A Geometrical Note. *By* R. Tucker, M.A.

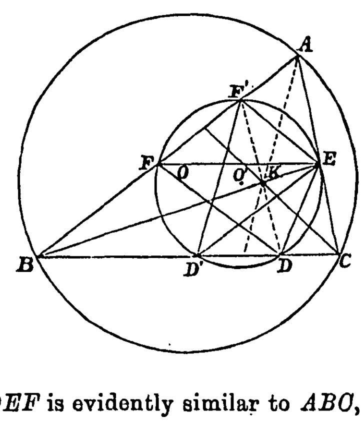
Read December 8th, 1892.

1. If ABC is a triangle, there is one in-triangle which has its sides parallel, and one in-triangle which has its sides antiparallel, to the sides of ABC, and these triangles have a common circumcircle, viz., the nine-point circle of ABC.

Again, there are two in-triangles which have their sides respectively positive and negative oblique isoscelians, and these triangles have a common circumcircle, viz., the sine-triple-angle circle.

2. It is the object of the following note to consider some properties of the six in-triangles, circumscribed in pairs by three circles, which have their sides one a parallel, one an antiparallel, and the remaining one a positive or negative isoscelian.

3. In the figure, EF is parallel to BC, ED antiparallel to AB with respect to C, and DF a positive isoscelian to LB.



The triangle DEF is evidently similar to ABC, and, if its sides are a', b', c', we have

$$\frac{b}{c} = \frac{b'}{c'} = \frac{DE}{EF};$$

but

$$\frac{DE}{CE} = \frac{c}{a} \quad \text{and} \quad \frac{EF}{AE} = \frac{a}{b},$$

therefore $$\frac{AE}{CE} = \frac{c^2}{a^2},$$

i.e., BE is the symmedian through B.

4. Now, because $$\frac{EF}{a} = \frac{AE}{b} = \frac{c^2}{a^2+c^2},$$

therefore modulus of similarity is

$$ac/(a^2+c^2) = ac/b, \text{ say}.$$

Hence
$$a' = a^2c/b_1 = DF,$$
$$b' = abc/b_1 = DE,$$
$$c' = ac^2/b_1 = EF.$$

Again,
$$\left.\begin{array}{l} BD = 2a^2c\cos B/b_1 \\ CE = a^2b/b_1 \\ AF = c^3/b_1 \end{array}\right\} \quad \left.\begin{array}{l} CD = ab^2/b_1 \\ AE = bc^2/b_1 \\ BF = ca^2/b_1 \end{array}\right\} \text{ ....(1).}$$

5. The trilinear coordinates are for

$$\left.\begin{array}{llll} D, & (0, & b, & 2a\cos B) \\ E, & (a, & 0, & c) \\ F, & (ab, & c^2, & 0) \end{array}\right\} \text{mod.} \equiv b_1 \text{ ....(2).}$$

6. Now let us consider the triangle $D'E'F'$, in which $E'D'$ is parallel to AB, $E'F'$ the antiparallel to BC with respect to A, and $D'F'$ a negative isoscelian to LB.

Then we can show, as before, that $D'E'F'$ is similar to ABC, that BE' is the symmedian through B, *i.e.*, E and E' coincide, and that the modulus of similarity is ac/b_1, as before.

Hence $$ED' = a', \quad EF' = b', \quad F'D' = c';$$

and
$$\left.\begin{array}{l} BD' = ac^2/b_1, \\ AF' = b^2c/b_1, \end{array}\right\} \quad \left.\begin{array}{l} CD' = a^3/b_1, \\ BF' = 2ac^2\cos B/b_1. \end{array}\right\}$$

7. Now $$BD.BD' = 2c^3a^3\cos B/b_1^2 = BF.BF',$$

and $$CD.CD' = a^4b^2/b_1^2 = CE^2;$$

therefore the two triangles have a common circumcircle which touches AC at E.

It follows therefore that there are two other circles circumscribed to pairs of triangles, similar to ABC, and which have contact respectively with BC, AB, at the points where they are cut by the symmedians through A, C.

8. The trilinear coordinates are for

$$\left.\begin{array}{ll} D', & (0,\ a^2,\ bc) \\ E', & (a,\ 0,\ c) \\ F', & (2c\cos B,\ b,\ 0) \end{array}\right\} \text{mod.} \equiv b_1 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

9. From (2), (3), we get the equations to DF', $D'E$, viz.,

$$ab\alpha - 2ca\cos B\beta + bc\gamma = 0,$$

$$ca\alpha + bc\beta - a^2\gamma = 0;$$

these intersect on $$c\beta = b\gamma,$$

i.e., on the symmedian through A, in p_2 say.

In like manner, DF' and EF can be shown to intersect on the symmedian through C, in r_2 say; hence the triangle Ep_2r_2 is in perspective with ABC, and has K for the centre of perspective, and for its symmedian point. It follows, then, that the circles we are considering are Tucker circles (*cf.* Milne's *Companion*, p. 136).

