7. THE I/O IT:

The Input/Output Impedance Theorem

How to find them directly from the Gain, thereby saving almost two-thirds of the work

Why do we need to deal with input and output impedances?

1. They may be part of the specifications.

2. They describe the interaction between two system blocks, and are therefore components of the Divide and Conquer approach, specifically incorporated in the Chain Theorem.

Definitions of "input" and "output:"

Input and output impedances are transfer functions (TFs), just as is the gain.

A TF is a ratio of one signal in a circuit to another, so the most general definition of "input" and "output" is that the "input" is the signal in the denominator, and the "output" is the signal in the numerator:

$$\frac{\text{"output"}}{\text{"input"}} = \text{transfer function TF}$$

If numerator and denominator are both voltages or currents, the TF is a voltage gain or a current gain; if the numerator is a voltage and the denominator is a current, the TF is a transimpedance (and vice versa for a transadmittance).

If the numerator is the voltage across the same port into which the denominator current flows, the TF is a self-impedance.

The denominator of a TF *is not necessarily* an independent excitation; the independent excitation may be elsewhere.

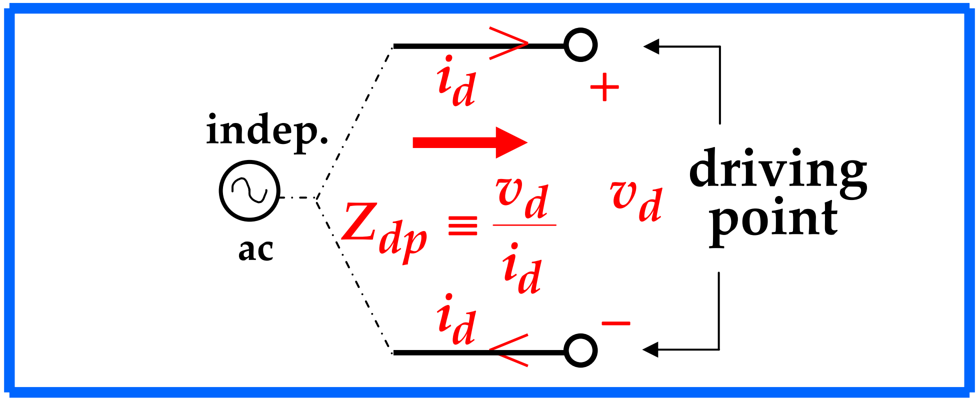
Thus, there are three kinds of "input":

1. **A signal at a port designated as "input"**
2. **An independent excitation**
3. **The denominator of a TF**

Driving Point Impedance

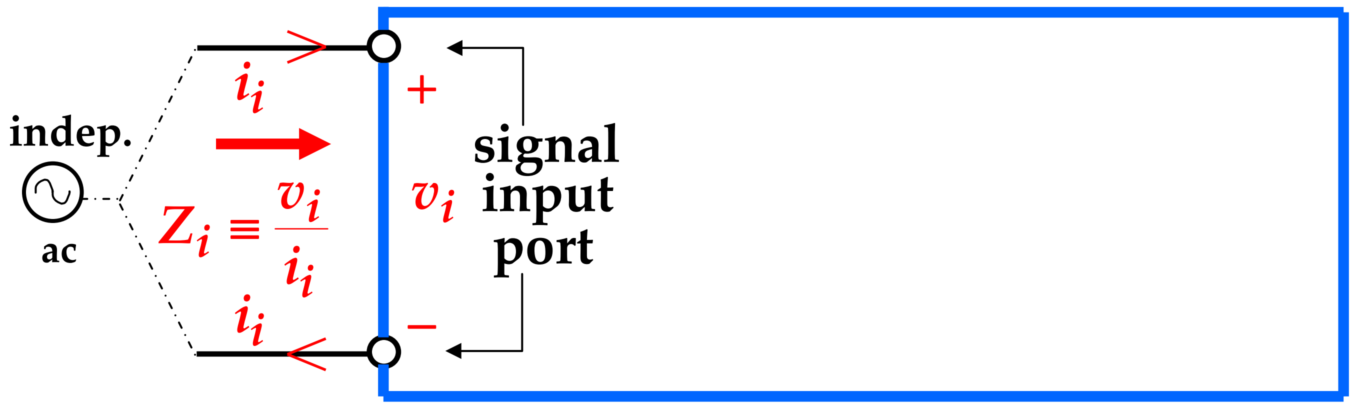
The port at which a circuit is driven is the driving point.

One of the many TFs of interest is the driving point impedance, which is the self-impedance "seen" at the driving point:



A system usually has designated signal "input" and "output" ports:

Input Impedance

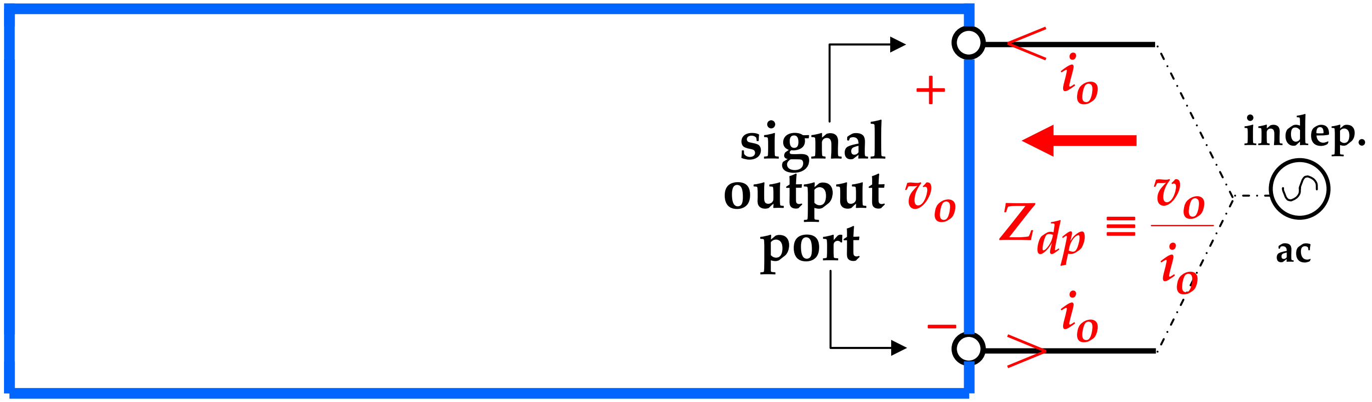


$$Z_{dp} \equiv \frac{v_i}{i_i} = Z_i \equiv \text{input impedance}$$

at the signal input port

Note: the "input" signal for the input impedance TF is i_i, although the input signal for the gain TF may be v_i or i_i, depending upon the definition of the gain.

