ART. V.—*On the law of Double Refraction in Iceland Spar*; by CHARLES S. HASTINGS.

THE law of double refraction in uniaxial crystals, first discovered by Huyghens, was supposed for a time to be definitively established by Fresnel's deriving it from principles of molecular mechanics. It was soon recognized, however, that a fundamental hypothesis in his reasoning does not bear critical inspection; namely, that the elastic forces brought into play by distortions due to the passage of waves are the same in kind as those produced by the displacement of a single particle. In short, Fresnel assumed that the velocity of a light wave is independent of the direction of propagation and depends only upon the direction of vibration. There have been many notable efforts to get rid of this difficulty in the theory of double refraction by a general treatment. Cauchy, Mac Cullagh, Neumann and Green are those whose names are most closely connected with the interesting history of investigation in this field of mathematical physics. All of these investigations have the feature in common, that the natural interpretation of the equations makes the direction of vibration in plane polarized light lie in the plane of polarization. To adapt the solutions to the contrary assumption, which is almost certainly the only one which can be reconciled to the known phenomena of optics, requires the most artificial restrictions in the relations of the constants involved. By such forced interpretations of formulas having a large number of constants, it is possible to derive a law for double refraction, even in Iceland spar, which does not differ from Huyghens's construction by an amount discoverable by observation; but an agreement between observation and theory extorted in this way cannot be regarded as satisfactory.

Intimately bound up with this question of double refraction is the question as to whether the differing velocities of light in vacuum and in a dense medium are due to differing densities or differing rigidities. Of these two views, equally probable *à priori*, only the first can possibly be brought into agreement with the observed phenomena of reflection. But in the case of a velocity of propagation dependent on the direction of wave-motion, which is the case of double refracting media, the difficulty is to conceive of a *density* as dependent upon direction. Rankine made the ingenious suggestion that this difficulty might be avoided by assuming that the molecules of a crystalline solid move in a frictionless fluid, and thus that their effective masses might depend upon the direction of motion. The special interest of this view from our standpoint is that it led Stokes to the first careful investigation of the accuracy of Huyghens's construction.*

In these investigations Professor Stokes found that the error in the construction could hardly exceed a unit in the fourth place of decimals, which was quite sufficient to disprove Rankine's hypothesis. This study, the details of which have not been published, remains unexcelled to the present time; for the investigations since made by Abria, Glazebrook and Kohlrausch, whether by the prism method or by total reflection, do not present a closer accordance between theory and observation. The results of earlier observers, cited in most treatises on double refraction, are all of quite inferior accuracy.

Of all these investigations, Glazebrook's, given in the Trans. Roy. Soc., vol. clxxi, 1880, is the most extensive. His method was to measure the deviations produced by four different prisms, so cut from the same piece of Iceland spar that the directions of the propagation of the light varied from an angle of $-3°$ to $+94°$ to the crystalline axis, the relation of this axis for each prism to its faces being determined by reference to planes of cleavage. The observations were made with considerable accuracy, indicating a probable error in the deduced indices of refraction considerably less than fifty units in the sixth place of decimals. The reductions, however, show a systematic deviation from Huyghens's construction, varying from 100 to 200 in the sixth decimal in the three hydrogen lines observed—the wave-surface for the more refrangible ray deviating most widely. This result would be of great theoretical interest if the values derived from observation were not vitiated by an important oversight in the details of the experiment, which the author himself points out. In view of this source of error the conclusion from the investigation is, that Huyghens's construction is true within the limit of error of these observations.

Briefly, then, the state of the case is this. The law of double refraction in Iceland spar as given by Huyghens is known to be true to within about one part in ten thousand, but no reason, dependent on the theories of elasticity, can be assigned why it should be as accurate as this, or how much more accurate we may expect to find it. The labor of testing the law to the last degree of refinement possible with modern instrumental means seems well worth while; for, except its simplicity, there is no reason in the world why it should not break down just at the limit assigned by Stokes's observations. I am quite willing to admit, also, that the systematic deviations of Glazebrook's observations, so near the limit of magnitude set

* Proceedings of the Royal Society, June, 1872; quoted by Sir Wm. Thomson in his Baltimore Lectures, p. 273.

by Stokes, and so difficult to explain by any plausible hypothesis as to their cause, suggested a not too remote probability that they indicated a physical reality.

