5. On Professor Tait's Problem of Arrangement. By Thomas Muir, M.A.

The problem in question is—To find the number of possible arrangements of a set of n things, subject to the conditions that the first is not to be in the last or first place, the second not in the first or second place, the third not in the second or third place, and so on.

A little consideration serves to show that we may with advantage shift the ground of the problem to the theory of determinants. For the sake of definiteness take the case of *five* things, A, B, C, D, E. Here A may be in the second, third, or fourth places only; B in the third, fourth, or fifth places only; and similarly of the others—a result which we may tabulate thus :—

$$\begin{array}{ccccc} . & A & A & A & . \\ . & . & B & B & B \\ C & . & . & C & C \\ D & D & . & . & D \\ E & E & E & . & . \end{array}$$

an A being written in the places which it is possible for A to occupy, and a dot signifying that the letter found in the same line with it may not occupy its place. Hence, to obtain the various arrangements, we see that for the first place we may have any letter that is in the first column; for the second place any letter that is in the second column, provided it be not in the same line with the letter taken from the first column; for the third place, any letter that is in the third column, provided it be not in the same line with either of the letters previously taken, and so on. This law of formation, however, is identical with that in accordance with which the terms of a determinant are got from the elements of the matrix; so that the problem we are concerned with is transformed into this: Find the number of terms of the determinant of the nth order of the form,—

$$\begin{vmatrix} \cdot & * & * & * & \cdot \\ \cdot & \cdot & * & * & * \\ * & \cdot & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{vmatrix}$$

where the dots and asterisks denote zero elements and non-zero elements respectively.

In the solution of this, determinants of a set of other forms require to be considered, viz.,—

$$\begin{vmatrix} \cdot & * & * & * & * \\ \cdot & \cdot & * & * & * \\ * & \cdot & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{vmatrix}, \quad \begin{vmatrix} \cdot & * & * & * & * \\ * & \cdot & * & * & * \\ * & \cdot & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{vmatrix}, \quad \begin{vmatrix} \cdot & * & * & * & * \\ \cdot & \cdot & * & * & * \\ * & * & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{vmatrix}, \text{ \&c.}$$

in all of which the elements of the main diagonal are zero; the elements of the adjacent minor diagonal being in the first also all zero, in the second all zero except the first element, in the third all zero except the second element, and so on. Let us denote the number of terms in these when they are of the nth order by

$$\chi_0(n)\,,\ \chi_1(n)\,,\ \chi_2(n)\,,\ \ldots .$$

and let the number of terms sought be

$$\Psi(n)\,.$$

Further, let any determinant-form with dots and asterisks stand for *the number of terms* in such a determinant; the four forms above, for example, being thus symbols equivalent to $\Psi(5)$, $\chi_0(5)$, $\chi_1(5)$, $\chi_2(5)$, respectively.

Beginning with the first of the χ forms we see it to be transformable into

$$\begin{vmatrix} \cdot & \ast & \ast & \ast & \cdot \\ \cdot & \cdot & \ast & \ast & \ast \\ \ast & \cdot & \cdot & \ast & \ast \\ \ast & \ast & \cdot & \cdot & \ast \\ \ast & \ast & \ast & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & \ast & \ast \\ \ast & \cdot & \cdot & \ast \\ \ast & \ast & \cdot & \cdot \\ \ast & \ast & \ast & \cdot \end{vmatrix}$$

and the second term of this may in like manner be changed into

$$\begin{vmatrix} \cdot & \cdot & \ast & \ast \\ \ast & \cdot & \cdot & \ast \\ \ast & \ast & \cdot & \cdot \\ \cdot & \ast & \ast & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \ast & \ast \\ \cdot & \cdot & \ast \\ \ast & \cdot & \cdot \end{vmatrix}$$

and the second term of this again into

$$\begin{vmatrix} \cdot & \ast & \cdot \\ \cdot & \cdot & \ast \\ \ast & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot \\ \ast & \cdot \end{vmatrix}$$

the second term of which is zero. Hence we have the result,

$$\chi_0(n) = \Psi(n) + \Psi(n-1) + \Psi(n-2) + \ldots + \Psi(3), \quad . \quad . \quad . \qquad (\alpha)$$

Of the other χ forms it is clear to begin with that

$$\chi_1 = \chi_{n-1}, \qquad \chi_2 = \chi_{n-2}, \quad \ldots .$$

and treating the distinct cases as we have treated χ_0, we equally readily see that

$$\begin{aligned} \chi_1(n) &= \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) \\ \chi_2(n) &= \chi_0(n) + \chi_0(n-1) + \chi_1(n-2) \\ \chi_3(n) &= \chi_0(n) + \chi_0(n-1) + \chi_2(n-2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \chi_{n-2}(n) &= \chi_0(n) + \chi_0(n-1) + \chi_1(n-2) \\ \chi_{n-1}(n) &= \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) \end{aligned}$$

which, on eliminating χ_1, χ_2, ... from the right-hand members, become

$$\chi_1(n) = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2)$$
$$\chi_2(n) = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4)$$
$$\chi_3(n) = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) + \chi_0(n-5) + \chi_0(n-6)$$
....
$$\chi_{n-2}(n) = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4)$$
$$\chi_{n-1}(n) = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2)\,.$$