Output Impedance



$$Z_{dp} \equiv \frac{v_o}{i_o} = Z_o \equiv \textbf{output impedance}$$

at the signal output port

Conventional Approach

Calculate the gain H

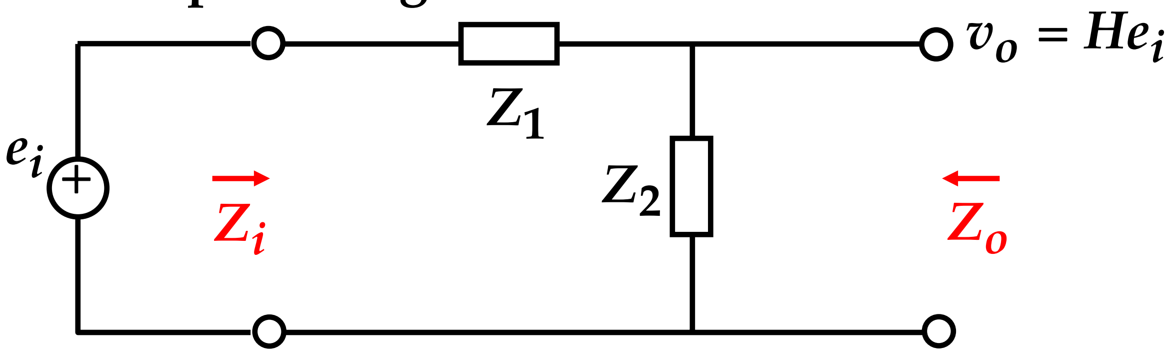
Calculate the output impedance Z_o

Calculate the input impedance Z_i

Usually these are done separately, each starting from scratch, and they may be equally lengthy analyses (especially if there is feedback present).

However, much of the analysis is the same in each case, so there is motivation to find a short cut that avoids the repetitions.

Consider the simple voltage divider:



The three analyses lead to:

$$H = \frac{Z_2}{Z_1 + Z_2} \qquad Z_i = Z_1 + Z_2 \qquad Z_o = Z_1 \| Z_2$$

The "hard part" in each case is calculation of $Z_1 + Z_2$.

However, Z_i and Z_o can be written in terms of H:

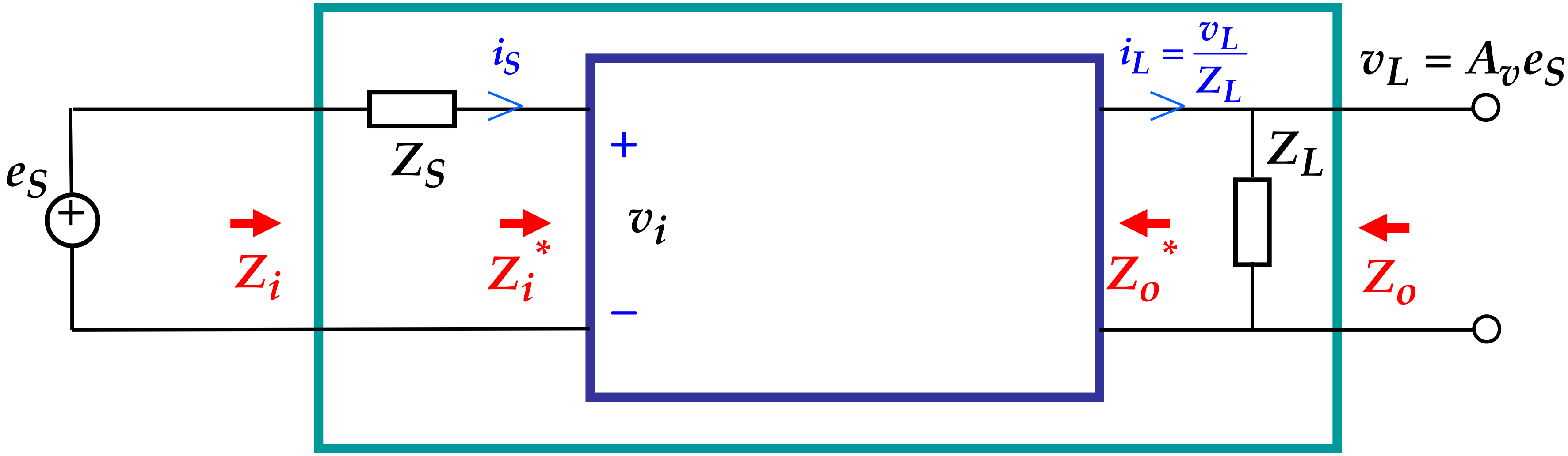
$$Z_i = \frac{Z_2}{H} \qquad Z_o = Z_1 H$$

Thus, the sum $Z_1 + Z_2$ need be calculated only once to find H, and then Z_i and Z_o can be found as products or quotients of H.

This trick doesn't work for more complex circuits, but there is still motivation to find a way to calculate Z_i and Z_o from H instead of starting from scratch.

