

ON THE UNIQUE EXPRESSION OF BINARY AND TERNARY FORMS

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The present paper furnishes simple proofs of the two following theorems :—

I. If $x_1 + x_2 + x_3 = 0$, any rational integral homogeneous function of degree n in any two of x_1, x_2, x_3 is a linear function of terms each involving two of x_1, x_2, x_3 , and in each term one of the two letters involved has an index $\geq \frac{2}{3}n$ (Jordan).

II. If $x_1 + x_2 + x_3 + x_4 = 0$, any rational integral homogeneous function of degree n in any three of x_1, x_2, x_3, x_4 is a linear function of terms each involving three of x_1, x_2, x_3, x_4 , and in each term one of the three letters involved has an index $\geq \frac{1}{2}n$.

The first theorem is due to Camille Jordan* : his proof depends on the evaluation of a determinant whose elements are binomial coefficients. The second theorem is new, and to prove it in the same manner as Jordan proved the first would require the evaluation of a determinant whose elements are trinomial coefficients. Another proof of the first theorem has been given by Stroht, and an extension of his method is open to the same objections. Neither Jordan's nor Stroh's proof is inductive.

Mr. J. H. Grace† has given an inductive proof of a more general result, from which Stroh's form of the first theorem may be deduced as a special case. This result is :—

Any binary form of order n can be expressed uniquely in the form $a_x^{n-\alpha} A + b_x^{n-\beta} B + c_x^{n-\gamma} C + \dots$, where a_x, b_x, c_x, \dots are r distinct linear forms, A, B, C, \dots are of orders $\alpha, \beta, \gamma, \dots$ respectively, and

$$\alpha + \beta + \gamma + \dots = n - r + 1.$$

* *Liouville's Journal*, 1876.

† *Math. Ann.*, Bd. xxxi.

‡ *Algebra of Invariants*, Grace and Young, Camb. Univ. Press, 1903.

The present paper consists of an essential modification of this proof with an obvious extension to the case of four variables—the proof of the second theorem being simpler than that of the first.

Just as the Jordan lemma owes its importance to its application to the fundamental identity $(bc)a_x + (ca)b_x + (ab)c_x \equiv 0$, arising in the symbolical treatment of covariants of binary forms (especially in perpetuants* where this identity may be used freely), so it seems probable that the second theorem may be of value in the symbolical treatment of ternary forms, where the fundamental identity is

$$(bcd)a_x + (cad)b_x + (abd)c_x \equiv (abc)d_x.$$

It will be seen that the method of this paper does not admit of direct application to the case of five or more variables.

I. The Jordan Lemma.

1. We shall first establish the result :—

If λ_x, μ_x, ν_x be three distinct linear binary forms, we cannot find binary quantities p_x, q_x, r_x satisfying any one of the three following relations for any value of m :—

total order $3m+2$,

$$\lambda_x^{2m+2} p_x^m + \mu_x^{2m+2} q_x^m + \nu_x^{2m+2} r_x^m \equiv 0, \quad (\text{i.})$$

total order $3m+1$,

$$\lambda_x^{2m+2} p_x^{m-1} + \mu_x^{2m+1} q_x^m + \nu_x^{2m+1} r_x^m \equiv 0, \quad (\text{ii.})$$

total order $3m$,

$$\lambda_x^{2m+1} p_x^{m-1} + \mu_x^{2m+1} q_x^{m-1} + \nu_x^{2m} r_x^m \equiv 0. \quad (\text{iii.})$$

The proof will be inductive : assuming that the relation of total order n requires each of the quantities involved to be identically zero, then the same will be proved true for the relation of total order $(n+1)$. Three cases must be considered, according as n is of the form $3m+1$, or $3m$, or $3m-1$.

