On the theory of Riemann's Integrals.

By

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On the fundamental integral functions.

The Riemann surface considered is represented by an equation of the form

$$f(s, z) = s^n + s^{n-1}(z, 1)_{\mu_1} + s^{n-2}(z, 1)_{\mu_2} + \cdots + (z, 1)_{\mu_n} = 0,$$

wherein s is an integral function of z, that is, does not become infinite except where z is infinite. At any value of z, $z = a$, I conceive the surface as consisting of $\varkappa$ branchings — superposed, the number of sheets that wind at these windingpoints being respectively

$$w_1 + 1, \ w_2 + 1, \ldots, w_\varkappa + 1$$

so that

$$w_1 + w_2 + \cdots + w_\varkappa + \varkappa = n,$$

and the number of branch points thus arising is $n - \varkappa$. The most ordinary case is when $\varkappa = n$ and

$$w_1 = w_2 = \cdots = w_\varkappa = 0.$$

The ordinary „Verzweigungspunkt" arises when

$$\varkappa = n - 1, \ w_1 = 0 = w_2 = \cdots = w_i - 1 = \cdots = w_n.$$

The case of a „sich aufhebender Verzweigungspunkt", at which two sheets just touch (as having the same value for z and s) without further connection, arises when

$$\varkappa = n, \ w_1 = w_2 = \cdots = w_n = 0$$

and is not distinguished in this description from an ordinary point.

A point on the surface which gives rise to a cusp (Rückkehrpunkt) on the corresponding plane curve $f(y, x) = 0$, is one at which two sheets not only wind but also touch as at a „sich aufhebender Verzweigungspunkt". This is given in the description here by

$$\varkappa = n - 1, \; w_1 = 0 = \cdots = w_i - 1 - \cdots = w_n$$

and is not distinguished from an ordinary branch point.

These examples will make the description clear. I say that each of the $\varkappa$ windings given by

$$w_1 + 1, \; w_2 + 1, \ldots, w_\varkappa + 1$$

constitutes a '*place*'. At these places dz is infinitesimal respectively of the orders

$$w_1, \; w_2, \; \ldots, \; w_\varkappa$$

namely, if in the neighbourhood of these places we write

$$z - a = t_1^{w_1+1}, \; t_3^{w_2+1}, \; \ldots, \; t_\varkappa^{w_\varkappa+1}$$

$t_1, t_2, \ldots, t_\varkappa$ will be infinitesimal of the first order.

Similarly we describe the character of the surface at $z = \infty$ by saying that at $z = \infty$ we may write

$$z = t_1^{-(w_1+1)}, \; \ldots, \; t_\varkappa^{-(w_\varkappa+1)}.$$

Kronecker (Crelle 91) shews that every integral algebraic function on the surface can be written in the form

$$(z, 1)_{\lambda_0} + (z, 1)_{\lambda_1} g_1 + \cdots + (z, 1)_{\lambda_{n-1}} g_{n-1},$$

where

$$g_i = \frac{s^i + s^{i-1}(z, 1)_{\nu_1} + \cdots}{(z, 1)^{\varrho_i}}$$

is an integral function.

Consider now any integral function g. Let its orders of infinity in the $\varkappa$ places at $z = \infty$ be

$$r_1, \; r_2, \; \ldots, \; r_\varkappa.$$

Let $L\left(\frac{r_i}{w_i+1}\right)$ denote the integer actually less than the number $\frac{r_i}{w_i+1}$, whether this number $\frac{r_i}{w_i+1}$ be integral or not, and let $L\left(\frac{r}{w+1}\right)$ denote the greatest one of the integers

$$L\left(\frac{r_1}{w_1+1}\right), \; L\left(\frac{r_2}{w_2+1}\right), \; \ldots, \; L\left(\frac{r_\varkappa}{w_\varkappa+1}\right).$$

I call $L\left(\frac{r}{w+1}\right)$ the *rank* of the integral function g.

Then we have the

Proposition. The sum of the ranks of the Kronecker functions

$$g_1, \; g_2, \; \ldots, \; g_{n-1}$$

is in all cases p, the „Geschlecht“ of the surface.

For, consider an integral algebraic function which is to be infinite at the $\varkappa$ places at $z = \infty$ respectively to orders

$$m(w_1+1),\ m(w_2+1),\ \ldots,\ m(w_\varkappa+1)$$

or as near below these as may be possible: m being a large enough integer to allow our regarding these

$$m(w_1+1),\ m(w_2+1),\ \ldots$$

places as independent.

The form of the function is necessarily

$$(z,1)_\lambda + g_1(z,1)_\mu + g_2(z,1)_\nu + \cdots$$

where

g_1 is such as to be infinite at $z=\infty$ respectively to orders $r_1, r_2, \ldots, r_\varkappa$,
g_2 is such as to be infinite at $z=\infty$ respectively to orders $t_1,\ t_2,\ \ldots,\ t_\varkappa$

and so on.