10. The equation to the circle DEF is

$$(a^2+c^2)^2 \Sigma(\alpha\beta\gamma) = (\Sigma a\alpha)\left[bc^3\alpha + 2c^2a^2\cos B\beta + a^3b\gamma\right] \ldots\ldots (4).$$

If O_2 be the centre of this circle, and ρ_2 [$= Rca/b_1$] its radius, then the coordinates of O_2 are

$$\rho_2\cos(A-B),\quad \rho_2,\quad \rho_2\cos(B-C);$$

this evidently lies on the line

$$\Sigma\left[bc(b^2-c^2)\alpha\right] = 0;$$

i.e., on the circum-Brocardal axis, as it should do, by § 9.

In like manner, O_3, O_1, the other centres, are

$$O_3,\quad \cos(C-A),\ \cos(B-C),\ 1;$$

$$O_1,\quad 1,\ \cos(A-B),\ \cos(C-A).$$

11. The coordinates of the "nine-point centre" of ABC are given by

$$\cos(B-C), \quad \cos(C-A), \quad \cos(A-B),$$

and therefore its inverse point by

$$1/\cos(B-C), \quad 1/\cos(C-A), \quad 1/\cos(A-B);$$

hence B_1O_2 and this inverse point are collinear. Similar results hold for O_3, O_1; hence we may see that AO_1, BO_2, CO_3 cointersect in the inverse of the nine-point centre.

12. The equation to the Brocard ellipse is

$$\Sigma(b^2c^2a^2) = 2abc\,\Sigma(a\beta\gamma),$$

and from § 9 we know that the circle DEF has double contact with the ellipse, viz., at E and at b'', given by

$$a(a^2-b^2)^2, \quad b(c^2-a^2)^2, \quad c(b^2-c^2)^2.$$

Similarly c'' is $\quad a(c^2-a^2)^2, \quad b(b^2-c^2)^2, \quad c(a^2-b^2)^2;$

and a'', $\quad a(b^2-c^2)^2, \quad b(a^2-b^2)^2, \quad c(c^2-a^2)^2.$

13. The equation to Eb'', which is, of course, parallel to the join of the Brocard points, because this last is the major axis of the ellipse, is

$$bca(c^2-a^2)+ca\beta(c^2+a^2-2b^2)-ab\gamma(c^2-a^2) = 0.$$

A slight consideration of the figure shows that the triangle $a''b''c''$ is congruent to the triangle formed by joining the feet of the symmedians of ABC.

Further, Aa'', Bb'', Cc'' cointersect in

$$a/(b^2-c^2)^2, \quad b/(c^2-a^2)^2, \quad c/(a^2-b^2)^2.$$

14. The equations to the tangents at a'', b'', c'', are

$$\frac{bca}{b^2-c^2}+\frac{ca\beta}{a^2-b^2}+\frac{ab\gamma}{c^2-a^2} = 0 \text{ ....(A)},$$

$$\frac{bca}{a^2-b^2}+\frac{ca\beta}{c^2-a^2}+\frac{ab\gamma}{b^2-c^2} = 0 \text{ ....(B)},$$

$$\frac{bca}{c^2-a^2}+\frac{ca\beta}{b^2-c^2}+\frac{ab\gamma}{a^2-b^2} = 0 \text{ ....(C)}.$$

These tangents intersect two and two in

$$a/(c^2-a^2),\quad b/(b^2-c^2),\quad c/(a^2-b^2) \;\;\ldots\ldots\ldots(\text{A, B});$$
$$a/(b^2-c^2),\quad b/(a^2-b^2),\quad c/(c^2-a^2) \;\;\ldots\ldots\ldots(\text{B, C});$$
$$a/(a^2-b^2),\quad b/(c^2-a^2),\quad c/(b^2-c^2)\ldots\ldots\ldots\ldots(\text{C, A}).$$

15. Since the equations to DF, $D'F'$ are

$$2c^2\cos B\alpha-2ab\cos B\beta+b^2\gamma=0,$$
$$b^2\alpha-2bc\cos B\beta+2a^2\cos B\gamma=0,$$

it follows that the join of B to their intersection, and the like joins for the other angles, cointersect in

$$a/(b^2-c^2),\quad b/(c^2-a^2),\quad c/(a^2-b^2).$$

16. The figure shows that $D'F$ is antiparallel to AC with reference to B; hence the circle DEF circumscribes an in-triangle with two sides EF, ED', parallel to BC, AB, and the third side the antiparallel $D'F$.

The Dioptrics of Gratings. *By* J. Larmor, F.R.S.

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When a beam of light falls upon a ruled or striated surface, a considerable portion of it is inevitably scattered and lost by the inequalities of the surface; and the residue is reflected or refracted in the ordinary manner. But when the striation varies from point to point in a continuous and fairly uniform way, there is sifted out from the incident beam, in addition to the *débris* of scattered light, a series of regular secondary beams, which are propagated onwards in directions inclined to that of the principal one.

The origin of such a diffracted beam, by the union of the diffracted parts from the different striae which arrive at its front in the same phase, was fully explained by Thomas Young, as also was the very perfect separation of the different chromatic constituents of a regular compound beam by a good grating of this kind. In the few pregnant