(i.) Operate on (i.) with $(\lambda_1 \partial / \partial x_2 - \lambda_2 \partial / \partial x_1)$: the resulting relation is of the form

$$\lambda_x^{2m+2} p_x^{m-1} + \mu_x^{2m+1} q_x^m + \nu_x^{2m+1} r_x^m \equiv 0$$

and therefore by hypothesis $p_x^{m-1} \equiv q_x^m \equiv r_x^m \equiv 0$. Now q_x^m requires $(\lambda_2 \partial / \partial x_1 - \lambda_1 \partial / \partial x_2) [\mu_x^{2m+2} q_x^m] \equiv 0$, and therefore $q_x^m \equiv 0$, since λ_x and μ_x are distinct ; and similarly $r_x^m \equiv 0$ requires $r_x^m \equiv 0$, since λ_x and ν_x are distinct : but, if $q_x^m \equiv r_x^m \equiv 0$ in the relation (i.), then $p_x^m \equiv 0$ also.

* Grace : *Proc. London Math. Soc.*, Vol. xxxv.

(ii.) Now operate on (ii.) with $(\mu_2 \partial / \partial x_1 - \mu_1 \partial / \partial x_2)$; we get a relation of the same form as (iii.), and the quantics in this relation are by hypothesis identically zero. We may show in the same way that the quantics p_x^{m-1} , r_x^m in (ii.) are each identically zero, and therefore $q_x^m \equiv 0$ also.

(iii.) Finally operate on (iii.) with $(\nu_2 \partial / \partial x_1 - \nu_1 \partial / \partial x_2)$: it gives a relation of the same form as (i.) when $(m-1)$ is written in that relation for m —in other words, it leads to a relation of total order $3m-1$, wherein by hypothesis each quantic involved is identically zero. As before, this requires that each of the quantics p_x^{m-1} , q_x^{m-1} of (iii.) is identically zero, and therefore $r_x^m \equiv 0$ also.

Hence for all values of n , if the relation of total order n is impossible, so also is the relation of total order $(n+1)$, and it is therefore sufficient to show that the relation of total order unity requires each of the quantics (constants) involved to be identically zero. (Where a quantic of negative order appears the corresponding term is of course absent.)

Putting $m = 0$ in (ii.), we have $(\mu_1 x_1 + \mu_2 x_2) Q + (\nu_1 x_1 + \nu_2 x_2) R \equiv 0$, and therefore $\mu_1 Q + \nu_1 R = 0$, $\mu_2 Q + \nu_2 R = 0$, and hence $Q \equiv R \equiv 0$, since $\mu_1 / \mu_2 \neq \nu_1 / \nu_2$.

The result is therefore universally true.

2. From this result we can deduce the theorem:—

Any binary form F_x^n of order n in any two of the variables x_1, x_2, x_3 is *uniquely* expressible in one of the following ways:—

$$n = 3m+2: \quad F_x^{3m+2} \equiv y_1^{2m+2} a_{1y}^m + y_2^{2m+2} a_{2y}^m + y_3^{2m+2} a_{3y}^m, \quad (\text{iv.})$$

$$n = 3m+1: \quad F_x^{3m+1} \equiv y_1^{2m+2} a_{1y}^{m-1} + y_2^{2m+1} a_{2y}^m + y_3^{2m+1} a_{3y}^m, \quad (\text{v.})$$

$$n = 3m: \quad F_x^{3m} \equiv y_1^{2m+1} a_{1y}^{m-1} + y_2^{2m+1} a_{2y}^{m-1} + y_3^{2m} a_{3y}^m, \quad (\text{vi.})$$

where y_1, y_2, y_3 are x_1, x_2, x_3 in some definite order and

$$y_1 + y_2 + y_3 \equiv x_1 + x_2 + x_3 \equiv 0.$$

There is a single expression for every way of taking y_1, y_2, y_3 as x_1, x_2, x_3 .