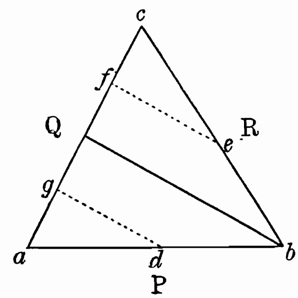
With these ends in view, all methods except those based upon prismatic refraction were practically excluded. Again, since it is impossible to get cleavage faces which admit of very accurate determinations of their angles of inclination, e. g., to within a second of arc, it seemed necessary to arrange the experiment so as to be independent of such accurate determinations. The method chosen, then, was to measure the various angles involved in an equilateral prism of Iceland spar in which one face was normal to the crystalline axis, the other two as nearly equally inclined to the axis as possible, and all three refracting edges as perfectly at right angles to the axis as practicable. Such a prism restricts the range of wave velocities which can be observed, but on the other hand, it enables us to find the direction of the crystalline axis from the observations themselves by mere considerations of symmetry, wholly independently of all assumptions of the law of double refraction.

(2) *Description of Prism.*

Since the accuracy of a determination of a refractive index depends largely on the character of the prism used, and especially in this case of extraordinary refraction, it may be worth while to describe the method employed to secure satisfactory results.

After selecting a good block of spar, a wooden model of the largest prism of desired orientation which could be obtained from the block was made. As this model represented the cleavage faces as well as the prism faces, it served as a guide as to how far any process of grinding should be carried. One of the obtuse trihedral angles was ground down, so that when the block rested upon this ground surface under a fixed telescope nearly perpendicular to it, the images of a distant object reflected by the three opposite cleavage faces could be brought to the crosswires of the telescope by merely rotating the block on the ground surface. This admitted of securing a face, P in the accompanying figure, very nearly perpendicular to the crystalline axis. The limit of accuracy was restricted only by the character of the reflection from the cleavage faces. The size of the face was determined by reference to the model. The next step was the formation of the surface, $d\,b\,e\,f\,g$ of the figure, to serve as a base for the prism and a rough guide for the other two faces of the prism. It was ground perpendicular to P, and, by a process similar to that used in fixing the direction of P, equally inclined to the cleavage planes $a\,b\,Q$

and $b\,c\,Q$. Then R was ground so that it made equal angles with the cleavage surfaces $a\,b\,Q$ and $a\,d\,g$, and an angle of 60° with P. As it was desirable to make this last angle tolerably accurate in order to eliminate all errors of the circle in a determination of the refracting angle, or, in other words, so that a repetition of the angle three times would bring the circle back to the same position within the range of the reading microscopes, the surface P was polished sufficiently to yield a good reflection, and then the angle at b was adjusted until it was equal to that of a glass prism known to be accurately 60°. Q was determined in a precisely similar way. The three surfaces were then polished to as close approximations to planes as possible. In this process most interesting differences in the physical properties of the surfaces were found, as might have been expected. R worked almost as readily as glass, except that its departure from flatness tended toward cylindrical surfaces instead of spherical. It was not difficult to make P flat, but the slightest carelessness in handling would produce tetrahedral pits in it. The surface Q, being inclined only 15° to the direction of cleavage, gave by far the most trouble, because it did not seem possible to get it very smooth by grinding. After carrying this process to its limit of accuracy, determined more, perhaps, by the extraordinary thermal properties of the material, than by purely technical difficulties in working, the faces were cut away until only circular areas were left on the three prism faces. These round faces were then modified, by methods which would only have an interest for the practical optician, until they were optically flat; that is, until their departures from their average planes was not more than a tenth of a wavelength of light. The test of flatness was the colors produced when white light was reflected nearly normally from the surface brought closely in contact with a surface of glass known to be plane. The diameters of the surfaces, in order of lettering, were:



2·8 cm., 2·8 cm. and 2.6 cm.

