ON SEQUENCES WHICH DETERMINE THE n^{TH} ROOT OF A RATIONAL NUMBER

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1. Dedekind in his *Tract on Continuity and Irrational Numbers* gives a relation from which two sequences of rational numbers may be found, one sequence consisting of numbers whose square is less than a certain rational number D, and the other of numbers whose square is greater than D. The first sequence has no greatest number, and the second sequence has no least number, provided there is no rational number which has its square equal to D.

The two sequences together, or, as will afterwards be shown, either of them separately, define the irrational number which we call the square root of D.

Dedekind's formula for getting one member of a sequence from the preceding one is

$$y = \frac{x^3+3Dx}{3x^2+D}, \tag{I.}$$

which gives $y-x = \dfrac{2x(D-x^2)}{3x^2+D}$ and $y^2-D = \dfrac{(x^2-D)^3}{(3x^2+D)^2}$;

whence it follows that, if we take a member x of the lower sequence, such that $x^2 < D$, then we get another rational number y, such that $y^2 < D$ and $y > x$.

Similarly, if $x^2 > D$, it follows that $y^2 > D$ and $y < x$. Prof. M. J. M. Hill has observed that Dedekind's result (I.) can be obtained by writing out the binomial expansion for $(y-x)^3$, namely, $y^3-3y^2x+3yx^2-x^3$, substituting D for y^2 wherever it occurs, and equating to zero.

We then get $yD-3Dx+3yx^2-x^3 = 0$, which is identical with (I.).

The first section of this paper deals with a set of formulæ which effect nothing more than Dedekind's formula (I.), but include it as a particular case. They were suggested by the above mentioned remark of Prof. Hill. The second section of the paper deals with a generalization of the formulæ when n-th powers are considered instead of squares.

SECTION I.

2. The results of this part of the paper were originally proved by a rather long process, which is now replaced by short proofs kindly placed at my disposal by Mr. A. E. Western. These proofs are given below. If we write down the binomial expansion for $(x-y)^n$, replace all even powers of y (y^{2r}) by powers of D (D^r) and all odd powers of y (y^{2r+1}) by the product of y and powers of D (yD^r), the equation $(x-y)^n = 0$ then becomes a simple equation for y, viz.,

$$x^n - {}_nC_1 x^{n-1} y + {}_nC_2 x^{n-2} D + \ldots + {}_nC_{2r} x^{n-2r} D^r - {}_nC_{2r+1} x^{n-2r-1} D^r y + \ldots + \begin{pmatrix} D^{\frac{1}{2}n} \text{ if } n \text{ be even} \\ yD^{\frac{1}{2}(n-1)} \text{ if } n \text{ be odd} \end{pmatrix} = 0.$$

Putting
$$y_1 = x^n + {}_nC_2 x^{n-2} D + \ldots + {}_nC_{2r} x^{n-2r} D^r + \ldots,$$
$$y_2 = {}_nC_1 x^{n-1} + {}_nC_3 x^{n-3} D + \ldots + {}_nC_{2r+1} x^{n-2r+1} D^r + \ldots,$$

the equation becomes $y = y_1/y_2$. Now

$$y_1 + y_2 D^{\frac{1}{2}} = (x + D^{\frac{1}{2}})^n, \qquad y_1 - y_2 D^{\frac{1}{2}} = (x - D^{\frac{1}{2}})^n;$$

therefore
$$y_1^2 - y_2^2 D = (x^2 - D)^n$$

or
$$y^2 - D = \frac{(x^2-D)^n}{y_2^2}. \qquad \text{(II.)}.$$

Again, we can write

$$2y_1 = (x + D^{\frac{1}{2}})^n + (x - D^{\frac{1}{2}})^n, \qquad 2y_2 D^{\frac{1}{2}} = (x + D^{\frac{1}{2}})^n - (x - D^{\frac{1}{2}})^n;$$

therefore $2D^{\frac{1}{2}}(y_1 - y_2 x) = (D - x^2)\left[(x + D^{\frac{1}{2}})^{n-1} - (x - D^{\frac{1}{2}})^{n-1}\right]$,

which becomes

$$y - x = \frac{(D - x^2)\left\{{}_{n-1}C_1 x^{n-2} + \ldots + {}_{n-1}C_{2r+1} x^{n-2r-2} D^r + \ldots\right\}}{y_2}. \qquad \text{(III.)}$$

Both the denominator and the co-factor of $D - x^2$ in the numerator are essentially positive.

