

"Programm des Gymnasiums zu Recklinghausen," liv. Schuljahr 1883—4. ("Untersuchungen über ähnliche Punktreihen auf den Seiten eines Dreiecks und auf deren Mittelsenkrechten, sowie über Kongruente Strahlenbüschel aus den Ecken desselben;" ein Beitrag zur Geometrie des Brocardschen Kreises vom Oberlehrer A. Artzt.)

On a Subsidiary Elliptic Function $\text{pm}(u, k)$.*

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[Read May 8th, 1884.]

I think there is an advantage in using a subsidiary elliptic function, $\text{pm}(u, k)$, as I propose to write it, in conjunction with the ordinary $\text{am}(u, k)$.

The theory of transformation leads in a very remarkable way to addition-theorems as regards both these functions.

For example, in a recent communication to the Society, it was shown by me that

$$\begin{aligned} \text{am}(3u, k) = 3 \text{am}(u, k) - \text{am}(Mu, \lambda) + \text{am}(M_1u, \lambda_1) \\ + \text{am}(M_2u, \lambda_2) + \text{am}(M_3u, \lambda_3); \end{aligned}$$

where $\lambda, \lambda_1, \lambda_2, \lambda_3$ are the roots of the modular equation in the cubic transformation. This being so, it can be proved that there is the correlative addition-theorem

$$\text{pm}(3u, k) = 3 \text{pm}(u, k) - \text{pm}(M_1u, \lambda) + \dots^*$$

1. *Definition of* $\text{pm}(u, k)$.

The new function is derived from $\text{am}(u, k)$ by the equation

$$\sin \text{pm}(u, k) = k \sin \text{am}(u, k) = k \text{sn}(u, k),$$

or, say,

$$\text{sm}(u, k) = k \text{sn}(u, k).$$

Differentiating this with regard to u , we have

$$\cos \text{pm } u \frac{d}{dt} \text{pm } u = k \text{cn } u \text{dn } u,$$

or, since $\cos \text{pm } u = \text{dn } u$,

$$\frac{d}{dt} \text{pm } u = k \text{cn } u.$$

* The notation " $\text{pm}(u, k)$ " has been adopted at Prof. Cayley's suggestion. Since $\sin \text{pm}(u, k) = \sin \text{am}\left(ku, \frac{1}{k}\right)$, he thinks that $\text{pm}(u, k) = \text{am}\left(ku, \frac{1}{k}\right)$ is the best definition of $\text{pm}(u, k)$.

If $\sin \phi = k \sin \theta$, it follows at once that

$$\frac{d\phi}{\sqrt{1 - \frac{1}{k^2} \sin^2 \phi}} = k \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}};$$

hence, putting $\theta = \text{am } u$, we have

$$\sin \text{am} \left(ku, \frac{1}{k} \right) = k \sin \text{am} (u, k) = k \text{sn} (u, k) = \text{sm} (u, k),$$

where

$$u = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The definition, then, gives the following relations, viz.,

$$\left. \begin{aligned} \text{sn} \left(ku, \frac{1}{k} \right) &= \text{sm} (u, k) = k \text{sn} (u, k) \\ \text{sn} (u, k) &= \text{sm} \left(ku, \frac{1}{k} \right) \\ \text{cn} \left(ku, \frac{1}{k} \right) &= \text{cm} (u, k) = \text{dn} (u, k) \\ \text{cn} (u, k) &= \text{cm} \left(ku, \frac{1}{k} \right) = \text{dn} \left(ku, \frac{1}{k} \right) \end{aligned} \right\}.$$

Again, since

$$\int_0^{\pi-\phi} \frac{d\phi}{\sqrt{k^2 - \sin^2 \phi}} = \int_0^{\pi} \frac{d\phi}{\sqrt{k^2 - \sin^2 \phi}} - \int_0^{\phi} \frac{d\phi}{\sqrt{k^2 - \sin^2 \phi}},$$

$$\begin{aligned} \text{and } \int_0^{\pi} \frac{d\phi}{\sqrt{k^2 - \sin^2 \phi}} &= 2 \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{k^2 - \sin^2 \phi}} = 2 \int_0^{\sin^{-1}(\frac{1}{k})} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \\ &= 2 (K + iK'), \end{aligned}$$

it follows that

$$\pi - \text{pm} (u, k) = \text{pm} (2K + 2iK' - u, k).$$

It will be presently shown that there is an expression for $\text{pm} \frac{2K}{\pi} x$, in terms of q -functions, viz.,

$$\sin^{-1} \left\{ \text{sm} \frac{2K}{\pi} x \right\} = 2\aleph (-)^{m-1} \tan^{-1} \left\{ \frac{2q^{m-1} \sin x}{1 - q^{2m-1}} \right\}, \quad \left(\begin{aligned} q &= e^{-\frac{\pi K'}{K}} \\ x &= \frac{u}{2K} \end{aligned} \right).$$

One of the minor advantages of the above notation is that

$$\text{dn } u \equiv \cos \text{pm } u$$

is clearly seen to be a kind of cosine function.