Hence considering first the place $z=\infty$ where z is infinite of order w_1+1

$$\lambda(w_1+1) \overline{\gtrless} m(w_1+1), \quad \mu(w_0+1)+r_1 \overline{\gtrless} m(w_1+1),$$
$$\nu(w_1+1)+t_1 \overline{\gtrless} m(w_1+1), \ldots.$$

Considering next the place $z=\infty$ where z is infinite of order w_2+1

$$\lambda(w_2+1) \overline{\gtrless} m(w_2+1), \quad \mu(w_2+1)+r_2 \overline{\gtrless} m(w_2+1),$$
$$\nu(w_2+1)+t_2 \overline{\gtrless} m(w_2+1), \ldots$$

and so on: there being $\varkappa$ such rows of conditions.

The first column of these conditions gives $\lambda = m$, shewing that such a function as postulated is certainly possible. The second column gives

$$\mu \overline{\gtrless} m - \frac{r_i}{w_i+1} \overline{\gtrless} m - 1 - L\left(\frac{r_i}{w_i+1}\right) \quad \text{for} \quad i = 1, 2, 3, \ldots, n,$$

and gives therefore, in the sense defined above,

$$\mu = m - 1 - L\left(\frac{r}{w+1}\right) = m - 1 - \text{rank of function } g_1$$

so

$$\nu = m - 1 - L\left(\frac{t}{w+1}\right) = m - 1 - \text{rank of function } g_2$$

Hence the number of arbitrary coefficients in our function, being

$$(\lambda+1)+(\mu+1)+(\nu+1)+\cdots$$

is

$$1 + nm - \left\{L\left(\frac{r}{w+1}\right) + L\left(\frac{t}{w+1}\right) + \cdots\right\}.$$

But, by the Riemann-Roch-Satz, since m is sufficiently large, the number of these arbitrary coefficients should be $1 + Q - p$, where Q is the number of infinities of the function, viz

$$Q + m(w_1 + 1 + w_2 + 1 + \cdots) = mn.$$

Hence

$$p = L\left(\frac{r}{w+1}\right) + L\left(\frac{t}{w+1}\right) + \cdots$$

as stated. (Cf. also Abel. Oeuvres comp. 1881, p. 173. Equation 80).

Of the expression of algebraic functions which are infinite only at an arbitrary place.

Consider the places $z = a$, the surface being here characterised by

$$w_1 + 1, \ldots, w_\varkappa + 1.$$

Let g be an integral function and r the least integer such that $\frac{g}{(z-u)^{r+1}}$ is not infinite at $z = \infty$. For this, if the orders of infinity of g be, in the $\varkappa$ places $z = \infty$, respectively

$$r_1, r_2, \ldots, r_\varkappa,$$

$$(r+1)(w_i+1) \overline{\overline{>}} r_i, \quad \text{viz } r \overline{\overline{>}} \frac{r_i}{w_i+1} - 1, \quad \text{viz } r \overline{\overline{>}} L\left(\frac{r_i}{w_i+1}\right)$$

$$\text{for } i = 1, 2, \ldots, n.$$

Hence

$$r = L\left(\frac{r}{w+1}\right).$$

viz = rank of function g.

If then K be an algebraic function only infinite at $z = a$ and such that $K(z-a)^{m+1}$ is just *not* infinite at $z = a$ and is therefore an integral function, we must have

$$(1) \quad K(z-a)^{m+1} = (z-a, 1)_{\lambda_0} + (z-a, 1)_{\lambda_1} g_1 + (z-a, 1)_{\lambda_2} g_2 + \cdots$$
$$\cdots + (z-a, 1)_{\lambda_{n-1}} g_{n-1}.$$

Put

$$z - a = \frac{1}{\zeta} \quad \text{and} \quad h_i = \frac{g_i}{(z-a)^{\tau_i+1}},$$

where τ_i is the rank of g_i.

Then

$$(2) \quad K = (1, \zeta)_{\lambda_0} \zeta^{m+1-\lambda_0} + (1, \zeta)_{\lambda_1} \zeta^{m-\tau_1-\lambda_1} h_1 + (1, \zeta)_{\lambda_2} \zeta^{m-\tau_2-\lambda_2} h_2 + \cdots.$$

But the equation (1) gives, since K is not infinite at $z = \infty$ and contains therefore in its expression, as I assume, no terms which become infinite at $z = \infty$,

$$\lambda_0 \overline{\overline{<}} m + 1, \quad \lambda_1(w_i + 1) + r_i \overline{\overline{<}} (m+1)(w_i + 1),$$
$$\lambda_2(w_i + 1) + t_i \overline{\overline{<}} (m+1)(w_i + 1), \ldots$$

where $r_1, r_2, \ldots, r_\varkappa$ are the orders of infinity of g_1 at $z = \infty$, $t_1, \ldots, t_\varkappa$ are the orders of infinity of g_2 at $z = \infty$ etc.

Hence

$$\frac{r_i}{w_i+1} \overline{\overline{<}} m+1-\lambda_1, \quad L\left(\frac{r_i}{w_i+1}\right) \overline{\overline{<}} m-\lambda_1, \quad m-\tau_1-\lambda_1 \overline{\overline{>}} 0,$$

$$\frac{t_i}{w_i+1} \overline{\overline{<}} m+1-\lambda_2, \quad L\left(\frac{t_i}{w_i+1}\right) \overline{\overline{<}} m-\lambda_2, \quad m-\tau-\lambda_2 \overline{\overline{>}} 0$$

etc.