For any binary form of order n may be expressed linearly in terms of $(n+1)$ linearly independent binary forms of the same order, and the number of arbitrary constants on the dexter sides of (iv.), (v.), (vi.) are $3m+3, 3m+2, 3m+1$ respectively: moreover, putting $\lambda_x \equiv y_1$, $\mu_x \equiv y_2$, $\nu_x \equiv y_3$, it follows from the preceding theorem that the forms on the dexter

sides of (iv.), (v.), (vi.) respectively are linearly independent, and that the expression is in each case unique; for otherwise there would arise relations of the forms (i.), (ii.), (iii.) respectively which have been proved to be impossible.

Now suppose a_{1y} to be written in y_1y_2 , a_{2y} in y_2y_3 , and a_{3y} in y_3y_1 respectively; then on replacing the y 's by the x 's in any manner we obtain the theorem I. (the Jordan lemma on covariants of degree 3).

II. Analogue of the Jordan Lemma for Four Variables.

3. We shall first show that:—

We cannot find ternary quantics p_x, q_x, r_x, s_x satisfying either of the following relations for any value of m :—

total order $2m+1$:

$$x_1^{m+1} p_x^m + x_2^{m+1} q_x^m + x_3^{m+1} r_x^m + (x_1 + x_2 + x_3)^{m+2} s_x^{m-1} \equiv 0, \quad (\text{i.})$$

total order $2m$:

$$x_1^{m+1} p_x^{m-1} + x_2^{m+1} q_x^{m-1} + x_3^{m+1} r_x^m + (x_1 + x_2 + x_3)^{m+1} s_x^{m-1} \equiv 0. \quad (\text{ii.})$$

As before, the proof is inductive, assuming that such a relation of total order n requires each of the quantics involved to be identically zero; then the same will be proved to be true of the relation of total order $(n+1)$: two cases must be considered, according as n is even or odd.

(i.) Operate on (i.) with $\partial/\partial x_3$: the resulting identity is of the form

$$x_1^{m+1} p_x^{m-1} + x_2^{m+1} q_x^{m-1} + x_3^m r_x^m + (x_1 + x_2 + x_3)^{m+1} s_x^{m-1} \equiv 0,$$

and therefore, by hypothesis,

$$p_x^{m-1} \equiv q_x^{m-1} \equiv r_x^m \equiv s_x^{m-1} \equiv 0.$$

Now $r_x^m \equiv 0$ implies $\frac{\partial}{\partial x_3} (x_3^{m+1} r_x^m) \equiv 0$, and therefore $r_x^m \equiv 0$; and

$s_x^{m-1} \equiv 0$ implies $\frac{\partial}{\partial x_3} \{(x_1 + x_2 + x_3)^{m+2} s_x^{m-1}\} \equiv 0$, and so $s_x^{m-1} \equiv 0$; further

$\frac{\partial}{\partial x_3} (p_x^m) \equiv 0$ and $\frac{\partial}{\partial x_3} (q_x^m) \equiv 0$, and therefore p_x^m, q_x^m , are each either zero or independent of x_3 .

But, since $r_x^m \equiv s_x^{m-1} \equiv 0$, if either of p_x^m, q_x^m is identically zero, so is the other; and, if both are independent of x_3 , the relation (i.) is

$$x_1^{m+1} p_x^m + x_2^{m+1} q_x^m \equiv 0,$$

wherein the variable x_3 does not occur: this also requires $p_x^m \equiv q_x^m \equiv 0$; for p_x^m, q_x^m cannot have factors x_2^{m+1}, x_1^{m+1} respectively.

(ii.) Operate on (ii.) with $(\partial/\partial x_1 - \partial/\partial x_2)$: we get a relation of the form

$$x_1^m p_x^{m-1} + x_2^m q_x^{m-1} + x_3^m r_x^{m-1} + (x_1 + x_2 + x_3)^{m+1} s_x^{m-2} \equiv 0,$$

and this of the same form as (i.) with $(m-1)$ in place of m .