2. *Application of the foregoing results to transformation formulæ.*

The first transformation is

$$\sin^{-1} \{ \operatorname{sn} (Mu, \lambda) \} = \sin^{-1} \{ \operatorname{sn} (u, k) \} \\ + 2\Sigma \tan^{-1} \left\{ \operatorname{dn} \left(\frac{4sK}{n}, k \right) \cdot \frac{\operatorname{sn} (u, k)}{\operatorname{cn} (u, k)} \right\};$$

change λ into $\frac{1}{\lambda}$, k into $\frac{1}{k}$, M into $M \frac{\lambda}{k}$, u into ku , and K into kK ,

then

$$\sin^{-1} \left\{ \operatorname{sn} \left(M\lambda u, \frac{1}{\lambda} \right) \right\} \\ = \sin^{-1} \left\{ \operatorname{sn} \left(ku, \frac{1}{k} \right) \right\} + 2\Sigma \tan^{-1} \left\{ \operatorname{dn} \left(\frac{4skK}{n}, \frac{1}{k} \right) \cdot \frac{\operatorname{sn} \left(ku, \frac{1}{k} \right)}{\operatorname{cn} \left(ku, \frac{1}{k} \right)} \right\},$$

i.e., from the above results,

$$\sin^{-1} \{ \operatorname{sm} (Mu, \lambda) \} \\ = \sin^{-1} \{ \operatorname{sm} (u, k) \} + 2\Sigma \tan^{-1} \left\{ \operatorname{cn} \left(\frac{4sK}{n}, k \right) \cdot \frac{k \operatorname{sn} (u, k)}{\operatorname{dn} (u, k)} \right\},$$

or $\operatorname{pm} (Mu, \lambda) = \operatorname{pm} (u, k) + 2\Sigma \tan^{-1} \left\{ k \operatorname{cn} \left(\frac{4sK}{n}, k \right) \cdot \frac{\operatorname{sn} (u, k)}{\operatorname{dn} (u, k)} \right\}.$

Differentiating this in regard to u , and taking

$$\frac{d}{du} \operatorname{pm} (u, k) = k \operatorname{cn} (u, k), \\ \frac{d}{du} \frac{\operatorname{sn} (u, k)}{\operatorname{dn} (u, k)} = \frac{\operatorname{cn} (u, k)}{\operatorname{dn}^2 (u, k)},$$

we have

$$M \frac{\lambda}{k} \cdot \operatorname{cn} (Mu, \lambda) = \operatorname{cn} (u, k) \left\{ 1 + 2\Sigma \frac{\operatorname{cn} \frac{4sK}{n}}{1 - k^2 \operatorname{sn}^2 \frac{4sK}{n} \operatorname{sn}^2 (u, k)} \right\},$$

which is one of the ordinary formulæ.

Similarly, the second transformation is

$$(-)^{i(n-1)} \operatorname{pm} (M_1 u, \lambda_1) \\ = \operatorname{pm} (u, k) + 2\Sigma (-)^i \tan^{-1} \left\{ k \operatorname{cn} \left(\frac{2siK'}{n}, k \right) \cdot \frac{\operatorname{sn} (u, k)}{\operatorname{dn} (u, k)} \right\},$$

where $\operatorname{pm} (u, k) = \sin^{-1} \operatorname{sn} (u, k) = \sin^{-1} \{ k \operatorname{sn} (u, k) \}$, &c.

Here change λ_1 into k and k into λ , M into N_1 , then

$$(-)^{i(n-1)} \operatorname{pm} (N_1 u, k) \\ = \operatorname{pm} (u, \lambda) + 2\Sigma (-)^i \tan^{-1} \left\{ \lambda \operatorname{cn} \left(\frac{2si\Lambda'}{n}, \lambda \right) \cdot \frac{\operatorname{sn} (u, \lambda)}{\operatorname{dn} (u, \lambda)} \right\};$$

change u into Mu , where $M = \frac{n}{N_1}$, hence

$$\begin{aligned} & (-)^{t(n-1)} \text{pm}(nu, k) \\ &= \text{pm}(Mu, \lambda) + 2\Sigma (-)^t \tan^{-1} \left\{ \lambda \text{cn} \left(\frac{2si\Lambda'}{n}, \lambda \right) \cdot \frac{\text{sn}(Mu, \lambda)}{\text{dn}(Mu, \lambda)} \right\}. \end{aligned}$$