Hence we have the

Proposition. An algebraic function which is only infinite at $z = a$ can be written

$$K = (1, \zeta)_{\mu_0} + (1, \zeta)_{\mu_2} h_1 + (1, \zeta)_{\mu_2} h_2 + \cdots + (1, \zeta)_{\mu_{n-1}} h_{n-1}$$

where

$$\zeta = \frac{1}{z-a}, \quad h_i = \frac{g_i}{(z-a)^{\tau_i+1}},$$

and $\mu_0, \ldots, \mu_{n-1}$ are all $\overline{\overline{>}} 0$.

It is easy to see that the ranks of $h_1, h_2, \ldots, h_{n-1}$, considered as functions of ζ are respectively the same as those of $g_1, g_2, \ldots, g_{n-1}$.

Of the expression of integrals of the first, second and third kinds — and of the form of adjoint curves in general.

We introduce in what follows certain forms*) $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$, writing

$$g_i(s, z) = \frac{s^i + s^{i-1}(z, 1)_{\mu_1} + \cdots}{D_i},$$

where D_i is an integral polynomial in z. The function φ_i is of the form

$$\varphi_i(s, z) = [s^{n-1-i} + s^{n-2-i}(z, 1)_{\nu_1} + \cdots] D_i.$$

The exact expressions for $\varphi_0(s, z)$, $\varphi_1(s, z)$, ... may be defined by the *identity*

$$\text{(A)} \quad \varphi_0(s', z) + \varphi_1(s', z) g_1(s, z) + \varphi_2(s', z) g_2(s, z) + \cdots + \varphi_{n-1}(s', z) g_{n-1}(s, z)$$
$$= \frac{f(s', z) - f(s, z)}{s' - s}$$
$$= s'^{n-1} + s'^{n-2}\chi_1(s, z) + s'^{n-3}\chi_2(s, z) + \cdots + \chi_{n-1}(s, z)$$
$$= s^{n-1} + s^{n-2}\chi_1(s', z) + s^{n-3}\chi_2(s', z) + \cdots + \chi_{n-1}(s', z),$$

where writing

$$f(s, z) = s^n + Q_1 . s^{n-1} + Q_2 . s^{n-2} + \cdots + Q_n,$$

the forms χ_1, χ_2. etc. are those given by

*) Cf. Dedekind & Weber. Crelle 92. (Theor. d. Algeb. Fctnen. e. Var.) where the same forms are introduced and called „die zu g complementäre Basis". Also Hensel. Crelle 109.

$$\chi_1(s, z) = s + Q_1, \quad \chi_2(s, z) = s^2 + sQ_1 + Q_2, \ldots,$$
$$\chi_{n-1}(s, z) = s^{n-1} + s^{n-2}Q_1 + \cdots + Q_{n-1}.$$

By equating the coefficients of the same powers of s on the two sides of equation (A) we obtain the explicit forms of the functions

$$\varphi_0, \varphi_1, \ldots, \varphi_{n-1}.$$

For *instance*, if

$$g_1(s, z) = \frac{\chi_1(s, z)}{D_1}, \quad g_2(s, z) = \frac{\chi_2(s, z)}{D_2}, \quad \ldots, \quad g_{n-1}(s, z) = \frac{\chi_{n-1}(s, z)}{D_{n-1}},$$

and this is a case of common occurrence, then

$$\varphi_0(s, z) = s^{n-1}, \; \varphi_1(s, z) = D_1 s^{n-2}, \ldots, \varphi_{n-1}(s, z) = D_{n-1},$$

while in general if the equations giving $s, s^2, \ldots, s^{n-1}$ in terms of $g_1, g_2, \ldots, g_{n-1}$, be

$$\begin{aligned} 1 &= 1, \\ s &= a_{1,0} + a_{1,1} \cdot g_1, \\ s^2 &= a_{2,0} + a_{2,1} \cdot g_1 + a_{2,2} \cdot g_2, \\ &\ldots\ldots\ldots\ldots \\ s^{n-1} &= a_{n-1,0} + a_{n-1,1}\, g_1 + a_{n-1,2}\, g_2 + \cdots + a_{n-1,n-1} g_{n-1}, \end{aligned}$$

where

$$a_{1,0}, \; a_{1,1}, \; a_{2,0}, \; a_{2,1}, \ldots$$

are integral polynomials in z, then

$$\begin{aligned} \varphi_0 &= \chi_{n-1} + a_{1,0}\chi_{n-2} + \cdots + a_{n-2,0}\;\chi_1 + a_{n-1,0}, \\ \varphi_1 &= \qquad\quad a_{1,1}\chi_{n-2} + \cdots + a_{n-2,1}\;\chi_1 + a_{n-1,1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_{n-2} &= \qquad\qquad\qquad\qquad a_{n-2,n-2}\chi_1 + a_{n-1,n-2}, \\ \varphi_{n-1} &= \qquad\qquad\qquad\qquad\qquad\qquad a_{n-1,n-1}, \end{aligned}$$

namely, if we write

$$(1, s, s^2, \ldots, s^{n-1}) = \Omega(1, g_1, g_2, \ldots, g_{n-1})$$

where Ω in a matrix whose determinant is

$$a_{1,1}\, a_{2,2} \ldots a_{n-1,n-1} = D_1\, D_2\, D_3 \ldots D_{n-1},$$

then

$$(\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) = \overline{\Omega}\,(\chi_{n-1}, \chi_{n-2}, \ldots, \chi_1, 1),$$

where $\overline{\Omega}$ is the matrix determined from Ω by changing its rows into columns — is what we call the '*transposed*' of Ω.