Now, in order to get sequences of the kind required, it is necessary that (i.), when $x^2 < D$, $y > x$, and, when $x^2 > D$, $y < x$. The relation (III.) always satisfies these conditions. Also (ii.), when $x^2 < D$, $y^2 < D$, and $x^2 > D$, $y^2 > D$. The relation (II.) satisfies both of these conditions only if n be odd; it satisfies the latter condition only if n be either even or odd.

Hence the generalized form of Dedekind's expression will determine sequences defining an irrational square root, provided n be an odd integer.

SECTION II.

3. We will now investigate an expression for defining an irrational n-th root of a rational number.

The conditions to be satisfied may be put in the following form:—

$$x^n - D = (x-y)\psi, \qquad y^n - D = (x-y)\phi, \tag{IV.}$$

where ϕ and ψ are positive rational functions of x, D. Then we see that, if

$$\left.\begin{array}{l} \text{(i.) } x^n < D, \text{ then } y > x, \text{ and therefore } y^n < D \\ \text{(ii.) } x^n > D, \text{ then } y < x, \text{ and therefore } y^n > D \end{array}\right\}. \tag{A}$$

We have to make the two equations (IV.) consistent.

This will be the case if $x^n - y^n = (x-y)(\psi - \phi)$,

$$\psi - \phi = x^{n-1} + x^{n-2}y + \dots + x^{n-r-1}y^r + \dots + y^{n-1};$$

therefore
$$\phi = \psi - (x^{n-1} + x^{n-2}y + \dots + x^{n-r-1}y^r + \dots + y^{n-1})$$
$$= \psi - \frac{x^n + x^{n-1}y + \dots + x^{n-r}y^r + \dots + xy^{n-1}}{x}.$$

Now we must make ϕ positive.

Consider first the case $x^n < D$, $y > x$, $y^n < D$; then

$$\frac{x^n + x^{n-1}y + \dots + x^{n-r}y^r + \dots + xy^{n-1}}{x} < \frac{ny^n}{x} < \frac{nD}{x}.$$

Secondly, when $x^n > D$, $y < x$, then

$$\frac{x^n + x^{n-1}y + \dots + x^{n-r}y^r + \dots + xy^{n-1}}{x} < \frac{nx^n}{x}.$$

Hence in either case

$$\frac{x^n + x^{n-1}y + \dots + x^{n-r}y^r + \dots + xy^{n-1}}{x} < n\,\frac{x^n + D}{x}.$$

If we put $\psi = n\dfrac{x^n + D}{x}$, then ψ itself will be a rational positive function of x, D, and ϕ will also be rational and positive.

Also the two equations (IV.) will be consistent when

$$\phi = \frac{x^n + D}{x}n - \frac{x^n + x^{n-1}y + \dots + x^{n-r}y^r + \dots + xy^{n-1}}{x}.$$

Substituting for ψ in (IV.),

$$x^n - D = (x-y)\,\frac{n(x^n + D)}{x}, \tag{V.}$$

$$x - y = \frac{(x^n - D)x}{n(x^n + D)},$$

$$y = x\left\{\frac{D-x^n+nx^n+nD}{n(x^n+D)}\right\} = x\,\frac{(n-1)\,x^n+(n+1)\,D}{n\,(x^n+D)}$$

$$= \frac{(n-1)\,x^{n+1}+(n+1)\,Dx}{n\,(x^n+D)}. \qquad \text{(VI.)}$$

This will be the required expression.

There are two equations marked (IV.). The values selected for ϕ and ψ are such as to make these two equations the same, viz.,

$$x^n - D = (x-y)\,\frac{n\,(x^n+D)}{x}.$$

It would appear at first sight, then, that this equation could be transformed into both the forms of (IV.); but, on attempting to put it into the form $y^n - D = (x-y)$ (positive rational function of x, D), we see that we get, on multiplication by $x^{n-1}+x^{n-2}y+\ldots+y^{n-1}$,

$$(x^n-D)\,(x^{n-1}+x^{n-2}y+\ldots+y^{n-1}) = (x^n-y^n)\,\frac{n\,(x^n+D)}{x};$$

therefore $$y^n - x^n = \frac{D-x^n}{D+x^n}\,\frac{x^n+x^{n-1}+\ldots+xy^{n-1}}{n};$$

therefore $$y^n - D = \frac{(x^n-D)\,x}{n\,(x^n+D)}\left\{\frac{x^n+D}{x}\,n - \frac{x^n+x^{n-1}y+\ldots+xy^{n-1}}{x}\right\}. \qquad \text{(VII.)}$$

Call the part in $\{\ \}$ L.