Lastly, change u into $\frac{u}{n}$, and make n infinite, i.e., make

$$\lambda = 0, \quad \frac{M}{n} = \frac{\pi}{2K}, \quad \frac{\Lambda'}{n} = \frac{\pi K'}{2K}, \quad \text{pm}(u, 0) = 0;$$

also $\lambda \text{cn} \left(\frac{2si\Lambda'}{n}, \lambda \right) = i \text{dn} \frac{(2t-1)i\Lambda'}{n} \div \text{sn} \frac{(2t-1)i\Lambda'}{n},$

where t is an integer extending from 1 to $\frac{1}{2}(n-1)$; then we have, ultimately, the formula

$$\text{pm}(u, k) = \sin^{-1} \{ k \text{sn}(u, k) \} = 2\Sigma (-)^{m-1} \tan^{-1} \left\{ \frac{2q^{m-1} \sin \frac{\pi u}{2K}}{1 - q^{2m-1}} \right\},$$

if $q = e^{-\frac{\pi K'}{K}}.$

This is the correlative of Jacobi's expression in q -functions for $\text{am}(u, k)$.

If we differentiate the expression just obtained with regard to x ,

then $\text{cn} \frac{2Kx}{\pi} = \frac{2\pi}{kK} \cos x \Sigma \left\{ (-)^{m-1} \frac{q^{m-1}(1 - q^{2m-1})}{1 - 2q^{2m-1} \cos 2x + q^{4m-2}} \right\},$

which is one of Jacobi's formulæ.

In a similar way, if we are given that there is an addition-theorem for a transformation of an odd degree

$$\text{am}(nu, k) = \pm n \text{am}(u, k) + \Sigma \text{am}(Mu, \lambda),$$

the correlative one is obtained by changing the function am into pm .

3. The above examples are, I think, sufficient to show that the two functions are intimately connected together, and that there would be an advantage in using some such symbol as $\text{pm}(u, k)$.

I write down the quadric transformations in the new notation, viz.,

$$\begin{aligned} \text{am}(u, k) + \text{pm}(u, k) &= 2 \text{am} \left(\frac{\Gamma u}{2K}, \gamma \right) \\ \pi - \text{am}(u, k) + \text{pm}(u, k) &= 2 \text{am} \left(\Gamma - \frac{\Gamma u}{2K}, \gamma \right) \end{aligned} \Bigg\},$$

where $\Gamma = (1+k)K$ and $\gamma = \frac{2\sqrt{k}}{1+k}.$

The formula for a transformation from a modulus k to the com-

plementary one, k' , is

$$\text{pm}(K+iu, k) + \text{pm}(K-u, k') = \frac{\pi}{2}.$$

As regards transformation in general, the conclusion at which I have arrived is, that the formula for an odd prime number n , viz.,

$$\text{am}(nu, k) = \pm n \text{am}(u, k) + \Sigma \text{am}(Mu, \lambda),$$

together with the above quadric ones, is sufficient for a transformation of any degree whatever.

It may be mentioned that the quadric equations can be presented in different forms; one of these is

$$2 \text{am}(u, k) = \text{am}\left(\frac{1-k'}{1+k'}u, \frac{1-k'}{1+k'}\right) + \text{pm}\left(\frac{1+k'}{1-k'}u, \frac{1-k'}{1+k'}\right).$$

As an illustration of the use of these formulæ, I append the following result.

$$\text{Let } 2 \text{am}(6u, k) = \text{am}\left(3Nu, \frac{1-k'}{1+k'}\right) + \text{pm}\left(3Nu, \frac{1-k'}{1+k'}\right),$$

where

$$N = 2(1+k');$$

then, by the triplication formula, the right-hand side of the equation may be broken up into a series of am- and pm-functions of the type-form $\text{am}(Mu, \lambda)$, &c., where the modular relation is that of the cubic transformation, viz.,

$$\sqrt{\lambda k_1} + \sqrt{\lambda' k'_1} = 1, \text{ if } k_1 = \frac{1-k'}{1+k'} \text{ and } k'_1 = \frac{2\sqrt{k'}}{1+k'}.$$

The modular equation in question reduces to

$$\{(1+\lambda)(1+k')-2\}^4 = 64\lambda k' \lambda'^2 k^2,$$

and agrees with the result previously given by me from the transformation-equation $y = \sin(A+B+C)$, where

$$\sin A = \frac{(1+k')x\sqrt{1-x^2}}{\sqrt{1-k^2x^2}}, \quad \cos B = \frac{1-(1+\beta'^2)x^2}{1-\beta^2x^2}, \text{ \&c.}$$

Hence, for the analogue of the trigonometrical function $\sin 6u$, the modular equation has been deduced from the general formulæ as above, and in a similar way that corresponding to any number whatever may be arrived at from those of the primes into which the number can be resolved.