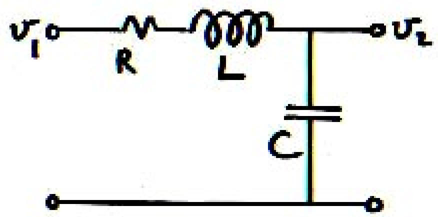
5. APPROXIMATIONS AND ASSUMPTIONS

How to build Low Entropy Expressions with minimum work

Double-pole low-pass LC filter



$$\frac{v_2}{v_1} = \frac{1}{1 + sRC + s^2LC}$$

$$= \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$= \frac{1}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

roots

in which

$$\omega_0 \equiv \frac{1}{\sqrt{LC}}$$ corner (resonant) frequency

$$Q \equiv \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{R_0}{R} \quad \text{where } R_0 \equiv \sqrt{\frac{L}{C}}$$

characteristic resistance

$Q < 0.5$: roots ω_1 and ω_2 are real

$Q > 0.5$: roots ω_1 and ω_2 are complex

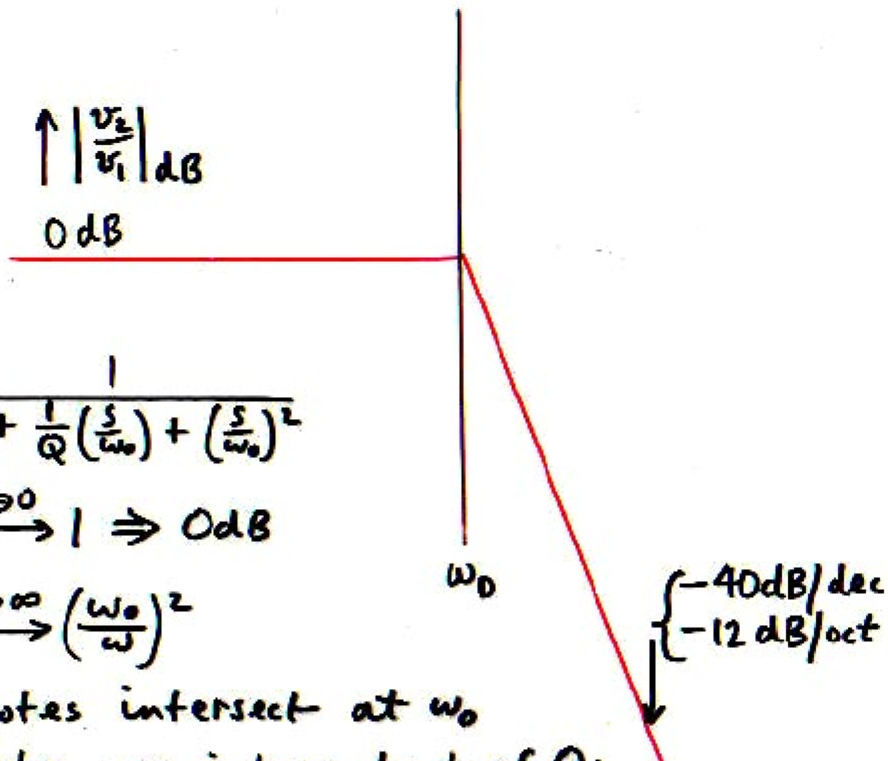
$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| \xrightarrow{\omega \to 0} 1 \Rightarrow 0\,dB$$

