$f(s, z) = 0$, there are $p+1$ cross-lines such that if one be taken to represent the rim, and the rest holes, of a flat plate, the surface may be dissected into one on which no irreducible contour is possible by the following process:—Cut the surface along curves a each of which goes round one of the cross-lines taken to represent holes, on one of the sheets of the surface which cross at that line. Connect each of these lines with the one taken to represent the rim by a cut b along a closed curve which crosses each of the two cross-lines once. Then connect the systems (ab) chainwise by $p-1$ cuts c.

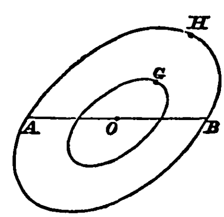
Geometrical Theorems relating to Mean Values.

By Prof. Crofton, F.R.S.

[*Read June 14th*, 1877.]

The following theorems are interesting as examples of the employment of the Theory of Probability to establish mathematical results:—

I. In any convex plane figure *which has a centre* O, let a straight line AOB revolve round O, and trace the locus of the centre of gravity G of either half into which it divides the area; let P be the perimeter of this locus; then $\frac{1}{4}$P *will be equal to the mean distance of the centre* O *from all points in the given area.*



For, let us consider the probability that, if a point be taken at random in the area, and also a straight line cross it at random,* the latter shall pass between the point and the centre O. This may be done in two ways:—

(1) Let the point X be fixed; the chance of the random line pass-

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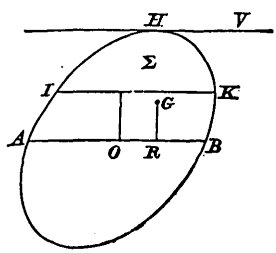
* By this is meant that an infinity of lines are drawn at random in the plane, and one is taken at random from among those which meet the area. This definition is equivalent to that stated by M. Bertrand:—The given area is supposed to be thrown at random on a floor ruled with parallel lines, so distant that it cannot fall on more than one; then the series of chords found by marking on the area the portion of one of the lines within it (whenever it does fall on a line), on each trial out of an infinite number of trials, will be the system of random chords one of which is chosen. (See Bertrand, *Calc. Int.*, p. 487; also *Phil. Trans.*, 1868.)

ing between X and O is $\frac{2\text{OX}}{\text{L}}$,* L being the perimeter of the figure. Hence, if X ranges all over the area, the required chance is

$$p = \frac{2}{\text{L}} \times \text{mean value of OX} = \frac{2\text{M}}{\text{L}}.$$

(2) Let IK be a fixed position for the random line; the chance of the point X falling on the side away from O is $\frac{\Sigma}{\Omega}$, where Σ is the segment IHK, Ω the given area. Hence the chance is, when the line occupies any of its possible positions,



$$p = \frac{1}{\Omega} \times \text{mean value of } \Sigma,$$

for all positions of IK.

Hence $$\frac{2\text{M}}{\text{L}} = \frac{1}{\Omega}\text{M}(\Sigma).$$

Now if p, ω be the perpendicular from O on IK, and its inclination to a given axis $$\text{M}(\Sigma) = \frac{\iint \Sigma\, dp\, d\omega}{\text{L}},\dagger$$

the integration extending to all positions of the chord IK; Σ always standing for the area of that segment which does not contain O.

To find the integral, let ω be constant, then we have to find $\int \Sigma\, dp$ for all positions of IK between a tangent HV and a diameter AB, both parallel to it; put C = chord IK,

$$\int \Sigma\, dp = p\Sigma - \int p\, d\Sigma,$$

also $$d\Sigma = -\text{C}\, dp.$$

Hence, between the above limits,

$$\int \Sigma\, dp = \int p\text{C}\, dp = \tfrac{1}{2}\Omega p_1,$$

where $p_1 = \text{GR}$, the distance from AB of the centre of gravity of AHB.

Hence $$\text{M} = \frac{1}{2\Omega}\iint \Sigma\, dp\, d\omega = \tfrac{1}{4}\int_0^{2\pi} p_1\, d\omega.$$

Now, as the diameter AB revolves, the tangent at G to the locus of

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* It is shewn (*Phil. Trans.*, 1868, p. 185; see also Williamson's *Integral Calculus*, 2nd edition, p. 329) that, if a plane is covered by an infinite number of random lines, the number of lines meeting any convex boundary is measured by L, the perimeter.