Inner and Outer Input and Output Impedances

Forward voltage gain $A_v = \frac{v_L}{e_S}$



There are two kinds of input and output impedances, depending on whether the system is defined to include the source and load impedances or not.

outer input impedance $\equiv Z_i$

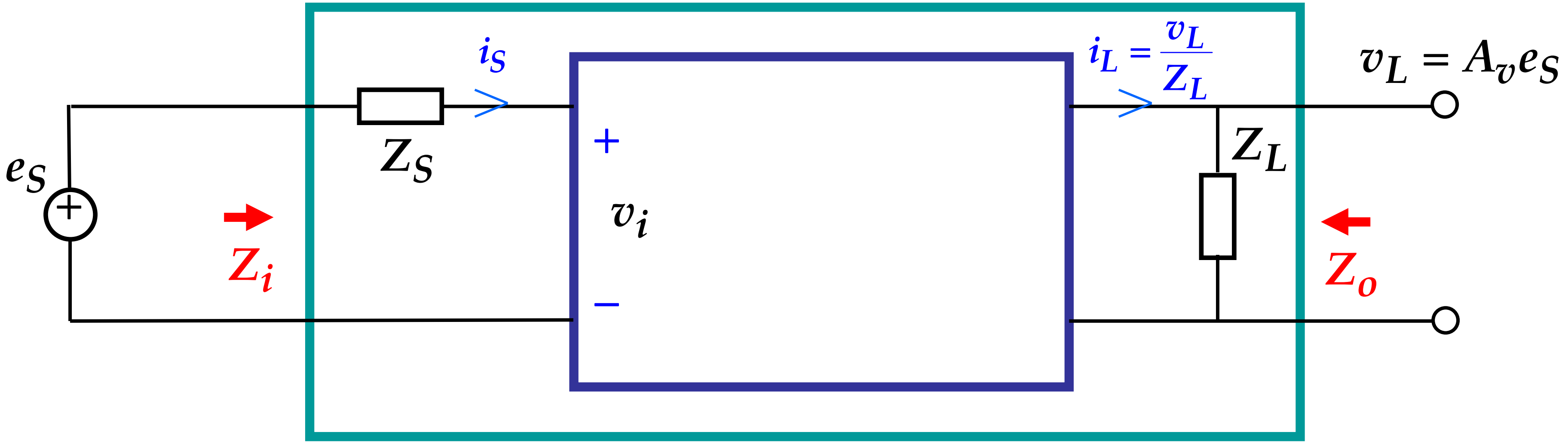
outer output impedance $\equiv Z_o$

inner input impedance $\equiv Z_i^*$

inner output impedance $\equiv Z_o^*$

Output Impedance Theorem

Forward voltage gain $A_v = \frac{v_L}{e_S}$



Forward transadmittance gain $Y_t = \frac{i_L}{e_S} = \frac{v_L}{Z_L e_S} = \frac{A_v}{Z_L}$

Short-circuit forward transadmittance gain $Y_t^{sc} = \left. \frac{A_v}{Z_L} \right|_{Z_L \to 0}$

Output Impedance Theorem

Forward voltage gain $A_v = \frac{v_L}{e_S}$

$$Z_o = \frac{\text{oc output voltage}}{\text{sc output current}} \quad \text{for the same } e_S \quad = \frac{A_v e_S}{Y_t^{sc} e_S}$$

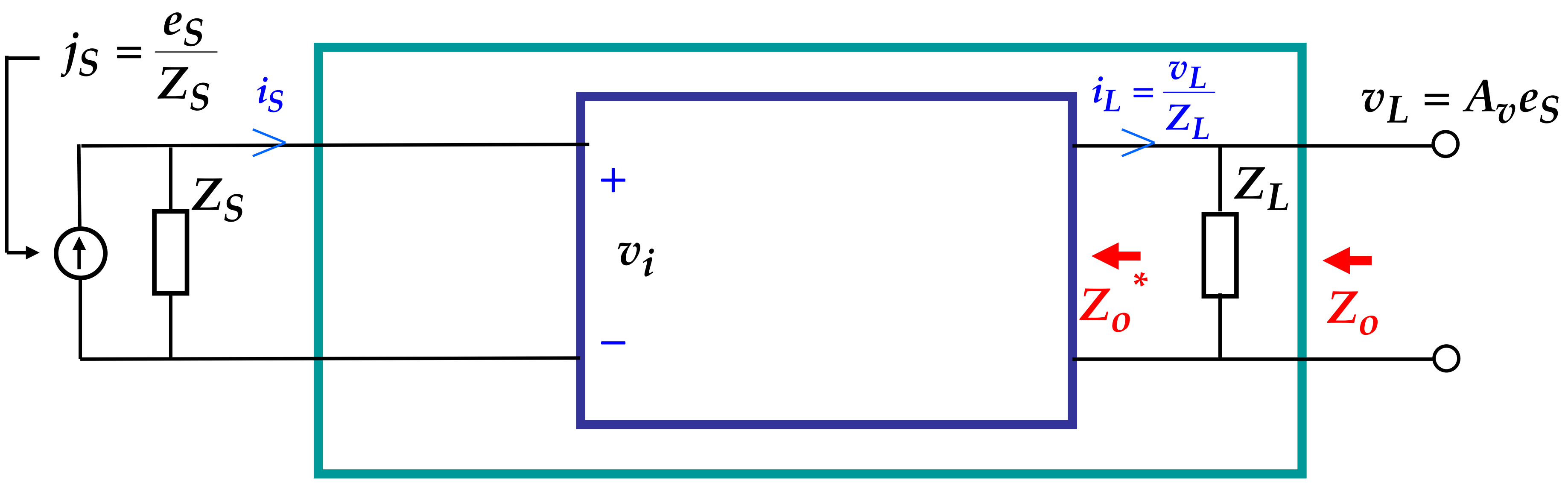
$$Z_o = \frac{A_v}{Y_t^{sc}} \quad = \frac{\text{fwd voltage gain}}{\text{sc fwd transadmittance}}$$

$$Z_o = \frac{A_v}{\left.\frac{A_v}{Z_L}\right|_{Z_L \to 0}}$$

This is the Output Impedance Theorem

Input Impedance Theorem

Convert the Thevenin independent source e_S, Z_S to a Norton equivalent:



Forward transimpedance gain

$$Z_t = \frac{v_L}{i_S} = \frac{v_L}{j_S\big|_{Z_S \to \infty}}$$

Input Impedance Theorem

Forward transimpedance gain

$$Z_t = \frac{v_L}{i_S} = \frac{v_L}{\left. j_S \right|_{Z_S \to \infty}} = \frac{v_L}{\left. \frac{e_S}{Z_S} \right|_{Z_S \to \infty}}$$