(3) *Spectrometer.*

The instrument with which the measures of the various angles were made has some features peculiar to it. The circle is of glass, 8 inches in diameter, and divided to single degrees, except in the case of the first degree, and three others separated from it and each other by quadrants, which are subdivided to tenths of a degree. The observing telescope may be moved independently or clamped to the circle; it is checked in its rotation only by the collimating telescope. It is obvious that by this construction it is always possible to measure an angle so that one end of the arc shall be at a degree mark and the other end fall within a subdivided degree; hence both ends of the arc are within the range of the reading microscopes. The great and manifest advantage of this construction is that every angle can be accurately measured after determining the absolute place of only 396 lines or 198 diameters.

The reading microscopes have micrometer screws of 80 threads to the inch, with heads divided into 100 parts, one revolution of the screw being equal to one minute of arc. The magnifying power is 220 diameters, doubtless unnecessarily high, but not found inconvenient, and a much lower power would have necessitated a notable change in the design, either finer micrometer screws or longer microscopes with correspondingly higher table and telescope carrier. The probable error of a single setting of the microscope was found to be less than $0''\cdot3$, or less than half a division of the micrometer head.

The errors of the circle were determined by means of two auxiliary microscopes clamped to the base-plate of the instrument at opposite sides. By bisections and trisections the absolute position was determined of each diameter at multiples of $5°$ from the initial diameter, to within a probable error of less than $1''$. As practically every such interval was involved in the observations several times, equations of condition were formed as checks upon the results; if a discrepancy as great as $1''$ was found the intervals were re-measured. A determination of any angle was thus reduced to a maximum of five repetitions, whence the true angle could be found, and, incidentally, the corrections of four other arcs. As an illustration of the precision of the method, I may state that in the only case where a suspicion of the accepted value led to a complete redetermination of all the constants involved, the correction deduced differed only $0''\cdot1$ from the former one. The origin of the suspicion was afterwards found to be a false temperature correction. This determination of the errors of the circle was the most laborious part of the whole investigation.

(4) *Angles of prism.*

The angles measured were those between the normals to the faces P and Q, Q and R, R and P, which were made with all attainable accuracy; those between the normal to P, and the normals to its three adjacent cleavage faces; the normal angle between R and the narrow cleavage face at b; and finally, the normal angles between the cleavage faces $a\,b\,Q$ and $b\,c\,Q$ respectively. The precision of all the measures involving reflection from a surface of cleavage is of course much inferior to those made upon the polished surfaces. The first group gives the refracting angles, and the others only serve to determine the direction of the crystalline axis, a datum not used in the final reduction but useful as a check on the work.

The general method of determining these angles was as follows: The telescope replaced the fixed collimator which was removed. By means of a plate of plane parallel glass and a *quasi* collimating eyepiece* the axis of the telescope was rendered strictly perpendicular to the axis of rotation of the instrument. The focal adjustment of the telescope could be made at the same time with great precision: magnifying power used, twenty diameters. Following this adjustment the glass plate was replaced by the prism, which was so adjusted that the line of collimation fell close to the center of each face when in position for observation. That this condition, a most important one, was fulfilled, was determined by removing the ocular and looking at the prism through the telescope tube.

TABLE I.—*Angles of prism* $= \alpha = 60$.