† See Williamson, *ub. sup.*

G is always parallel to AB, since, for an infinitesimal rotation of AB, the displacement of G is parallel to the line joining the centre of gravity of the infinitesimal triangle *cut off* from AHB, with that of the equal triangle *added on* to AHB; hence p_1 = perpendicular from O on that tangent, ω = inclination of p_1; hence, by Legendre's theorem, on rectification, $\int_0^{2\pi} p_1\, d\omega = P$, the perimeter of that locus;

$$\therefore \quad M = \tfrac{1}{4}P.$$

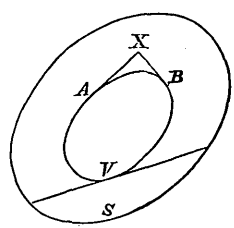
Another proof of this result may be obtained by first proving the following theorem:—

If ρ be the radius of curvature at G of the locus of G,

$$\rho = \frac{AB^3}{6\Omega}.$$

This result, which easily follows from considering the displacement of G for a small rotation of AB, gives us the *intrinsic* equation of the locus of G, if AB is known as a function of ω.

II. Let there be any two convex boundaries so related that a tangent at any point V to the inner, cuts off a constant segment S from the outer (*e. g.*, two concentric similar ellipses); let the annular area between them be called A; from a point X taken at random on this annulus draw tangents XA, XB to the inner. Find the mean value of the arc AB.



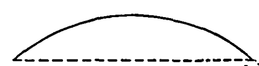
We shall find $M(AB) = L\dfrac{S}{A}$,

L being the whole length of the inner curve ABV.

We will first prove the following lemma:—

If there be any convex arc AB, and if N_1 be (the measure of) the number of random lines which meet it *once*, N_2 the number which meet it *twice*,

$$2 \text{ arc } AB = N_1 + 2N_2.$$



For draw the chord AB, the number of lines meeting the convex figure so formed is

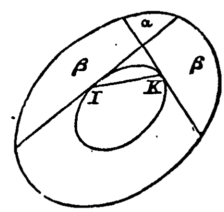
$$N_1 + N_2 = \text{arc} + \text{chord (the perimeter)};$$

but N_1 = number of lines meeting the chord = 2 . chord,

$$\therefore \quad 2 \text{ arc} + N_1 = 2N_1 + 2N_2, \quad \therefore \quad 2 \text{ arc} = N_1 + 2N_2.$$

Now fix the point X, and draw XA, XB. If a random line cross the boundary L, and p_1 be the probability that it meets the arc AB once, p_2 that it does so twice,

$$\frac{2AB}{L} = p_1 + 2p_2;$$



and if the point X range all over the annulus, and p_1, p_2 are the same probabilities for all positions of X,

$$\frac{2}{L} M(AB) = p_1 + 2p_2.$$

Let now IK be any position of the random line; drawing tangents at I, K, it is easy to see that it will cut the arc AB *twice* when X is in the space marked α, and *once* when X is in either space marked β; hence, for this position of the line,

$$p_1 + 2p_2 = \frac{2\alpha + 2\beta}{A} = \frac{2S}{A}, \text{ which is constant; hence } \frac{M(AB)}{L} = \frac{S}{A}.$$

Hence the mean value of the arc is the same fraction of the perimeter that the constant area S is of the annulus.

If L be not related as above to the outer boundary,

$$\frac{M(AB)}{L} = \frac{M(S)}{A},$$

M(S) being the mean area of the segment cut off by a tangent at a random point on the perimeter L.

The above result may be expressed as an integral. If s be the arc AB included by tangents from any point (x, y) on the annulus,

$$\iint s\, dx\, dy = LS.$$

It has been shewn (*Phil. Trans.*, 1868, p. 191) that, if θ be the angle between the tangents XA, XB,

$$\iint \theta\, dx\, dy = \pi (A - 2S).$$

The mean value of the tangent XA or XB may be shewn to be

$$M(XA) = \frac{S}{2A} P,$$

where P = perimeter of locus of centre of gravity of the segment S. Theorem I. is a case of this.

III. If C be the length of a chord crossing any convex area Ω; Σ, Σ' the areas of the two segments into which it divides the area; and p, ω be the *coordinates* of C, viz., the perpendicular on C from any fixed pole, and the angle made by p with any fixed axis; then

$$\iint C^4\, dp\, d\omega = 6 \iint \Sigma\Sigma'\, dp\, d\omega,$$

both integrations extending to all possible values of p, ω which give a line meeting the area.