If (Q) denote the matrix

$$\begin{pmatrix} Q_{n-1} & Q_{n-2} & \ldots & Q_1 & 1 \\ Q_{n-2} & Q_{n-3} & \ldots & 1 & 0 \\ \ldots & & & & \\ Q_1 & 1 & 0 \ldots & & \\ 1 & 0 & \ldots & & \end{pmatrix}$$

whose determinant is ± 1, then

$$(\chi_{n-1}, \chi_{n-2}, \ldots, \chi_1, 1) = (Q)(1, s, s^2, \ldots, s^{n-1})$$

and we may write

$$\text{(B)} \qquad (\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) = \overline{\Omega}(Q)\,\Omega(1, g_1, g_2, \ldots, g_{n-1}).$$

The definition (A) leads to other forms for

$$\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$$

in general — thus. —

Let

$$s_1, s_2, \ldots, s_{n-1}, s_n$$

denote the values of s arising from $f(s, z) = 0$ for any value of z. Denote $\varphi_0(s_i, z)$, $g(s_i, z)$ by $\varphi_0^{(i)}$, $g^{(i)}$ etc.

Then

$$\varphi_0^{(1)} + \varphi_1^{(1)} g_1^{(1)} + \varphi_1^{(1)} g_2^{(1)} + \cdots + \varphi_{n-1}^{(1)} g_{n-1}^{(1)} = f'(s_1) = \left[\frac{\partial f(s, z)}{\partial s}\right]_{s=s_1},$$

$$\varphi_0^{(1)} + \varphi_1^{(1)} g_1^{(i)} + \varphi_2^{(1)} g_2^{(i)} + \cdots + \varphi_{n-1}^{(1)} g_{n-1}^{(i)} = 0, \qquad (i=2, 3, \ldots, n).$$

Hence if

$$c_0 \varphi_0^{(1)} + c_1 \varphi_1^{(1)} + c_2 \varphi_2^{(1)} + \cdots + c_{n-1} \varphi_{n-1}^{(1)} = \varphi^{(1)},$$

$$\begin{vmatrix} c_0 & c_1 & c_2 & \ldots & c_{n-1} & \varphi^{(1)} \\ 1 & g_1^{(1)} & g_2^{(1)} & \ldots & g_{n-1}^{(1)} & f'(s_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & g_1^{(i)} & g_2^{(i)} & \ldots & g_{n-1}^{(i)} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0$$

namely

$$\text{(C)} \qquad \frac{\varphi^{(1)}}{f'(s_1)} \begin{vmatrix} 1 & g_1^{(1)} & g_2^{(1)} & \ldots & g_{n-1}^{(1)} \\ 1 & g_1^{(2)} & g_2^{(2)} & \ldots & g_{n-1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & g_1^{(n)} & g_2^{(n)} & \ldots & g_{n-1}^{(n)} \end{vmatrix} = \begin{vmatrix} c_0 & c_1 & \ldots & c_{n-1} \\ 1 & g_1^{(2)} & \ldots & g_{n-1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & g_1^{(n)} & \ldots & g_{n-1}^{(n)} \end{vmatrix}.$$

It is in this form we shall use the functions $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ — limiting ourselves then in the firt instance to values of z where

$$s_1, s_2, \ldots, s_{n-1}, s_n$$

are different. We remark that by multiplying both sides of equation (C) by the determinant which occurs on the left, we obtain

$$\frac{\varphi_0}{f'(s)}, \frac{\varphi_1}{f'(s)}, \ldots, \frac{\varphi_{n-1}}{f'(s)},$$

expressed as rational functions of s and z, and in a form identical with those called by Hensel (Crelle 109) $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n$.

Let now $P^z_{z_1, z_2}$ be an integral of the third kind, infinite only at two places where

$$z = z_1, \quad z = z_2$$

and in fact like $\int_{z_2}^{z_1} \frac{dt}{t}$, so that $\frac{dP}{dz}$ is an algebraic function, infinite like $\frac{1}{z - z_1}$ at the first, and like $-\frac{1}{z - z_2}$ at the second place — infinite moreover at every winding-place of the surface, as for instance where $z = a$, but such that $(z - a)\frac{dP}{dz}$ is there zero of the first order. Thus if

$$\alpha = (z - a_1)(z - a_2) \ldots,$$

be the integral polynomial in z which vanishes at all the finite branch points of the surface, it being supposed in the first instance that neither z_1 or z_2 are branch points, and g be any integral algebraic function, then

$$\left(g\,\alpha(z - z_1)(z - z_2)\frac{dP}{dz}\right)_1$$

$$+ \left(g\,\alpha(z - z_1)(z - z_2)\frac{dP}{dz}\right)_2 + \cdots + \left(g\,\alpha(z - z_1)(z - z_2)\frac{dP}{dz}\right)_n,$$

where the suffixes indicate the values of the function in the n sheets of the surface for any value of z, is a symmetrical function of these n values, and is therefore a rational function of z alone, is moreover only infinite for $z = \infty$, and vanishes for all values of z which make $\alpha = 0$, and is thus of the form αJ, where J is an integral polynomial in z. Dividing then the equation by α, writing

$$\frac{J}{(z - z_1)(z - z_2)} = \frac{\lambda_1(z - z_2) - \lambda_2(z - z_1) + (z - z_1)(z - z_2)K}{(z - z_1)(z - z_2)},$$