Input Impedance Theorem

Forward voltage gain $A_v = \dfrac{v_L}{e_S}$

Forward transimpedance gain

$$Z_t = \frac{v_L}{i_S} = \frac{v_L}{j_S\big|_{Z_S \to \infty}} = \frac{v_L}{\left.\frac{e_S}{Z_S}\right|_{Z_S \to \infty}} = \frac{v_L}{\left.\frac{v_L / A_v}{Z_S}\right|_{Z_S \to \infty}} = Z_S A_v\big|_{Z_S \to \infty}$$

Input Impedance Theorem

Forward voltage gain $A_v = \dfrac{v_L}{e_S}$

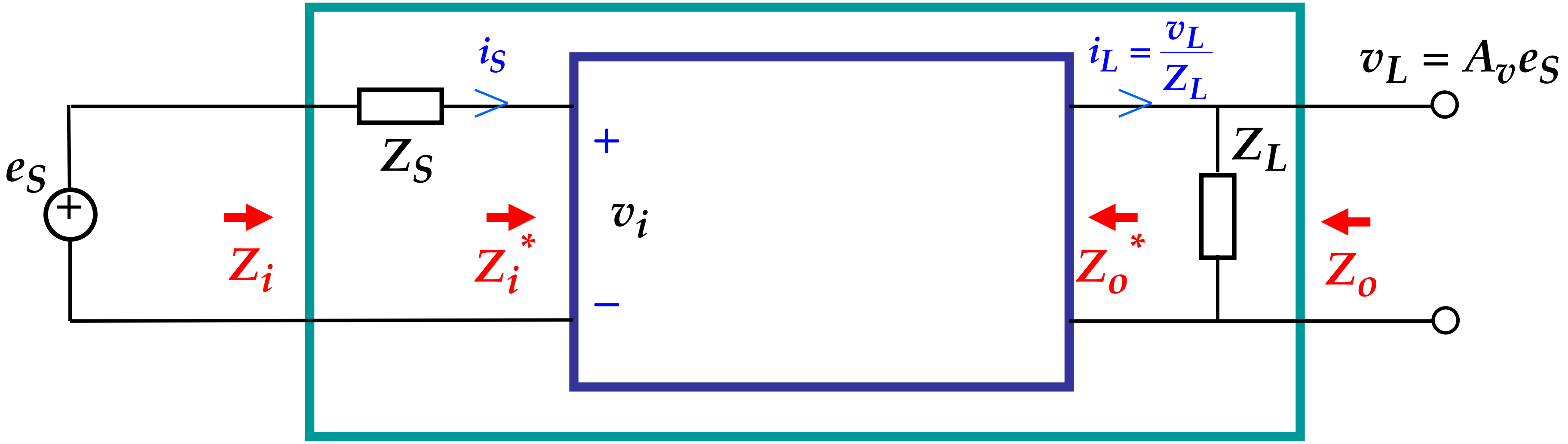
$$Z_i = \frac{\text{input voltage}}{\text{input current}} \quad \text{for the same } v_L \quad = \frac{v_L / A_v}{v_L / Z_t}$$

$$Z_i = \frac{Z_t}{A_v} = \frac{\text{fwd transimpedance}}{\text{fwd voltage gain}}$$

$$Z_i = \frac{Z_S A_v \big|_{Z_S \to \infty}}{A_v}$$

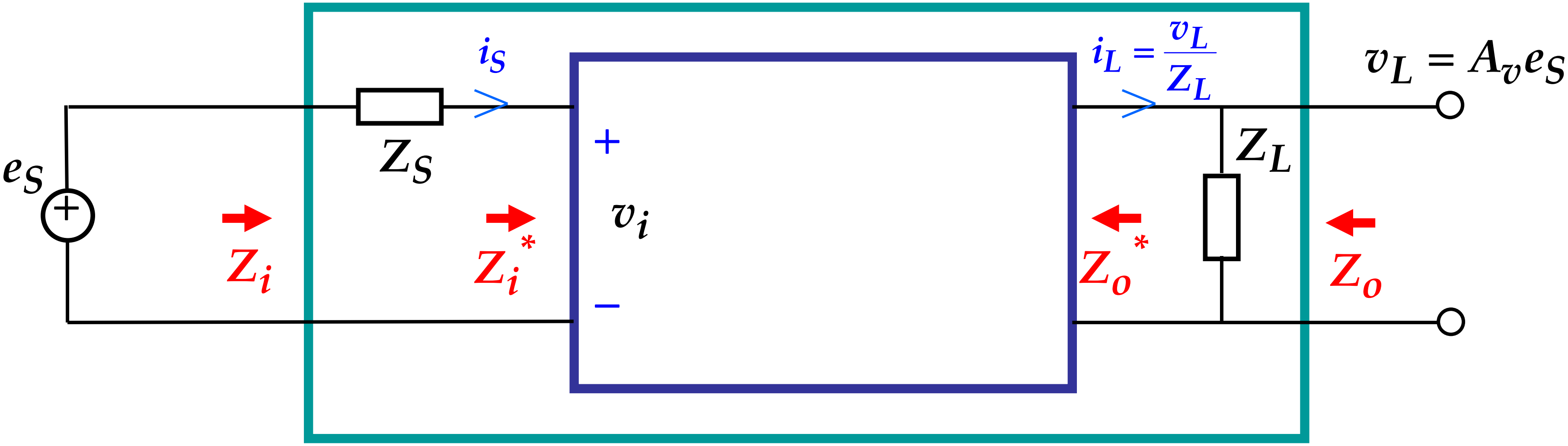
This is the Input Impedance Theorem

Inner and Outer Input and Output Impedances



The value of the formulas is that once the gain is known, only a simple limit with respect to either Z_L or Z_S need be calculated to find the outer output or input impedances Z_o or Z_i.