This identity will follow by proving that, if ρ be the distance between two points taken at random in the area, the mean value of ρ will be

$$M(\rho) = \frac{1}{\Omega^2} \iint \Sigma\Sigma' dp\, d\omega \quad \text{.......................} (1),$$

and also

$$M(\rho) = \frac{1}{6\Omega^2} \iint C^4 dp\, d\omega \quad \text{.......................} (2).$$

The first follows by considering that, if a random line crosses the area, the chance of its passing between the two points is $\frac{2}{L} M(\rho)$, L being the perimeter of Ω. Again, for any given position of the random line C, the chance of the two points lying on opposite sides of it is $\frac{2\Sigma\Sigma'}{\Omega^2}$; therefore, for all positions of C, the chance is $\frac{2}{\Omega^2} M(\Sigma\Sigma')$; but the mean value $M(\Sigma\Sigma')$, for all positions of the chord, is

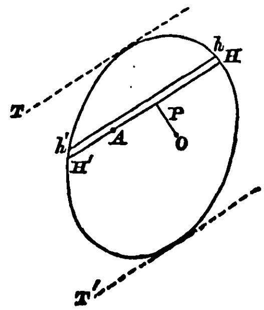
$$M(\Sigma\Sigma') = \frac{\iint \Sigma\Sigma' dp\, d\omega}{\iint dp\, d\omega} = \frac{1}{L} \iint \Sigma\Sigma' dp\, d\omega.$$

To prove equation (2), we remark that the mean value of ρ is found by supposing each of the points A, B to occupy in succession every possible position in the area, and dividing the sum of their distances in each case by the whole number of cases, the measure of which number is Ω^2. Confining our attention to the cases in which the inclination of the distance AB to some fixed direction *lies between* θ *and* $\theta + d\theta$, let the position of A be fixed, and draw through it a chord $HH' = C$, at the inclination θ: the sum of the cases found by giving B all its positions is

$$d\theta \int_0^r \rho \,.\, \rho\, d\rho + d\theta \int_0^{r'} \rho \,.\, \rho\, d\rho = \tfrac{1}{3}(r^3 + r'^3)\, d\theta,$$

where $r = AH$, $r' = AH'$. Now let A occupy successively all positions between HH′ and hh', a chord parallel to it at a distance $= dp$; the sum of all the cases so given will be

$$\tfrac{1}{3} d\theta\, dp \int_0^C (r^3 + r'^3)\, dr = \tfrac{1}{3} d\theta\, dp\, \tfrac{1}{4} 2C^4$$
$$= \tfrac{1}{6} d\theta\, dp\, C^4.$$



Now if A moves over the whole area, the sum of the cases will be

$$\tfrac{1}{6} d\theta \int C^4\, dp,$$

where p = perpendicular on C from any fixed pole O, and the integration extends to all parallel positions of C between two tangents T, T′ to the boundary, the inclination of which is θ. Removing now the restriction as to the direction of the distance AB, and giving it all values from 0 to π, the sum of all the cases is

$$\tfrac{1}{8}\int_0^\pi d\theta \int C^4\, dp\,;$$

or, if $\omega =$ inclination of p, $d\omega = d\theta$, and the sum is

$$\tfrac{1}{8}\iint C^4\, dp\, d\omega.$$

The mean value of the *reciprocal* of the distance AB of two points taken at random in a convex area is easily shewn to be

$$M\left(\frac{1}{\rho}\right) = \frac{1}{\Omega^2}\iint C^3\, dp\, d\omega.$$

Thus, for a circle,
$$M\left(\frac{1}{\rho}\right) = \frac{16}{3\pi r}.$$

It may also be shewn that the mean area of the triangle formed by taking three points A, B, C within any convex area is

$$M\,(ABC) = \Omega - \frac{1}{\Omega^3}\iint C^3\, \Sigma^2\, dp\, d\omega.$$

APPENDIX.

PROF. H. J. S. Smith, in his Presidential Address, frequently refers to the writings of the great German mathematician and physicist, Gauss. The centenary of his birth was celebrated on the 30th April, 1877, at Brunswick. A notice of his life and writings was printed in "Nature" (Vol. xv., No. 390, April 19th), much of the interest of which was due to an analysis of Gauss's powers as a mathematician by Prof. Smith. A list of a few works which were published in connexion with the celebration is given in "Nature" (June 14th, 1877).

Mr. Frankland's paper (p. 57) was printed in "Nature" (April 12th, 1877, No. 389). A letter from Mr. C. J. Monro, questioning some of