λ_1, λ_2 being constants and K a polynomial in z,

and remembering that

$$(z - z_1)\frac{dP}{dz} = 1$$

at the infinity where $z = z_1$, and

$$(z - z_2)\frac{dP}{dz} = -1$$

at the other infinity, we see that

$$\left(g\frac{dP}{dz}\right)_1 + \left(g\frac{dP}{dz}\right)_2 + \cdots + \left(g\frac{dP}{dz}\right)_n = \frac{g(s_1, z_1)}{z - z_1} - \frac{g(s_2, z_2)}{z - z_2} + (z, 1)^{\mu - 1},$$

where (s_1, z_1), (s_2, z_2) are the two infinities of $P^z_{z_1, z_2}$.

Further if τ be the rank of the integral function g, so that $\frac{g}{z^{\tau+1}}$ is just *not* infinite in any sheet at $z=\infty$, this equation gives

$$\left(\frac{g}{z^{\tau+1}}\frac{dP}{dz}\right)_1 + \left(\frac{g}{z^{\tau+1}}\frac{dP}{dz}\right)_2 + \cdots + \left(\frac{g}{z^{\tau+1}}\frac{dP}{dz}\right)_n$$
$$= \frac{g(s_1, z_1)}{z^{\tau+1}(z-z_1)} - \frac{g(s_2, z_2)}{z^{\tau+1}(z-z_2)} + \frac{(z, 1)^{\mu-1}}{z^{\tau+1}}.$$

Now at a place $z=\infty$ where z is infinite of order $w+1$, say $z = t^{-(w+1)}$

$$\frac{dP}{dz} = -\frac{t^{w+2}}{w+1}\frac{dP}{dt}$$

is zero to order $w+2$.

Hence

$$(\tau + 1 - \overline{\mu-1})(w+1) \overline{>} w+2,$$
$$\tau - \mu + 2 \overline{>} 1 + \frac{1}{w+1},$$
$$\mu \overline{<} \tau + 1 - \frac{1}{w+1}.$$

Hence μ is at most equal to the rank τ of the function g.

In particular

$$\left(\frac{dP}{dz}\right)_1 + \left(\frac{dP}{dz}\right)_2 + \cdots + \left(\frac{dP}{dz}\right)_n = \frac{1}{z-z_1} - \frac{1}{z-z_2}$$
$$g_1^{(1)}\left(\frac{dP}{dz}\right)_1 + g_1^{(2)}\left(\frac{dP}{dz}\right)_2 + \cdots + g_1^{(n)}\left(\frac{dP}{dz}\right)_n = \frac{g_1(s_1, z_1)}{z-z_1} - \frac{g_1(s_2, z_2)}{z-z_2} + (z, 1)^{\tau_1'-1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$g_{n-1}^{(1)}\left(\frac{dP}{dz}\right)_1 + g_{n-1}^{(2)}\left(\frac{dP}{dz}\right)_2 + \cdots + g_{n-1}^{(n)}\left(\frac{dP}{dz}\right)_n = \frac{g_{n-1}(s_1, z_1)}{z-z_1} - \frac{g_{n-1}(s_2, z_2)}{z-z_2} + (z, 1)^{\tau_{n-1}'-1},$$

where $\tau_1', \tau_2', \ldots, \tau_{n-1}$ are respectively not greater than the ranks $\tau_1, \tau_2, \ldots, \tau_{n-1}$. Solving these equations for $\left(\frac{dP}{dz}\right)_i$, and then removing the suffix, we have, in accordance with the definitions (C),

$$f'(s)\frac{dP}{dz} = (z, 1)^{\tau'-1}\varphi_1 + (z, 1)^{\tau_2'-1}\varphi_2 + \cdots + (z, 1)^{\tau_{n-1}'-1}\varphi_{n-1}$$
$$+ \frac{\varphi_0 + \varphi_1 g_1(s_1, z_1) + \cdots + \varphi_{n-1} g_{n-1}(s_1, z_1)}{z-z_1}$$
$$- \frac{\varphi_0 + \varphi_1 g_1(s_2, z_2) + \cdots + \varphi_{n-1} g_{n-1}(s_2, z_2)}{z-z_2}$$

where φ_i stands for $\varphi_i(s, z)$.

In this 'necessary' form of $\frac{dP}{dz}$ there enter at the most

$$\tau_1 + \tau_2 + \cdots + \tau_{n-1} + 1$$

arbitrary homogeneous coefficients: namely, in accordance with what was proved, at most $p+1$. But we know that the most general form of $\frac{dP}{dz}$ involves such $p+1$ terms, being in fact

$$\lambda_1 \frac{dv_1}{dz} + \lambda_2 \frac{dv_2}{dz} + \cdots + \lambda_p \frac{dv_p}{dz} + \left(\frac{dP}{dz}\right)_0$$

where $v_1, v_2, \ldots$ are the normal integrals of the first kind and $\left(\frac{dP}{dz}\right)_0$ a special form of $\frac{dP}{dz}$.