Then in the case $x^n < D$ the first equation of (IV.) shows that $y > x$, and the relation (VII.) shows that $y^n < D$, provided L be positive.

But L has been shown to be positive only if we first assume that $y^n < D$. Hence it requires some other method to show that $y^n < D$ when $x^n < D$.

If, however, $x^n > D$, then, as before, $y < x$, and L is now positive whether $y^n >$ or $< D$; hence the relation (VII.) shows that $y^n > D$.

It is not necessary, therefore, in this latter case to verify that the conditions are satisfied by the relation (VI.), provided n be integral. An independent mode of proof has been adopted.

4. Before proceeding to this we will give the verification of the result in the cases when $n = 2, 3, 4$.

(a) When $n = 2$.—

$$y = \frac{x^3+3Dx}{2\,(x^2+D)}, \qquad y - x = \frac{(D-x^2)\,x}{2\,(x^2+D)},$$

$$y^2 - D = \frac{(x^2-D)\,(x^4+3x^2D+4D^2)}{4\,(x^2+D)^2}.$$

(β) When $n = 3$.—

$$y = \frac{2x^4+4Dx}{3(x^3+D)}, \quad y-x = \frac{D-x^3}{3(x^3+D)}x,$$

$$y^3-D = \frac{(x^3-D)(8x^9+29x^6D+44x^3D^2+27D^3)}{27(x^3+D)^3}.$$

(γ) When $n = 4$.—

$$y = \frac{3x^5+5Dx}{4(x^4+D)}, \quad y-x = \frac{D-x^4}{4(x^4+D)}x,$$

$$y^4-D = \frac{(x^4-D)(81x^{16}+365x^{12}D+691x^8D^2+655x^4D^3+256D^4)}{256(D+x^4)^4}.$$

In all three cases the conditions (A) are satisfied.

5. Proceeding to the proof of the general case, we have

$$y = x\left[1-\frac{x^n-D}{n(x^n+D)}\right];$$

therefore $$y^n-D = x^n\left[1-\frac{x^n-D}{n(x^n+D)}\right]^n-D.$$

Now we wish to show that (i.), when $x^n < D$, $y^n < D$; (ii.), when $x^n > D$, $y^n > D$.

Case (i.). When $x^n < D$.—We must show that

$$x^n\left[1+\frac{D-x^n}{n(x^n+D)}\right]^n-D < 0.$$

Put $\dfrac{D-x^n}{D+x^n} = a$, where $0 < a < 1$. Then $\dfrac{D}{x^n} = \dfrac{1+a}{1-a}$. Hence we must show that

$$\left(1+\frac{a}{n}\right)^n - \frac{1+a}{1-a} < 0 \quad \text{or} \quad \frac{1+a}{1-a} > \left(1+\frac{a}{n}\right)^n.$$

Now we have, if a and b be positive and m a positive quantity > 1,

$$ma^{m-1}(a-b) > a^m-b^m > mb^{m-1}(a-b). \qquad \text{(VIII.)}$$

(See Chrystal's *Algebra*, chapter xxiv., § 7.)

Putting $a = 1+\dfrac{a}{y}$, $b = 1$, $m = \dfrac{y}{z}$, where $y > z$,

$$\left(1+\frac{a}{y}\right)^{y/z}-1 > \frac{y}{z}\,\frac{a}{y};$$

therefore $\left(1+\dfrac{a}{y}\right)^{y/z} > 1+\dfrac{a}{z}$, $\left(1+\dfrac{a}{y}\right)^y > \left(1+\dfrac{a}{z}\right)^z$, where $y > z$.

Hence the expression $(1+a/n)^n$ continually increases as we increase n.