PQ			PR			QR		
Obs.	t	Red.	Obs.	t	Red..	Obs.	t	Red.
+1′·205	17·2	+1′·285	−2′·516	17·0	−2′·519	+1′·330	16·5	+1′·303
·242	17·1	·280	·492	17·0	·519	·298	16·5	·303
·273	17·0	·276	·521	17·0	·519	·254	16·6	·293
·407	19·8	·421	·521	19·55	·394	·067	19·2	·046
·421	20·0	·412	[·652]	19·65	----	1·141	19·25	·041
·432	20·0	·412	·418	19·75	·384	0·908	19·55	1·013
·391	20·0	·412	·350	20·7	·338	·895	21·0	0·875
·469	20·1	·417	·315	21·0	·323	·883	21·0	·875
·453	20·0	·412	·225	21·5	·299	·858	21·5	·828
·390	20·0	·412	·235	21·3	·308	·836	21·1	·866
475	20·1	·417	·350	21·1	·318	·842	21·1	·866
·514	20·1	·417	·354	21·1	·318	·873	21·2	·856
·470	21·0	·457	·306	21·2	·313	741	22·5	·732
·484	21·2	·466	·231	20·8	·333	·703	22·6	·723
·481	21·1	·462	·242	22·9	·230	+0·695	22·8	+0·704
·453	21·7	·489	·247	23·0	·225			
·427	21·7	·489	−2 ·247	23·0	−2 ·225			
·478	21·6	·485						
·519	21·6	·485						
·445	21·6	·485						
·435	21·8	·494						
·511	23·0	·548						
·511	23·0	·548						
+1 ·528	23·0	+1 ·548						

* This is described in the paper "On the influence of temperature on the optical constants of glass." This Jour.. III, vol. xv, p. 271.

In the case of the prism angles each was repeated three times, whence, since they were all quite close to 60°, not only were all errors of graduation eliminated, but the absolute values of the instrumental arcs 120° and 240° determined with great accuracy. The influence of temperature on the magnitudes of the angles becomes evident even in comparatively rude observations. Table I gives *all* the measures of these angles. Of course the angles given are the supplements of those directly observed; they are also corrected for circle errors. Following the column containing the observed angle is given the temperature of the prism, and then the value reduced to a temperature of 20° C. The method by which the last column was calculated will be given farther on.

The observation of PR enclosed in brackets is rejected. Two or three others might have been rejected without changing the results except to give them smaller probable errors.

In order to find the values of the angles a standard temperature (20° C.) was chosen as the standard, observation equations of the form

$$M = m + n\ (t-20),$$

whence normal equations of the form

$$\Sigma\alpha^2.\ m + \Sigma\alpha\beta.\ n - \Sigma\alpha.\ M = 0,$$
$$\Sigma\alpha\beta.\ m + \Sigma\beta^2.\ n - \Sigma\beta.\ M = 0,$$

gave the means of finding m and n. The values of the coefficients of the normal equations are as follows:

	$\Sigma\alpha^2$	$\Sigma\alpha\beta$	$\Sigma\beta^2$	$\Sigma\alpha M$	$\Sigma\beta.\ M$
PQ	24	13·7	72·67	34·514	22·647
PR	16	7·9	64·6	—37·570	–15·588
QR	15	2·4	66·42	14·324	— 3·980

The observed values of a from these equations are:

For	PQ	60° 1′. 412±0′·006+0′·0454 (t—20°)
	PR	59 57′. 628±0′·009+0′·0489 (t—20°)
	QR	60 0′. 970±0′·008—0′·0950 (t—20°)

The probable errors of a single observation of an angle were found to be 0′·028, 0′·035 and 0′·032, respectively, and the probable errors of the coefficients of the terms containing the temperature 0′·0035, 0′·0045 and 0′·0039, respectively. The probable error of 2″ for a single observation seems large, considering the refinement of the method used, and indeed it would be for a glass prism; but regarding the enormous change from temperature and the extreme difficulty of determining that of the prism, it must, I think, be regarded as satisfactory.