Hence we can infer

(1) The most general form of integral of the first kind is

$$\int^z \frac{dz}{f'(s)} \left[(z, 1)^{\tau'-1} \varphi_1(s, z) + (z, 1)^{\tau_2'-1} \varphi_2(s, z) + \cdots \right.$$
$$\left. + (z, 1)^{\tau_{n-1}'-1} \varphi_{n-1}(s, z)\right]$$

where $\tau_i' \leqq \tau_i$, and the coefficients in $(z, 1)^{\tau_i'-1}$ are arbitrary.

(2) A special and actual form of integral of the third kind logarithmically infinite like $\log t_1 - \log t_2$, where at the first place

$$z - z_1 = t_1^{w_1+1},$$

and at the second

$$z - z_2 = t_2^{w_2+1},$$

is

$$\int^z \frac{dz}{f'(s)} \left[\frac{\varphi_0(s, z) + \varphi_1(s, z) g_1(s_1, z_1) + \cdots + \varphi_{n-1}(s, z) g_{n-1}(s_1, z_1)}{z - z_1}\right.$$
$$\left. - \frac{\varphi_0(s, z) + \varphi_1(s, z) g_1(s_2, z_2) + \cdots + \varphi_{n-1}(s, z) g_{n-1}(s_2, z_2)}{z - z_2}\right]$$

or

$$\int^z \frac{dz}{f'(s)} \int_{z_2}^{z_1} d\zeta \, \frac{d}{d\zeta} \left[\frac{\varphi_0(s, z) + \varphi_1(s, z) g_1(\sigma, \zeta) + \cdots + \varphi_{n-1}(s, z) g_{n-1}(\sigma, \zeta)}{z - \zeta}\right].$$

We can prove in quite a similar way that one form of an integral of the second kind which is once algebraically infinite at an *ordinary* place where $z = \zeta$ like $\frac{1}{\zeta - z}$ is given by

$$Z_\zeta^{z,c} = \int_c^z \frac{dz}{f'(s)} \frac{d}{d\zeta} \left[\frac{\varphi_0(s, z) + \varphi_1(s, z) g_1(\sigma, \zeta) + \cdots + \varphi_{n-1}(s, z) g_{n-1}(\sigma, \zeta)}{z - \zeta}\right],$$

and we can quite easily modify these forms to the case when ζ is a branch point.

Remarks and Examples.

A comparison of the methods and results of this note, which was suggested to me by Hensel's paper, Crelle 109, with his papers (Crelle 109, 111), where the integral of the third kind, though probably in contemplation, is not mentioned, will shew to what extent I am indebted to him. It appears to me that without an exact specification of the forms of $g_1, g_2, \ldots, g_{n-1}$, his paper (wherein, however, the results of Riemann's theory are not assumed) does not prove that in

$$\frac{dv}{dz} = (z, 1)^{\tau'-1}\varphi_1 + \cdots + (z, 1)^{\tau'_{n-1}-1}\varphi_{n-1}$$

all the coefficients in $(z, 1)^{\tau'_i-1}$ may be taken to be arbitrary: though it proves that this is a 'necessary' form of $\frac{dv}{dz}$. For it is not shewn that the equations which he obtains

$$u_1 a_{i1} + u_2 a_{i2} + \cdots + u_n a_{in} = P\bar{u}_i$$

lead necessarily to integral polynomials $u_1, \ldots, u_n$ for every integral form of $\bar{u}_i$. The *proof* here given that

$$\tau_1 + \tau_2 + \cdots + \tau_{n-1} = p,$$

an equation taken by him as definition of p, is designed to fill this Lücke, as I conceive it to be. Moreover his use of dimensions, founded on the *form* of algebraic functions, is apt perhaps to lead to misconception. — In illustration of this point consider the case

$$\begin{aligned} f(s, z) &= s^3 + s^2(z, 1)_2 + sz(z, 1)_1 + Az^2 = 0 \\ &= s^3 + s^2 Q_1 + sQ_2 + Q_3 = 0 \end{aligned}$$

say. Since, by writing $y = \frac{z}{s}$, the equation becomes

$$z + y(z, 1)_2 + y^2(z, 1)_1 + Ay^3 = 0,$$

we see that a fundamental set of integral functions is $1, y, y^2$ and for the original curve, is, therefore

$$1, \frac{z}{s}, \frac{z^2}{s^2},$$

where

$$\frac{z}{s} = -\frac{s^2 + sQ_1 + Q_2}{Az} = -\frac{\chi_2(s, z)}{Az} = -\frac{g_2(s, z)}{A}$$

say

$$\frac{z^2}{s^2} = -\frac{z}{s}\frac{(z, 1)_1}{A} - (s + Q_1) = g_2\frac{(z, 1)_1}{A^2} = \chi_1(s, z).$$