So, by hypothesis, $p_x^{m-1} \equiv q_x^{m-1} \equiv r_x^{m-1} \equiv s_x^{m-2} \equiv 0$; and in the same way this requires that $p_x^m \equiv q_x^m \equiv 0$, and each of r_x^m, s_x^{m-1} in the relation (ii.) is either zero or involves x_1 or x_2 only in the connection $x_1 + x_2$. If either of r_x^m, s_x^{m-1} is identically zero, so also is the other: otherwise we have a relation like

$$x_3^m r_x^m + \{(x_1 + x_2) + x_3\}^{m+1} s_x^{m-1} \equiv 0,$$

where r_x^m, s_x^{m-1} are binary quantics in the two variables $(x_1 + x_2)$ and x_3 ; but this also requires $r_x^m \equiv s_x^{m-1} \equiv 0$; for r_x^m, s_x^{m-1} cannot have factors $(x_1 + x_2 + x_3)^{m+1}$ and x_3^m respectively.

Hence for all values of n , if the relation of total order n is impossible, so also is the relation of total order $(n+1)$, and we need, therefore, only consider the relation of order unity, where the quantics involved are constants.

Putting $m = 0$ in (i.), we have

$$x_1 P + x_2 Q + x_3 R \equiv 0 \quad \text{and therefore} \quad P \equiv Q \equiv R \equiv 0.$$

The result is therefore true in general.

4. From this result we deduce the following theorem:—

Any ternary form F_x^m of order n in any three of the four variables x_1, x_2, x_3, x_4 is uniquely expressible in one of the following ways:—

$n = 2m + 1$:

$$F_x^{2m+1} \equiv y_1^{m+1} a_{1y}^m + y_2^{m+1} a_{2y}^m + y_3^{m+1} a_{3y}^m + y_4^{m+2} a_{4y}^{m-1}, \quad (\text{iii.})$$

$n = 2m$:

$$F_x^{2m} \equiv y_1^{m+1} a_{1y}^{m-1} + y_2^{m+1} a_{2y}^{m-1} + y_3^{m+1} a_{3y}^{m-1} + y_4^m a_{4y}^m, \quad (\text{iv.})$$

where y_1, y_2, y_3, y_4 are x_1, x_2, x_3, x_4 in some definite order, and

$$y_1 + y_3 + y_3 + y_4 \equiv x_1 + x_2 + x_3 + x_4 = 0.$$

There will be four unique expressions for every form according as we take $y_4 = x_1, x_2, x_3$ or x_4 .

For the number of arbitrary constants involved in the quantics on the dexter side of (i.) is

$$\frac{3(m-1)(m+2)}{2} + \frac{m(m+1)}{2} = \frac{(m+1)}{2}[4m+6] = \frac{(2m+2)(2m+3)}{2}$$

and the number of arbitrary constants involved in the quantics on the dexter side of (ii.) is

$$\frac{3m(m+1)}{2} + \frac{(m+1)(m+2)}{2} = \frac{m+1}{2}[4m+2] = \frac{(2m+1)(2m+2)}{2}.$$

Now any ternary quantic of order n is expressible in terms of $\frac{1}{2}\{(n+1)(n+2)\}$ linearly independent ternary quantics of the same order; and from the preceding theorem it follows that the forms on the dexter sides of (iii.), (iv.) respectively are linearly independent, and that the expression is in each case unique; for otherwise there would arise relations of the forms (i.), (ii.) respectively, which have been proved to be impossible.

Express $a_{1y}, a_{2y}, a_{3y}, a_{4y}$ in terms of $y_1y_2y_3, y_2y_3y_4, y_3y_4y_1, y_4y_1y_2$ respectively. Then every term on the dexter side of (i.) involves three letters, and in every term one of these letters has an index $\geq (m+1)$, i.e. $\geq \frac{1}{2}n$; and every term on the dexter side of (ii.) involves three letters, and in every term one of these letters has an index $\geq m$, i.e. $\geq \frac{1}{2}n$.

This proves the second theorem stated at the beginning of this paper.