These constants derived directly from observation are subject to certain geometrical conditions which will modify them very slightly and reduce the probable errors. As it was found, in the course of the observations, that the normal to any one face is inclined only 12″ to the plane fixed by the other two normals, we have—

$$\Sigma\alpha = 180^\circ$$
$$n_1 + n_2 = -n_3.$$

But as observed,

$$\Sigma\alpha = 180^\circ\ 0'{\cdot}010 \pm 0'{\cdot}013.$$
$$n_1 + n_2 = -n_3 + 0'{\cdot}0007 \pm 0'{\cdot}006.$$

Adjusting the observed values in accordance with the equations of condition we have finally:

$$\alpha_{PQ} = 60^\circ\ \ 1'\ 24''{\cdot}59 \pm 0''{\cdot}29.$$
$$\alpha_{PR} = 59^\circ\ 57'\ 37''{\cdot}42 \pm 0''{\cdot}43.$$
$$\alpha_{QR} = 60^\circ\ \ 0'\ 57''{\cdot}98 \pm 0''{\cdot}39.$$
$$n_3 = -5''{\cdot}68 \pm 0''{\cdot}19.$$

The value of n_3 enables us to find at once the difference in the principal coefficients of thermal expansion, as well as the variations of the angles of the rhombohedron. By an obvious relation, if a_1 and a_2 are the coefficients in the axial direction and at right angles to it respectively, we deduce

$$a_1 - a_2 = 10^{-6}(31{\cdot} \pm 1{\cdot}).$$

The best value known is that of Fizeau, which is

$$10^{-6}(31{\cdot}6).$$

But the relations of immediate value to us are those of the temperature variations of the angles between the normals of *P* and an adjacent cleavage face, of *R* and the cleavage face *b*, and of the two faces *a b Q* and *b c Q*. They are, in the order named, if ϑ is the measured angle

$$\frac{\Delta\vartheta}{\Delta t} = +0'{\cdot}056.$$
$$= -0'{\cdot}103.$$
$$= -0'{\cdot}085.$$

(5) *Position of crystalline axis.*

The measures upon which this constant depends are subject to large errors on account of the imperfect reflections from the cleavage faces, especially from the edge *b*, which is only 1^{mm} wide and gives two images. The values given below are reduced to a temperature of 20° C.

Angle.	
Pb	$= 44^\circ\ 39'{\cdot}12 \pm 0{\cdot}50.$
P(abQ)	$= 44^\circ\ 36'{\cdot}57 \pm 0{\cdot}045.$
P(adg)	$= 44^\circ\ 37'{\cdot}05 \pm 0{\cdot}120.$
Rb	$= 75^\circ\ 25'{\cdot}00 \pm 0{\cdot}160.$
(abQ) (bcQ)	$= 105^\circ\ \ 4'{\cdot}88.$

Of these the first and fourth, giving them equal weights, yield

$$Pb = 44° \; 38'{\cdot}26$$

which, with the second and third, give

$$44° \; 37'{\cdot}29$$

as the angle between the crystalline axis and the normal to a cleavage plane. The last of the measured angles implies

$$44° \; 36'{\cdot}70$$

for the angle between the axis and the normal to a cleavage face. This value, however, rests upon two observations only and cannot therefore be regarded as of great weight. We may, perhaps, attribute to it a weight $\frac{1}{6}$ that of the value derived from the other measures, whence the accepted value becomes

$$44° \; 37'{\cdot}19.$$

This value gives, for the direction of the axis drawn from *P inward*, an inclination

$$\xi = 1' \; 4''{\cdot}1$$

from the normal to P towards Q, and

$$\eta = 0' \; 12'',$$

i. e., 12″ below the refracting plane of the prism QR; they can hardly be in error as much as 15″.

It is perhaps worth noting that the accepted value 44° 37′·19 gives 105° 5′·07 for the dihedral obtuse angle of the rhombohedron at 20° C., which is practically the value accepted by mineralogists.