Hence we may take as a fundamental set

$$1, g_1, g_2,$$

where

$$g_1 = \chi_1(s, z), \qquad g_2 = \frac{\chi_2(s, z)}{z}$$

and hence, from

$$\varphi_0' + \varphi_1' g_1 + \varphi_2' g_2 = s'^2 + s'\chi_1 + \chi_2$$

obtain

$$\varphi_0 = s^2, \quad \varphi_1 = s, \quad \varphi_2 = z.$$

Writing now in

$$f(s, z), \quad s = \frac{\eta}{\zeta^2}, \quad z = \frac{1}{\zeta},$$

so that

$$f(s, z) = \zeta^{-6}[\eta^3 + \eta^2(1, \zeta)_2 + \eta\zeta^2(1, \zeta)_1 + A\zeta^4] = \zeta^{-6}F(\eta, \zeta)$$

say,

$$g_1 = \zeta^{-2}[\eta + (1, \zeta)_2],$$

$$g_2 = \zeta^{-3}[\eta^2 + \eta(1, \zeta)_2 + \zeta^2(1, \zeta)_1]$$

have associated with them the indices 2 and 3 respectively, while

$$\frac{\varphi_0}{f'(s)} = \frac{s^2}{3s^2 + 2sQ_1 + Q_2} = \frac{\eta^2}{3\eta^2 + 2\eta(1, \zeta)_2 + \zeta^2(1, \zeta)_1},$$

$$\frac{\varphi_1}{f'(s)} = \frac{s}{f'(s)} = \frac{\eta\zeta^2}{3\eta^2 + 2\eta(1, \zeta)_2 + \zeta^2(1, \zeta)_1},$$

$$\frac{\varphi_2}{f'(s)} = \frac{z}{f'(s)} = \frac{\zeta^3}{3\eta^2 + 2\eta(1, \zeta)_2 + \zeta^2(1, \zeta)_1}$$

have, in accordance with Hensel's work, associated respectively with them the indices

$$0, \quad -2, \quad -3,$$

say

$$0, \quad -\mu_1, \quad -\mu_2.$$

Apparently then in accordance with his work the general integral of the first kind is

$$(z, 1)^0 \frac{\varphi_1}{f'(s)} + (z, 1)^1 \frac{\varphi_2}{f'(s)}$$

and

$$p = \mu_1 + \mu_2 - 3 + 1 = 3.$$

As a fact $p = 1$: and Hensel's results are based on the hypothesis that

$$F'(\eta) = \frac{\partial}{\partial \eta} F(\eta, \zeta)$$

does not vanish for $\zeta = 0$, as is the case in this example. — But it is not I think convenient to make this hypothesis — which would exclude from the direct application of the theory that most important case when the surface is given in Weierstrass's normal form in which all the sheets wind at $z = \infty$. In this example it is easy to prove that $\int \frac{\varphi_1}{f'(s)} dz$ is finite at every place $z = \infty$, but $\int \frac{\varphi_2}{f'(s)} dz$ is logarithmically infinite in two sheets at $z = \infty$. And this is included in the forms we have given: for it can be immediately shewn by considering $F(\eta, \zeta)$ and $F'(\eta)$ at $\zeta = 0$ that g_1 is of rank unity and g_2 of rank zero, and the only finite integral is therefore $\int \frac{\varphi_1}{f'(s)} dz$.

Of course these remarks are not intended to detract from the very great interest attaching to the results given by Hensel.

I give as a further example of the formula here obtained for the integral of the third kind, the application to the hyperelliptic case

$$s^2 - (z, 1)_{2p+2} = 0.$$

Here

$$g_1 = s,$$

$$\varphi_0 = s, \quad \varphi_1 = 1,$$

and the integral of the third kind is therefore

$$\int^z \frac{dz}{s}\left(\frac{s+\sigma_1}{z-\zeta_1} - \frac{s+\sigma_2}{z-\zeta_2}\right).$$

I proceed to verify

(1) That this form*) is obtainable when the integral of the third kind is built after the rules given in Clebsch and Gordan, Abel. Fctnen., see for example. Noether, Math. Ann. 37, 434.

(2) That for $p = 2$ this is equivalent to the covariant form given by Klein (e. g. Math. Ann. 32, 352).

(1) The straight line passing through the points of discontinuity

$$(s, z) = (\sigma_1, \zeta_1), \quad (s, z) = (\sigma_2, \zeta_2)$$

being written in the form

$$Az + \frac{s}{f_0(\zeta_1 - \zeta_2)} + C,$$

where

$$(z, 1)_{2p+2} = f(z) = f_0 z^{2p+2} + f_1 z^{2p+1} + \cdots.$$

If this straight line meet the curve again in

$$\zeta_3, \zeta_4, \ldots, \zeta_{2p+2},$$

we shall have

$$\frac{s^2}{f_0^2(\zeta_1-\zeta_2)^2} - (Az + C)^2$$

$$= \frac{1}{f_0(\zeta_1-\zeta_2)^2}(z-\zeta_1)(z-\zeta_2)(z-\zeta_3)\cdots(z-\zeta_{2p+2}).$$

And since

$$Az - \frac{s}{f_0(\zeta_1 - \zeta_2)} + C$$

is obtained from

$$Az + \frac{s}{f_0(\zeta_1 - \zeta_2)} + C$$

*) Also found in Schwarz. Formeln und Lehrsätze z. Gebr. Ellipt. Functionen pag. 88, (6).