Also $$\underset{n=\infty}{\mathrm{L}}\left(1+\frac{a}{n}\right)^n = e^a.$$

Hence, dealing only with finite values of n,

$$\left(1+\frac{a}{n}\right)^n < \underset{n=\infty}{\mathrm{L}}\left(1+\frac{a}{n}\right)^n$$
$$< e^a$$
$$< 1+a+\frac{a^2}{2!}+\ldots+\frac{a^r}{r!}+\ldots,$$

which is an absolutely convergent series $< 1+a+a^2+\ldots+a^r+\ldots$, which converges when $|a| < 1$. Hence $\left(1+\frac{a}{n}\right)^n < 1-a$, and, since $1+a > 1$,

therefore $$\left(1+\frac{a}{n}\right)^n < \frac{1+a}{1-a}.$$

Hence we have proved what was required.

Case (ii.). When $x^n > D$.—We must show that

$$x^n\left[1-\frac{x^n-D}{n(x^n+D)}\right]^n - D > 0.$$

Putting $\dfrac{x^n-D}{x^n+D} = a$, so that $0 < a < 1$, $\dfrac{D}{x^n} = \dfrac{1-a}{1+a}$. Now, from the formula (VIII.), putting $a = 1$, $b = 1-a/y$, $m = y/z$, where $y > z$,

$$\frac{y}{z}\,\frac{a}{y} > 1-\left(1-\frac{a}{y}\right)^{y/z};$$

therefore $\left(1-\dfrac{a}{y}\right)^{y/z} > \left(1-\dfrac{a}{z}\right)$, $\left(1-\dfrac{a}{y}\right)^y > \left(1-\dfrac{a}{z}\right)^z$, where $y > z$.

Hence the function $(1-a/n)^n$ continually increases as we increase n.

Hence $$(1-a/n)^n > 1-a, \quad \text{where} \quad n > 1;$$

and $1+a > 1$, therefore $$1-a > \frac{1-a}{1+a};$$

therefore $$\left(1-\frac{a}{n}\right)^n > \frac{1-a}{1+a}, \qquad \left(1-\frac{a}{n}\right)^n - \frac{1-a}{1+a} > 0,$$

which is the required result.

We have, therefore, shown that, if

$$y = \frac{(n-1)\,x^{n+1}+(n+1)\,Dx}{n(x^n+D)},$$

then (i.), if $x^n < D$, $y > x$ and $y^n < D$; (ii.), if $x^n > D$, $y < x$ and $y^n > D$. This has been proved for all cases in which n is any quantity greater than unity.

6. A more general expression for y may be given. For, if

$$\phi = \psi - \frac{x^n + x^{n-1}y + \ldots + x^{n-r}y^r + \ldots + xy^{n-1}}{x}$$

and ϕ is positive when $\psi = n\dfrac{x^n+D}{x}$, then clearly ϕ will be positive if $\psi = n\dfrac{x^n+D}{x}$ + any rational positive function of x, D. Call this function F; then $\psi = n\dfrac{x^n+D}{x}+F$.

Substituting in the first equation (IV.),

$$x^n - D = (x-y)\left[\frac{n(x^n+D)}{x}+F\right], \qquad y - x = \frac{(D-x^n)x}{n(x^n+D)+xF},$$

$$y = \frac{(n-1)x^{n+1}+(n+1)Dx+x^2F}{n(x^n+D)+xF}. \tag{IX.}$$

It may readily be shown, by the help of the results proved, that this value for y will satisfy the given conditions.

In the first place, if $x^n < D$, $y > x$, and, if $x^n > D$, $y < x$. Again,

$$y = x\left[1+\frac{D-x^n}{n(x^n+D)+xF}\right].$$

Case (i.) When $x^n < D$.—We wish to show that $y^n < D$. Now

$$1+\frac{D-x^n}{n(x^n+D)+xF} < 1+\frac{D-x^n}{n(x^n+D)};$$

therefore $$\left[1+\frac{D-x^n}{n(x^n+D)+xF}\right]^n < \left[1+\frac{D-x^n}{n(x^n+D)}\right]^n.$$

But we have shown that

$$x^n\left[1+\frac{D-x^n}{n(x^n+D)}\right]^n - D < 0;$$

therefore $$x^n\left[1+\frac{D-x^n}{n(x^n+D)+xF}\right]^n - D < 0;$$

therefore $y^n - D < 0$, $y^n < D$, which we wished to prove.