$$\xrightarrow{\omega \to \infty} \left(\frac{\omega_0}{\omega}\right)^2$$

Asymptotes intersect at ω_0

Asymptotes are independent of Q;
Q affects shape only in neighborhood of ω_0



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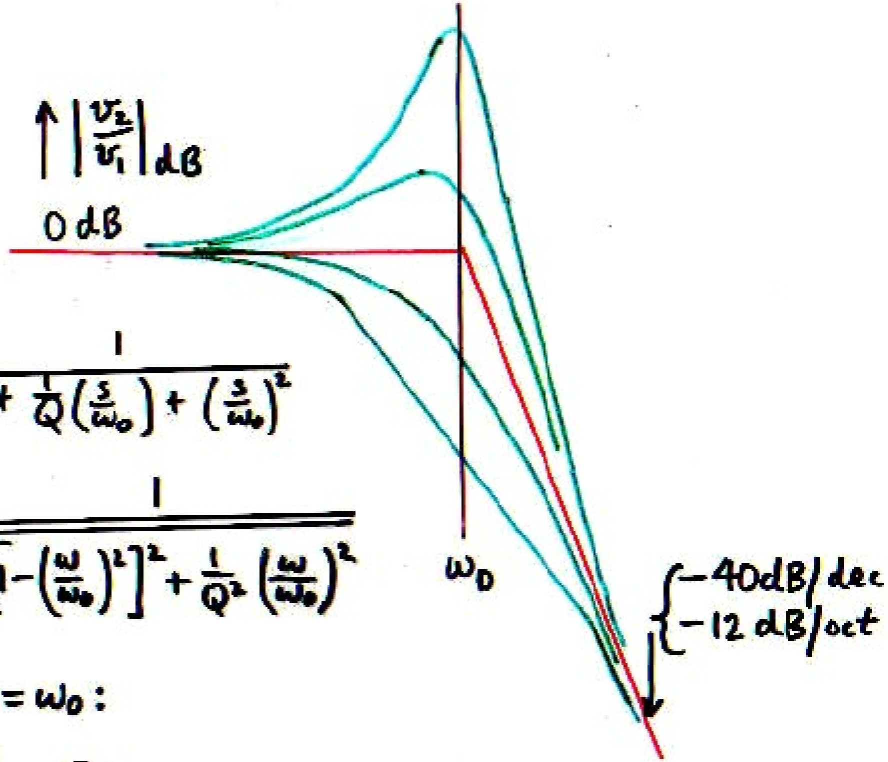
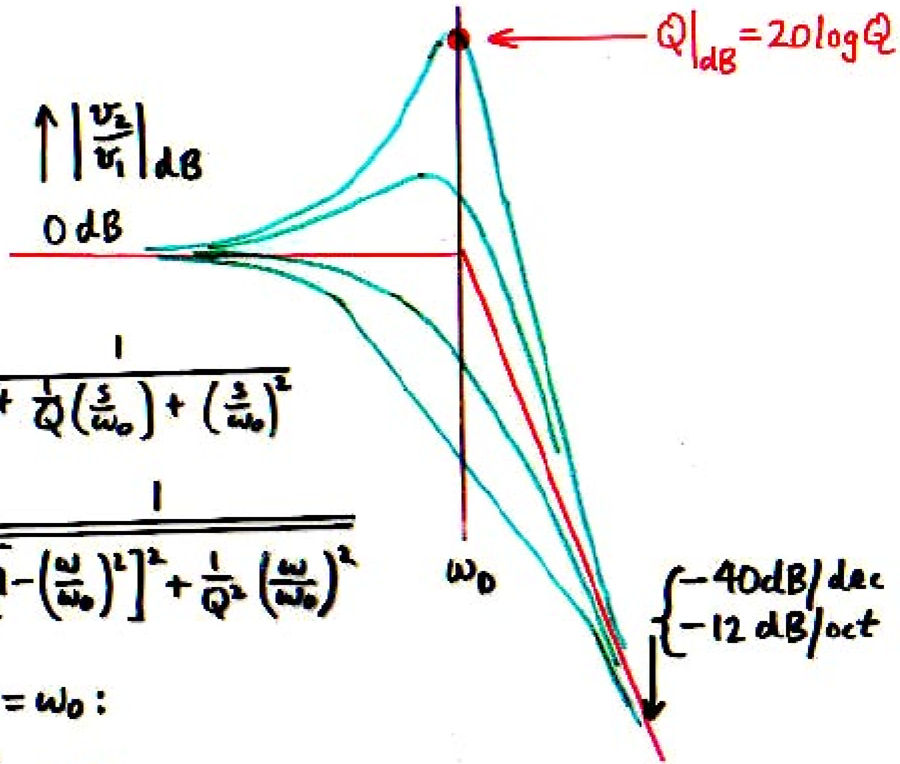