Inner Input and Output Impedances

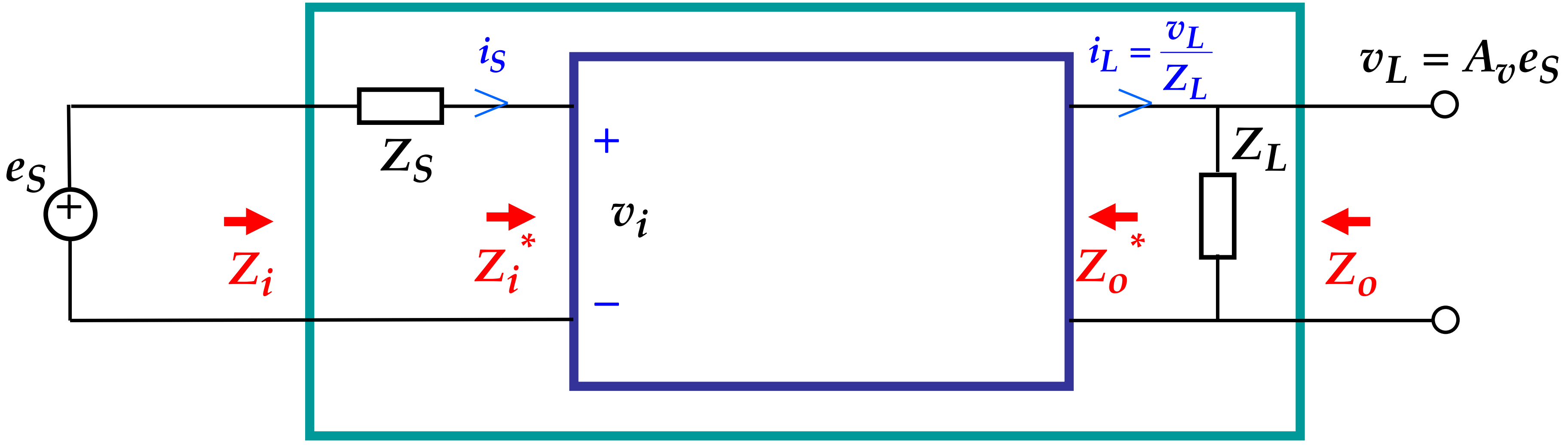


It is obvious that

$$Z_o^* = Z_o\Big|_{Z_L\to\infty} = \left.\frac{A_v}{\left.\frac{A_v}{Z_L}\right|_{Z_L\to 0}}\right|_{Z_L\to\infty} = \frac{A_v\big|_{Z_L\to\infty}}{\left.\frac{A_v}{Z_L}\right|_{Z_L\to 0}}$$

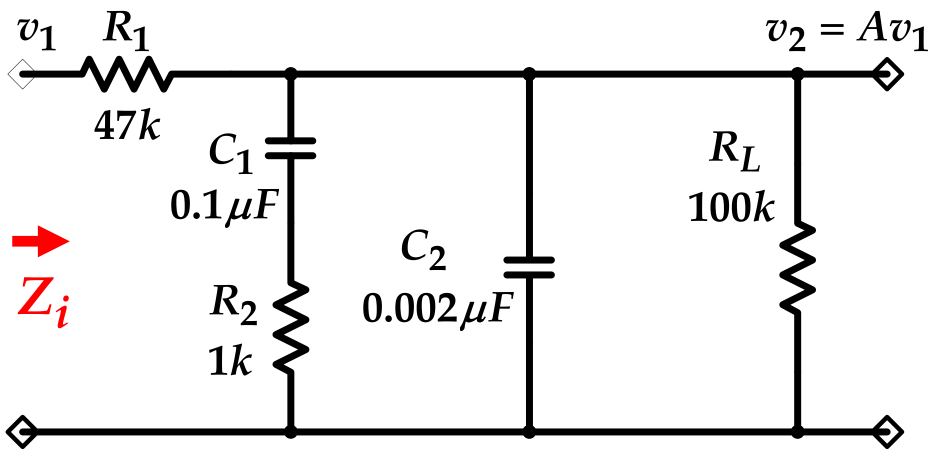
$$Z_i^* = Z_i\Big|_{Z_S\to 0} = \left.\frac{Z_S A_v\big|_{Z_S\to\infty}}{A_v}\right|_{Z_S\to 0} = \frac{Z_S A_v\big|_{Z_S\to\infty}}{A_v\big|_{Z_S\to 0}}$$

Inner Input and Output Impedances



Thus, all four input and output impedances can be found by taking simple limits upon the gain A_v.

This has been treated already. The result for the gain A is:



$$A = A_0 \frac{1+\frac{s}{\omega_z}}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$A_0 \equiv \frac{R_L}{R_1+R_L} \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2+R_1\|R_L\right)}$$

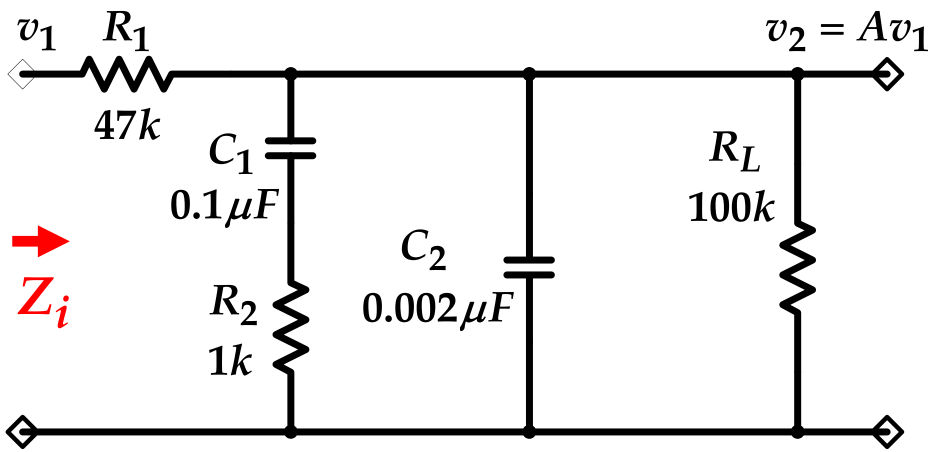
$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1\|R_2\|R_L\right)}$$