Case (ii.). $x^n > D$.—We wish to show that $y^n > D$.

$$y = x\left[1-\frac{x^n-D}{n(x^n+D)+xF}\right].$$

Now
$$1-\frac{x^n-D}{n(x^n+D)+xF}>1-\frac{x^n-D}{n(x^n+D)};$$

therefore $x^n\left[1-\frac{x^n-D}{n(x^n+D)+xF}\right]^n>\left[1-\frac{x^n-D}{n(x^n+D)}\right]^n$, if $n>1$.

But we have shown that
$$x^n\left[1-\frac{x^n-D}{n(x^n+D)}\right]^n-D>0;$$

therefore
$$x^n\left[1-\frac{x^n-D}{n(x^n+D)+xF}\right]^n-D>0;$$

therefore $y^n-D>0$, $y^n>D$, which we wished to prove.

7. Two sequences of rational numbers are therefore determined by the relation (VI.), one in ascending and the other in descending order of magnitude.

Let the ascending sequence be $x_1, x_2, \ldots, x_m, \ldots$, such that, if $0<x_1^n<D$, $x_1<x_2<\ldots<x_m$ and $x_m^n<D$; therefore $(x_1^n, x_2^n, \ldots)$ converges, and its limit $\leqslant D$. Similarly, let the descending sequence be $x_1', x_2', \ldots, x_m', \ldots$, where $(x_1'^n, x_2'^n, \ldots)$ converges and its limit $\geqslant D$. It will be shown that both sequences converge to the same limit.

If we start with x_1, we can find another number x_2 of the ascending sequence from the relation
$$x_2=\frac{(n-1)x_1^{n+1}+(n+1)Dx_1}{n(x_1^n+D)};$$

then
$$x_3=\frac{(n-1)x_2^{n+1}+(n+1)Dx_2}{n(x_2^n+D)},$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots,$$
$$x_{m+1}=\frac{(n-1)x_m^{n+1}+(n+1)Dx_m}{n(x_m^n+D)}.$$

Thus
$$x_{m+1}=x_m\left\{1+\frac{D-x_m^n}{n(x_m^n+D)}\right\};$$

therefore
$$x_{m+1}-x_m=\frac{D-x_m^n}{n(x_m^n+D)}x_m.$$

Let $x_{m+1}-x_m=\delta_m$. Since $x_{m+1}=x_1+\delta_1+\delta_2+\ldots+\delta_m$ and $x_{m+1}^n \not> D$, therefore $x_1+\delta_1+\delta_2+\ldots+\delta_m+\ldots$ ad inf. is a series which converges to some limit E, where, at present, all we know of the limit is that $E^n\leqslant D$.

Now it is a necessary condition for the convergence of the series $\sum_{m=1}^{\infty} \delta_m$ that after a certain term δ_m, where $m \geqslant \mu$, $|\delta_m| < \epsilon$, an arbitrarily small positive quantity. Hence we must have, after a certain term,

$$\left| \frac{D-x_m^n}{n(x_m^n+D)} x_m \right| < \epsilon,$$

$$|D-x_m^n| < \frac{|n(x_m^n+D)|}{|x_m|} \epsilon < \frac{2nD}{x_1} \epsilon;$$

therefore $|D-x_m^n| < \eta$, where η is arbitrarily small. Hence, provided we do not start with x_1 zero, x_m^n converges to the limit D as m is increased. Therefore the ascending sequence $x_1, x_2, \ldots, x_m, \ldots$ converges to the limit E, where $E^n = D$.

In the same way it may be shown that the descending sequence $x_1', x_2' \ldots, x_m', \ldots$ converges to this same limit E. We call E the n-th root of the rational number D. Hence the irrational number is completely defined by either of the two sequences.

8. I am indebted to Prof. Hill for a formula not included in any of the above, for the case $n = 3$, $y-x = \dfrac{x(D-x^3)}{2x^3+D}$. This gives

$$y^3-D = \frac{x^3-D)^3(x^3+D)}{(2x^3+D)^3},$$

which gives, if $x^3 < D$, $y > x$ and $y^3 < D$; but, if $x^3 > D$, $y < x$ and $y^3 > D$. And probably there are a great many other formulæ of a similar nature.