by changing the sign of s, it is equal to

$$\frac{1}{f_0(\zeta_1-\zeta_2)^2}\begin{vmatrix} z & s & 1 \\ \zeta_1 & -\sigma_1 & 1 \\ \zeta_2 & -\sigma_2 & 1\end{vmatrix}$$

$$=\frac{1}{f_0(\zeta_1-\zeta_2)^2}\begin{vmatrix} z-\zeta_1 & s+\sigma_1 \\ z-\zeta_2 & s+\sigma_2\end{vmatrix}=\frac{(\zeta-\zeta_1)(z-\zeta_2)}{f_0(\zeta_1-\zeta_2)^2}\left(\frac{s+\sigma_1}{z-\zeta_1}-\frac{s+\sigma_2}{z-\zeta_2}\right).$$

But since the general adjoint curve of order $n-2=2p$ is

$$(z,1)_{2p}+s(z,1)_{p-1}=0$$

the form of the integral of the third kind formed after Clebsch and Gordan, must be, save for additive terms which are integrals of the first kind,

$$\int\frac{(z-\zeta_3)\cdots(z-\zeta_{2p+2})}{Az+\frac{s}{f_0(\zeta_1-\zeta_2)}+C}\frac{dz}{s}$$

$$=\int\frac{\left[Az-\frac{s}{f_0(\zeta_1-\zeta_2)}+C\right](z-\zeta_3)\cdots(z-\zeta_{2p+2})}{\frac{1}{f_0(\zeta_1-\zeta_2)^2}(z-\zeta_1)(z-\zeta_2)(z-\zeta_3)\cdots(z-\zeta_{2p+2})}\frac{dz}{s}$$

$$=\int\left[\frac{s+\sigma_1}{z-\zeta_1}-\frac{s+\sigma_2}{z-\zeta_2}\right]\frac{dz}{s}=\int^{z}\frac{dz}{s}\int_{\zeta_2}^{\zeta_1}d\zeta\,\frac{d}{d\zeta}\left(\frac{s+\sigma}{z-\zeta}\right)$$

as stated.

(2) We have

$$\frac{d}{d\zeta}\left(\frac{s+\sigma}{z+\zeta}\right)=\frac{\frac{d\sigma}{d\zeta}(z-\zeta)+\sigma+s}{(z-\zeta)^2}$$

$$=\frac{\frac{1}{2}\frac{df(\zeta)}{d\zeta}(z-\zeta)+\sigma s+\sigma^2}{(z-\zeta)^2}\,\frac{1}{\sigma}$$

where

$$\sigma^2=f(\zeta)=(\zeta,1)_{2p+2}.$$

Now in fact for $p=2$, writing $f(z)=a_z^6$, it is easy to verify that

$$\frac{a_z^3a_\zeta^3}{(z-\zeta)^2}=\frac{\frac{1}{2}(z-\zeta)\frac{df}{d\zeta}+\sigma^2}{(z-\zeta)^2}+\frac{1}{120}(z-\zeta)\frac{d^3f}{d\zeta^3}+\frac{1}{10}\frac{d^2f}{d\zeta^2}$$

and hence obtain

$$\int_{\zeta_2}^{\zeta_1}d\zeta\,\frac{d}{d\zeta}\left(\frac{s+\sigma}{z-\zeta}\right)=\int_{\zeta_2}^{\zeta_1}\frac{d\zeta}{\sigma}\,\frac{a_z^3a_\zeta^3+\sigma s}{(z-\zeta)^2}-\frac{1}{10}\int_{\zeta_2}^{\zeta_1}\frac{d\zeta}{\sigma}\left[\frac{(z-\zeta)}{12}\frac{d^3f}{d\zeta^3}+\frac{d^2f}{d\zeta^2}\right]$$

and hence, writing

$$\frac{1}{120}\int_{\zeta_2}^{\zeta_1}\frac{d\zeta}{\sigma}\frac{d^3 f}{d\zeta^3}=\lambda,$$

$$\int_{\zeta_2}^{\zeta_1}\frac{d\zeta}{\sigma}\left(-\frac{1}{120}\zeta\frac{d^3 f}{d\zeta^3}+\frac{1}{10}\frac{d^2 f}{d\zeta^2}\right)=\mu$$

the equation

$$\int^{z}\left[\frac{s+\sigma_1}{z-\zeta_1}-\frac{s+\sigma_2}{z-\zeta_2}\right]\frac{dz}{s}$$

$$=\int^{z}\frac{dz}{s}\int_{\zeta_2}^{\zeta_1}\frac{d\zeta}{\sigma}\frac{a_z^3 a_\zeta^3+\sigma s}{(z-\zeta)^2}-\lambda\int^{z}\frac{z\,dz}{s}-\mu\int^{z}\frac{dz}{s}$$

which was desired, the form employed by Klein being

$$\int^{z}\frac{dz}{s}\int_{\zeta_2}^{\zeta_1}\frac{d\zeta}{\sigma}\frac{a_z^3 a_\zeta^3+\sigma s}{(z-\zeta)^2}.$$

Note. We may add in connection with (2) of page 10 that the integral infinite of the first order at a branch place ζ is

$$\int^{z}\frac{dz}{f'(s)}\,\frac{\varphi_1 g_1'+\cdots+\varphi_{n-1}g_{n-1}'}{z-\zeta}$$

where g_1' is the differential coefficient in regard to the infinitesimal at ζ, etc.

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