The formula for the outer input impedance is

$$Z_i = \frac{Z_S A\big|_{Z_S\to\infty}}{A}$$

Since R_1 is a surrogate Z_S,

$$Z_i = \frac{R_1 A\big|_{R_1\to\infty}}{A}$$



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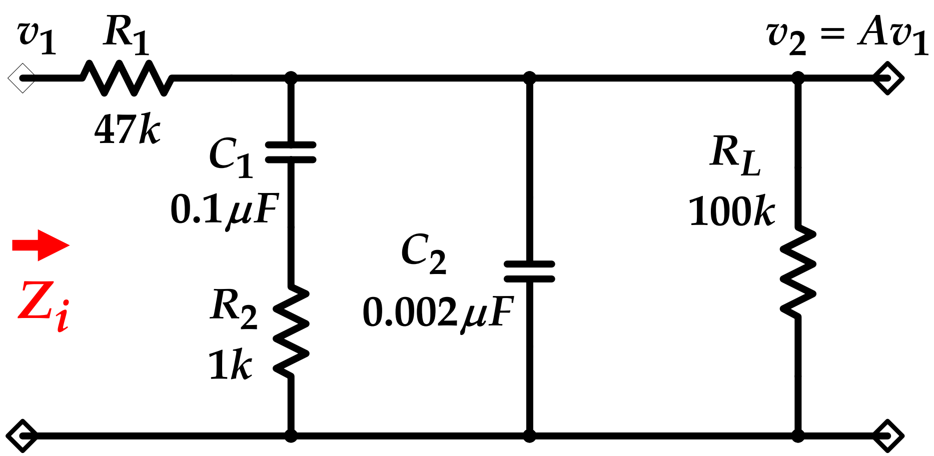
$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

The limit can be taken factor by factor:

$$Z_i = \frac{R_1 A\big|_{R_1\to\infty}}{A} = \frac{R_1 A_0\big|_{R_1\to\infty}}{A_0} \frac{\left(1+\frac{s}{\omega_z}\right)\Big|_{R_1\to\infty}}{\left(1+\frac{s}{\omega_z}\right)} \frac{\left(1+\frac{s}{\omega_1}\right)}{\left(1+\frac{s}{\omega_1}\right)\Big|_{R_1\to\infty}} \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_2}\right)\Big|_{R_1\to\infty}}$$



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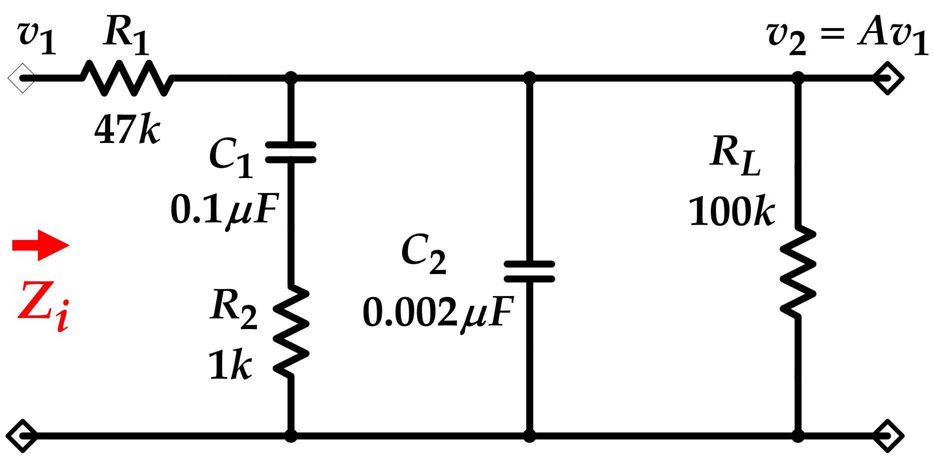
A huge simplification emerges: any factor in A that does not contain R_1 is unaffected by the limit and therefore cancels.



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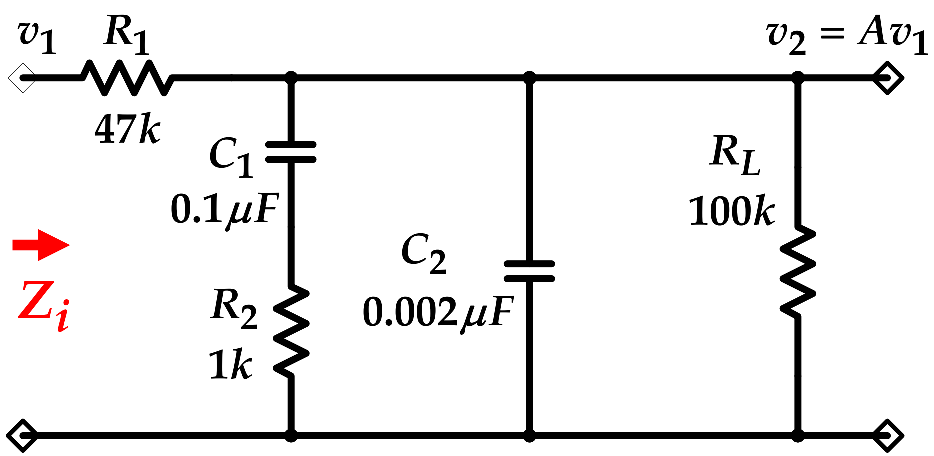

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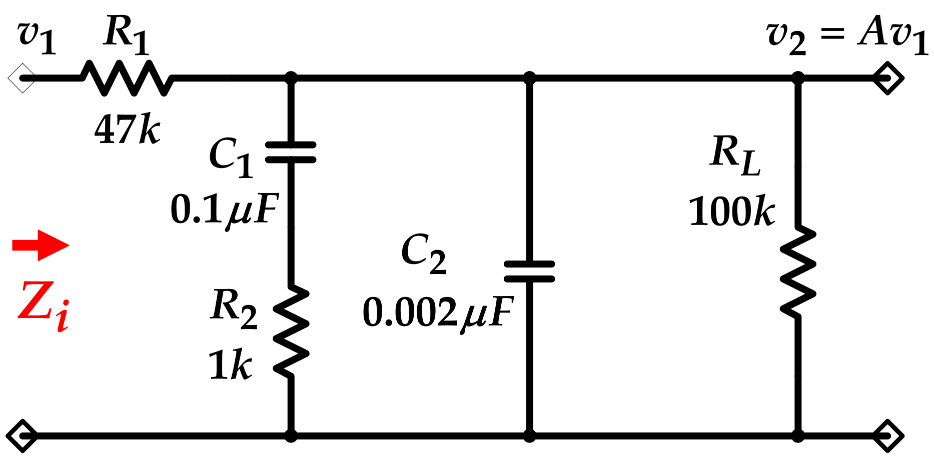
The limit can be taken factor by factor:

$$Z_i = \frac{R_1 A\big|_{R_1\to\infty}}{A} = \frac{R_1 A_0\big|_{R_1\to\infty}}{A_0} \frac{\left(1+\frac{s}{\omega_1}\right)}{\left(1+\frac{s}{\omega_1}\right)\Big|_{R_1\to\infty}} \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_2}\right)\Big|_{R_1\to\infty}}$$



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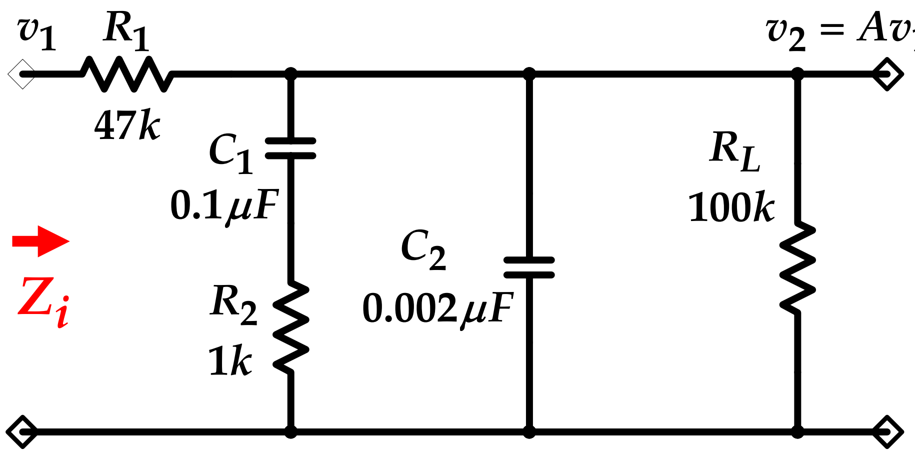
You can take advantage of this knowledge in advance, by highlighting R_1 in the parameter definitions and omitting these factors when substituting into the formula.

Rewrite the definitions highlighting R_1 :



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$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

$$Z_i = R_{i0} \frac{\left(1+\frac{s}{\omega_1}\right)}{\left(1+\frac{s}{\omega_1|_{R_1\to\infty}}\right)} \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_2|_{R_1\to\infty}}\right)}$$

where

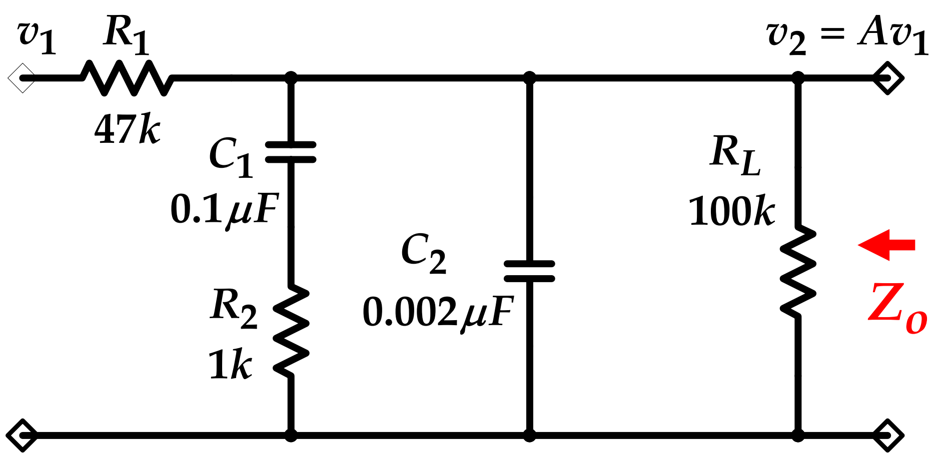
$$R_{i0} \equiv \frac{R_1 \left.\frac{1}{R_1+R_L}\right|_{R_1\to\infty}}{\frac{1}{R_1+R_L}} = R_1 + R_L$$

$$\omega_1|_{R_1\to\infty} \equiv \left.\frac{1}{C_1\left(R_2 + R_1 \| R_L\right)}\right|_{R_1\to\infty} = \frac{1}{C_1\left(R_2 + R_L\right)} < \omega_1$$

$$\omega_2|_{R_1\to\infty} \equiv \left.\frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}\right|_{R_1\to\infty} = \frac{1}{C_2\left(R_2 \| R_L\right)} < \omega_2$$

Exercise 7.1

Find the outer output impedance Z_o



$$A = A_0 \frac{1 + \frac{s}{\omega_z}}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

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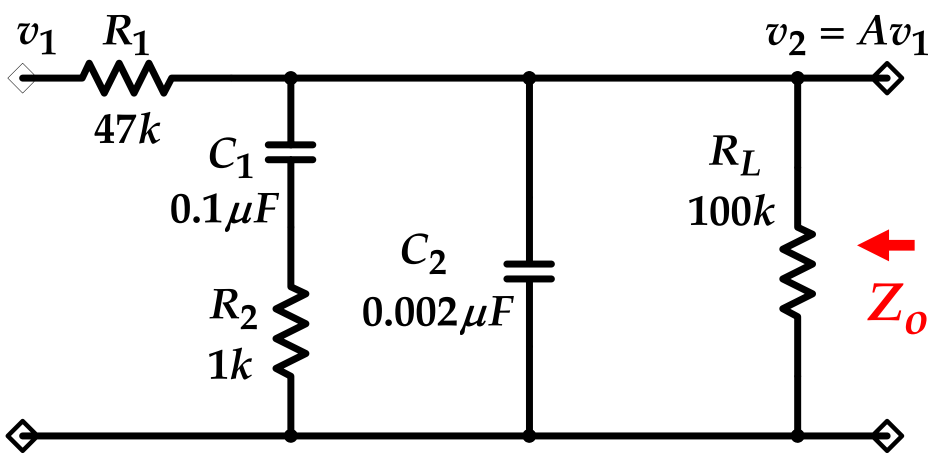
$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