(6) *Angles of deviation.*

Minimum angles of deviation were determined in each case; there are thus two angles for each prism-angle. The line pointed upon was the more refrangible component of the D line of the solar spectrum, except in the case of the extraordinary image by the faces Q R, of which the dispersion was too small to admit of easy separation, and, by mistake, in four pointings on the double deviations for the ordinary image by the same refracting angle when D_1 was observed on one side. Care was taken to adjust the collimator, telescope and prism, so that the axial ray passed through the center of the prism in both positions for minimum deviation, i. e., right and left. The lines of collimation were made at right angles to the axis of the circle and to the refracting faces by means of the plane glass plate and the collimating eyepiece. For observing the spectrum a magnifying power of 31 was employed. Table

II contains all the measures for the ordinary ray, then the temperature (t), the barometric height (Bar.), and the angle corrected to 30 inches barometric height. In the table, the mistakes mentioned, and which were confined to the four preceding the last, are corrected by adding 0′·285, the measured distance between D_1 and D_2.

TABLE II.—*Double angles of deviation for ordinary ray D_2.*

$2\Delta_0 = 104°$

PQ				PR				QR			
Obs.	t	Bar.	Cor.	Obs.	t	Bar.	Cor.	Obs.	t	Bar.	Cor.
+8′·454	20·3	29·85	+8′·423	−3′·772	19·4	30·1	−3′·752	+7·658	16·7	30·1	+7′·678
·472	20·4	29·85	·441	·701	19·8	30·1	·681	·546	17·1	30·1	·566
·497	20·7	29·85	·466	·661	19·9	30·1	·641	·531	17·2	30·1	·551
				·559	20·3	30·1	·539	·336	17·9	30·1	·356
				·560	20·2	30·1	·540	·134	18·8	30·1	·154
				·552	20·6	30·1	·532	+7′·085	19·0	30·1	·105
				·618	19·8	29·75	·669	+6′·983	19·1	30·1	+7′·003
				·479	20·3	29·75	·530	·921	19·7	30·1	+6′·941
				−3′·455	20·4	29·75	−3′·506	·914	20·1	30·1	·934
								·799	20·1	30·1	·819
								·839	20·2	30·1	·859
								·743	20·6	30·1	·700
								+6′·680	20·6	30·1	+6′·762

The observations for PR and QR were reduced by forming observation equations of the type

$$M = m + n(t - 20).$$

and, the temperature correction for PQ being assumed as the same as that for PR, the reduced values for Δ are, for

PQ	52°	4′	10″·20 ± 0″·54 + 6″·72 (t − 20)
PR	51°	58′	11″·52 ± 0″·18 + 6 ·72 (t − 20)
QR	52°	3′	26″·10 ± 0″·30 — 7″·17 (t − 20)

TABLE III.

$2\Delta_\varepsilon =$

PQ 94°				PR 94°				QR 72°			
Obs.	t	Bar.	Cor.	Obs.	t	Bar.	Cor.	Obs.	t	Bar.	Cor.
+7′·983	20·7	29·85	+7′·955	−2′·518	19·6	30·1	−2′·537	+3′·474	16·8	30·1	+3′·489
8′·045	20·7	29·85	8′·017	·555	19·7	30·1	·574	·454	16·9	30·1	·469
·101	20·8	29·85	·073	·454	19·9	30·1	·472	·456	17·4	30·1	·471
·110	20·9	29·85	·082	·430	19·9	30·1	·448	·360	17·7	30·1	·375
+8′·088	21·0	29·85	+8′·060	·436	20·5	30·1	·454	·356	18·9	30·1	·371
				·391	20·6	30·1	·409	·307	19·0	30·1	·322
				·429	20·0	29·75	·475	·331	19·2	30·1	·346
				−2′·361	20·1	29·75	−2′·407	·309	19·4	30·1	·324
								[3′·495]	20·1	30·1	----
								·294	20·4	30·1	·309
								·277	20·5	30·1	·392
								·251	20·6	30·1	+3′·266
								[3′·634]	20·6	30·1	----

Table III gives the double angles of deviation as measured for the extraordinary ray for each refracting angle. As has already been stated, the deviation is that belonging to D_2, except in the case of the edge QR, where, on account of the small dispersion, the sodium line was set upon as a single line.