Here the second of the series of right-hand members has two terms more than the first, the third two terms more than the second, and so on until we approach the middle of the series, when, if n be odd, the two middle right-hand members are found to be the same as the one preceding or the one following them, the whole four ending thus—

$$\ldots. + \chi_0(5) + \chi_0(4) + \chi_0(3)\,;$$

and if n be even, the one middle right-hand member is found to be greater by unity than the one preceding or the one following it, and to end thus—

$$\ldots. + \chi_0(5) + \chi_0(4) + \chi_0(3) + 1\,,$$

the 1 arising from the fact that the above process of reduction, in the case of $\chi_{\frac{1}{2}n}(n)$, leads us finally, not to

$$\begin{vmatrix} \cdot & * & * \\ \cdot & \cdot & * \\ * & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} * & * \\ \cdot & \cdot \end{vmatrix}, \text{ but to } \begin{vmatrix} \cdot & * & * \\ \cdot & \cdot & * \\ * & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & * \\ * & \cdot \end{vmatrix}$$

i.e., to $\chi_0(3) + 1$.

Returning now to the Ψ form, and taking

$$\begin{vmatrix} \cdot & * & * & * & \cdot \\ \cdot & \cdot & * & * & * \\ * & \cdot & \cdot & * & * \\ * & * & \cdot & \cdot & * \\ * & * & * & \cdot & \cdot \end{vmatrix}$$

we transform it into

$$\begin{vmatrix} \cdot & * & * & * \\ * & \cdot & * & * \\ * & \cdot & \cdot & * \\ * & * & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & * & * \\ * & \cdot & * & * \\ * & * & \cdot & * \\ * & * & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & * & * \\ * & \cdot & \cdot & * \\ * & * & \cdot & * \\ * & * & * & \cdot \end{vmatrix}$$

the middle term of which becomes by transposition of the first two rows, and the subsequent transposition of the first two columns,

$$\begin{vmatrix} \cdot & * & * & * \\ \cdot & \cdot & * & * \\ * & * & \cdot & * \\ * & * & \cdot & \cdot \end{vmatrix}$$

Consequently we have

$$\Psi(5) = \chi_1(4) + \chi_2(4) + \chi_3(4) ,$$

and it is easily seen that a similar transformation is possible in every case, giving

$$\Psi(n) = \chi_1(n-1) + \chi_2(n-1) + \chi_3(n-1) + \dots + \chi_{n-2}(n-1) .$$

Expressing $\chi_2, \chi_3, \dots$ in terms of χ_0 by means of what precedes, we have

$$\begin{aligned} \Psi(n) = {} & \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) \\ & + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) + \chi_0(n-5) \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) + \chi_0(n-5) \\ & + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) , \end{aligned}$$

and now using (a), to express χ_0 in terms of Ψ, we find

$$\begin{aligned}\Psi(n) = \quad & \Psi(n-1) + 2\Psi(n-2) + 3\{\Psi(n-3) + \ldots\ldots\ldots\ldots + \Psi(3)\} \\ & + \Psi(n-1) + 2\Psi(n-2) + 3\Psi(n-3) + 4\Psi(n-4) \\ & \qquad\qquad + 5\{\Psi(n-5) + \ldots\ldots\ldots\ldots + \Psi(3)\} \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \Psi(n-1) + 2\Psi(n-2) + 3\Psi(n-3) + 4\Psi(n-4) \\ & \qquad\qquad + 5\{\Psi(n-5) + \ldots\ldots\ldots\ldots + \Psi(3)\} \\ & + \Psi(n-1) + 2\Psi(n-2) + 3\{\Psi(n-3) + \ldots\ldots\ldots\ldots + \Psi(3)\}\end{aligned}$$

where, on the first line the coefficient of the third and all the following terms is 3, on the second line the coefficient of the fifth and all the following terms is 5, on the third line the coefficient of the seventh and all the following terms is 7, and so on, the middle term (when such occurs) having a 1 superadded.

Hence, for the determination of $\Psi(n)$ when $\Psi(n-1)$, $\Psi(n-2)$, ... are known, we have

$$\begin{aligned}\Psi(n) = (n-2)\Psi(n-1) + (2n-\ \ 4)\Psi(n-2) + (3n-\ \ 6)\Psi(n-3) \\ + (4n-10)\Psi(n-4) + (5n-14)\Psi(n-6) \\ + (6n-20)\Psi(n-6) + (7n-26)\Psi(n-7) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + \frac{1-(-1)^n}{2},\end{aligned}$$

where the coefficients proceed for two terms with the common difference $n-2$, for the next two terms with the common difference $n-4$, for the next two terms with the common difference $n-6$, and so on.

And as it is self-evident that $\Psi(2) = 0$, we obtain

$$\begin{aligned}\Psi(3) &= 1\Psi(2) + 1 &&= \quad 1 \\ \Psi(4) &= 2\Psi(3) &&= \quad 2 \\ \Psi(5) &= 3\Psi(4) + 6\Psi(3) + 1 &&= \quad 13 \\ \Psi(6) &= 4\Psi(5) + 8\Psi(4) + 12\Psi(3) &&= \quad 80 \\ \Psi(7) &= 5\Psi(6) + 10\Psi(5) + 15\Psi(4) + 18\Psi(3) + 1 &&= \quad 579 \\ \Psi(8) &= 6\Psi(7) + 12\Psi(6) + 18\Psi(5) + 22\Psi(4) + 26\Psi(3) &&= \quad 4738\end{aligned}$$

and so forth.