Exercise 7.1 - Solution

Rewrite the definitions highlighting R_L :



$$A = A_0 \frac{1+\frac{s}{\omega_z}}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

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$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

$$Z_o = \frac{A}{\left.\frac{A}{R_L}\right|_{R_L \to 0}} = R_{o0} \frac{\left(1+\frac{s}{\left.\omega_1\right|_{R_L \to 0}}\right)\left(1+\frac{s}{\left.\omega_2\right|_{R_L \to 0}}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

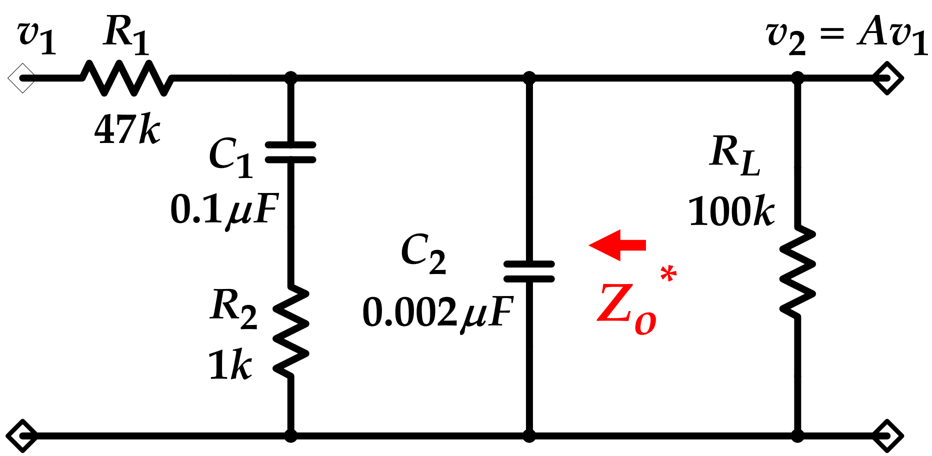
where

$$R_{o0} \equiv \frac{A_0}{\left.\frac{A_0}{R_L}\right|_{R_L \to 0}} = \frac{\frac{R_L}{R_1+R_L}}{\left.\frac{1}{R_L}\frac{R_L}{R_1+R_L}\right|_{R_L \to 0}} = R_1 \| R_L$$

$$\left.\omega_1\right|_{R_L \to 0} = \frac{1}{C_1 R_2} = \omega_z \qquad \left.\omega_2\right|_{R_L \to 0} = \infty$$

Exercise 7.2

Find the inner output impedance Z_o^*

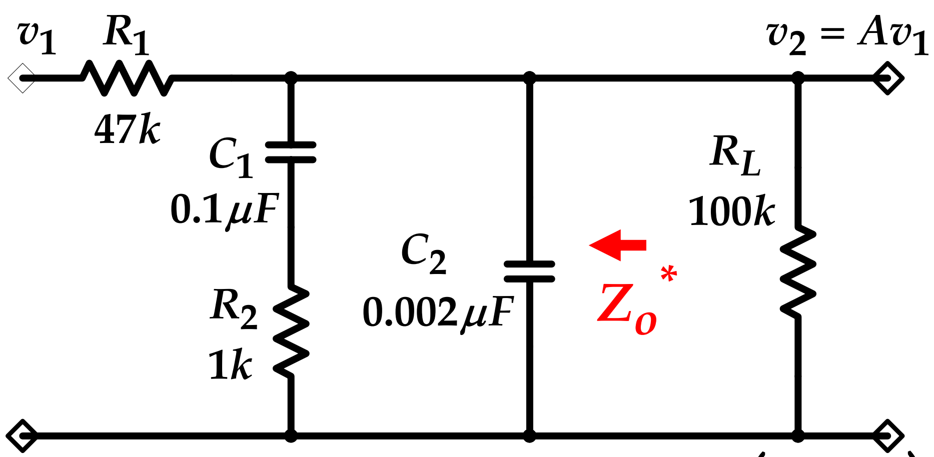


$$Z_o = R_{o0} \frac{\left(1 + \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

$$R_{o0} \equiv R_1 \| R_L \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2 + R_1 \| R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

Exercise 7.2 - Solution



$$Z_o = R_{o0}\frac{\left(1+\frac{s}{\omega_z}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$R_{o0} \equiv R_1 \| R_L \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2 + R_1 \| R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

$$Z_o^* = Z_o\Big|_{R_L\to\infty} = R_{o0}^*\frac{\left(1+\frac{s}{\omega_z}\right)}{\left(1+\frac{s}{\omega_1\big|_{R_L\to\infty}}\right)\left(1+\frac{s}{\omega_2\big|_{R_L\to\infty}}\right)}$$

where

$$R_{o0}^* \equiv R_{o0}\Big|_{R_L\to\infty} = R_1 \| R_L\Big|_{R_L\to\infty} = R_1$$

$$\omega_1\Big|_{R_L\to\infty} \equiv \frac{1}{C_1\left(R_2 + R_1 \| R_L\right)\Big|_{R_L\to\infty}} = \frac{1}{C_1\left(R_2 + R_1\right)} < \omega_1$$

$$\omega_2\Big|_{R_L\to\infty} \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)\Big|_{R_L\to\infty}} = \frac{1}{C_2\left(R_1 \| R_2\right)} < \omega_2$$

Bottom Line

The Input/Output Impedance Theorem allows you to find the input and output impedances of a circuit by taking simple limits upon the already known gain.

This saves almost two-thirds of the work required to obtain the three results separately.

Taking one limit upon the gain gives the outer impedances;

Taking two limits upon the gain gives the inner impedances.

A huge simplification occurs by anticipating that factors in the gain that do not contain the source or load impedance, do not appear in the result.