As before, the observations enclosed in brackets are rejected. These were reduced in quite the same way as were the deviations for the ordinary ray, with the following resulting values for Δ_ε:

PQ	47°	3′	$58''\cdot23 \pm 0''\cdot39 + 3''\cdot60(t-20°)$
PR	46°	58′	$45''\cdot69 \pm 0''\cdot24 + 3''\cdot60(t-20°)$
QR	36°	1′	$39''\cdot21 \pm 0''\cdot21 - 1''\cdot58(t-20°)$

In order to reduce the angle of deviation for QR to what it should be for D_2, the angular distance between D_1 and D_2 for the ordinary ray was determined, and half its product by $\frac{d\Delta_\varepsilon}{d\Delta_0}$ for this region of the spectrum, was taken as an additive correction. The value of the correction was found to be $3''\cdot85$, whence the deviation for the extraordinary ray D_2 for QR becomes

$$36° \quad 1' \quad 43''\cdot06 \pm 0''\cdot21$$

(7) *Principal indices of refraction.*

The crystalline axis has been found to make an angle of less than 1′ with the plane bisecting the refracting angle QR; hence we may apply the ordinary formula connecting the index of refraction with the angles of minimum deviation and refraction, namely,

$$\mu = \frac{\sin \frac{\Delta + \alpha}{2}}{\sin \frac{\alpha}{2}}$$

The resulting indices will be the principal indices for calcite at a temperature of 20° C.

	$\mu_0 =$
for PQ	$1\cdot658393 \pm 2$
PR	$1\cdot658387 \pm 2$
QR	$1\cdot658387 \pm 2$
	$1\cdot658389 \pm 1\cdot2$

The single value of μ_ε is

$$1\cdot486450 \pm 1\cdot4$$

(8) *Test of Huyghens's Law.*

We are now in a position to test the law of extraordinary refraction from the principal indices of refraction and the observed extraordinary deviations by the refracting angles PQ and PR. First, we have the well known law,

$$\left[\frac{1}{\mu'_\varepsilon}\right]^2 = \frac{\cos^2\vartheta}{\mu^2_o} + \frac{\sin^2\vartheta}{\mu^2_\varepsilon},$$

where μ_o and μ_ε are the reciprocals of the principal wave velocities as before, and μ'_ε is the reciprocal of the velocity of the extraordinary wave whose normal makes an angle ϑ with the crystalline axis. This enables us to compute μ'_ε, knowing ϑ. Second, we have the series of relations given by Professor Stokes (British Association Report, 1862),

$$\mu'_\varepsilon = \frac{\sin\varphi}{\sin\varphi'} = \frac{\sin\psi}{\sin\psi'}$$

$$\varphi + \psi = \varDelta + \alpha$$

$$\varphi' + \psi' = \alpha$$

$$\operatorname{tg}\frac{\varphi' - \psi'}{2} = \operatorname{tg}\frac{\alpha}{2}.\operatorname{tg}\frac{\varphi - \psi}{2}.\cot\frac{\varDelta + \alpha}{2},$$

where φ ψ are the angles of incidence and emergence respectively, and φ' ψ' the angles which the wave normal makes with the faces of the prism within it. These relations enable us to derive a value for μ'_ε from the observations, perfectly independently of any assumption as to the law of double refraction if we know either φ or ψ. They afford a much readier test than that of calculating the deviations for an assumed law.

We do not, it is true, know the values of φ for the extraordinary refractions by PQ and PR, but as the prism was always set for minimum deviation it is easy to find these values, either by taking advantage of the fact that Huyghens's law is already known to be nearly true, whence the angle of incidence for minimum deviation can be calculated, or, more simply, from the relation

$$\frac{\sin\varphi}{\sin\varphi'} = \frac{\sin\psi}{\sin\psi'},$$

and the two purely geometrical equations which follow this equation above.

