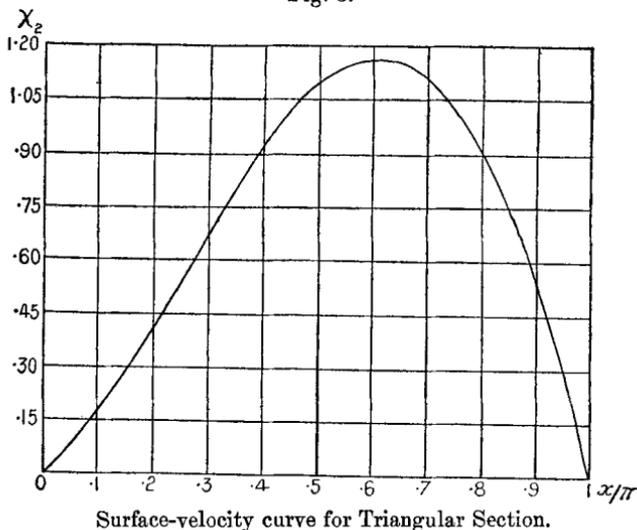* *Comptes Rendus*, t. lxxxviii. pp. 142–147 (1879).

Fig. 4.

Surface-velocity curve for Parabolic Section.

Fig. 5.

Surface-velocity curve for Triangular Section.

of the channel and their reflexion in the free surface, and I is the moment of inertia about its centroid of this section, then for the majority of simple cases κ has a value which lies between ·0228 and ·0260.   In the two present instances we have respectively

$$\kappa = ·0252, \quad \kappa = ·0232.$$