It is found by trial that for PQ, the light being incident on Q the value of φ which satisfies the condition is 50° 25′, and for PR and incidence on R, the value of φ is 50° 21′. A small change in these angles does not alter the difference between the observed and calculated values of μ'_ε, which affords the test of the law.

The substitution of these values in the equation of Stokes gives—

	μ'_ε
PQ	1·606114±1·6
PR	1·606103±1·6

It remains to calculate the values of μ'_ε from Huyghens's theory from the known values of φ' or ψ' and the assumed direction of the crystalline axis defined by ξ above, since η is so small that it can be regarded as zero. The measured value of ξ is $1' 4''$ with a considerable uncertainty, but I find that a value of $1' 6''$ will make the differences between observation and theory symmetrical. With this value we have

	ϑ	μ'_ε [calc.]
PQ	31° 19′ 45″·68	1·606109±1·8
PR	31 19 58 ·34	1·606099±1·8

where the probable errors are calculated without disregarding the fact that we have imposed the arbitrary condition that the differences shall be symmetrical.

The difference between a measured index of refraction in Iceland spar at an angle of 30° with the crystalline axis, and the index calculated from Huyghens's law and the measured principal indices of refraction, thus appears to be 4·5 units in the sixth place decimals, while, assuming the truth of the law we ought to expect, from the probable errors of the quantities involved, a difference of ±2·4, only about half as great. There is, however, one source of constant error in the observations which has not been alluded to, namely, the fact that the temperatures of the prism were measured by a different thermometer in the case of the angles of the prism and the angles of deviation. In the former a rather insensitive thermometer divided to single degrees and estimated to tenths was used, and in the latter a very sensitive thermometer divided to half-degrees. By reference to my notes I find that the two systems of temperatures are connected only by an eye comparison on a single day, so, although I believe that the error of comparison cannot be much over one tenth of a degree, it is by no means certain, or even improbable, that an error of this magnitude may enter. It was not thought in that stage of the investigation that such an error was of any significance. Unfortunately one of the thermometers has since been broken so that a direct comparison is out of the question. The observations of the ordinary indices contain implicitly, however, the desired correction as appears from the following reasoning:—

Let dt be the excess of the reading of the first thermometer, used in the prism-angle measures, over that of the second; then its most probable value is that which renders the probable error of the mean value of μ_o a minimum, when the three observed values are regarded as independently determined magnitudes.

Thus $$\frac{d\mu_o}{dt}=\frac{d\mu_o}{d\alpha}\cdot\frac{d\alpha}{dt}=4{\cdot}2\times5{\cdot}68 \text{ for QR}$$
$$=-4{\cdot}2\times2{\cdot}84 \text{ for PQ and PR},$$

the first differential coefficient being derived from the formula from which μ_o is calculated, and the second is given on p. 66. From these and the values of μ_o on p. 70 treated as independent determinations, we have

$$23{\cdot}9\ dt=-2$$
$$11{\cdot}9\ dt=\ \ 2$$
$$11{\cdot}9\ dt=-4,$$

whence

$$dt=-0^\circ{\cdot}084\pm0{\cdot}032.$$

From this it is obvious that such a correction is required. Supposing, then, that the angles of the prism given above correspond to a temperature of 19°·916 C. instead of 20° C. we have the following definitive values for the quantities involved:

	α	Δ_o	Δ_ε	μ_o
PQ	60° 1′ 24″·83	52° 4′ 10″·20	47° 3′ 58″·26	1·658392
PR	59 57 37·66	51 58 11·52	46 58 45·69	1·658387
QR	60 0 57·60	52 3 26·10	36 1 43·06	1·658389

whence $\mu_o=1{\cdot}658389$ μ_ε 1·486452

	μ'_ε	μ'_ε [calc.]
PQ	1·606113	1·606110
PR	1·606102	1·606100.

The conclusion is, that Huyghens's law is probably true to less than one part in five hundred thousand, and, consequently, that there is no known method by which we can hope to discover an error in it by observation alone.

New Haven, Nov., 1887.