$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

At $\omega = \omega_0$:

$$\left|\frac{v_2}{v_1}\right| = Q$$



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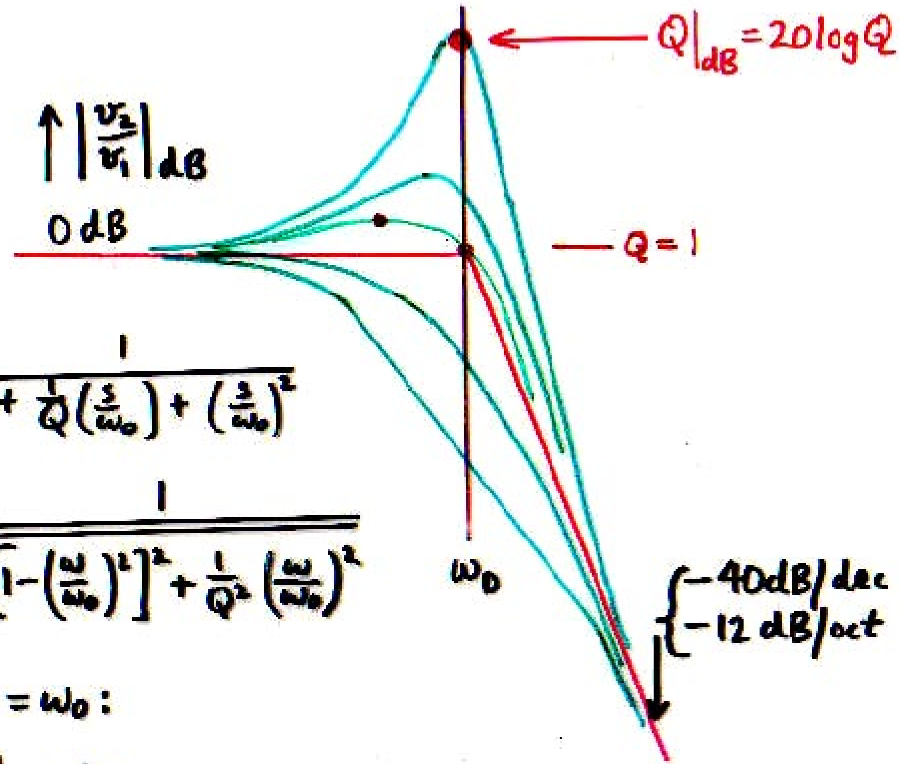

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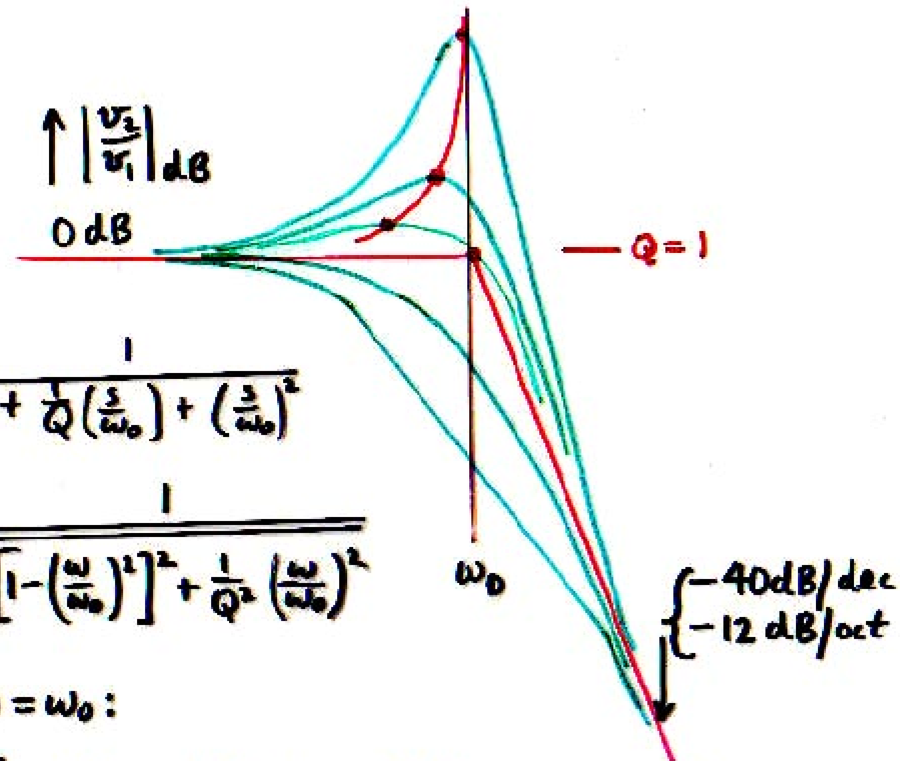

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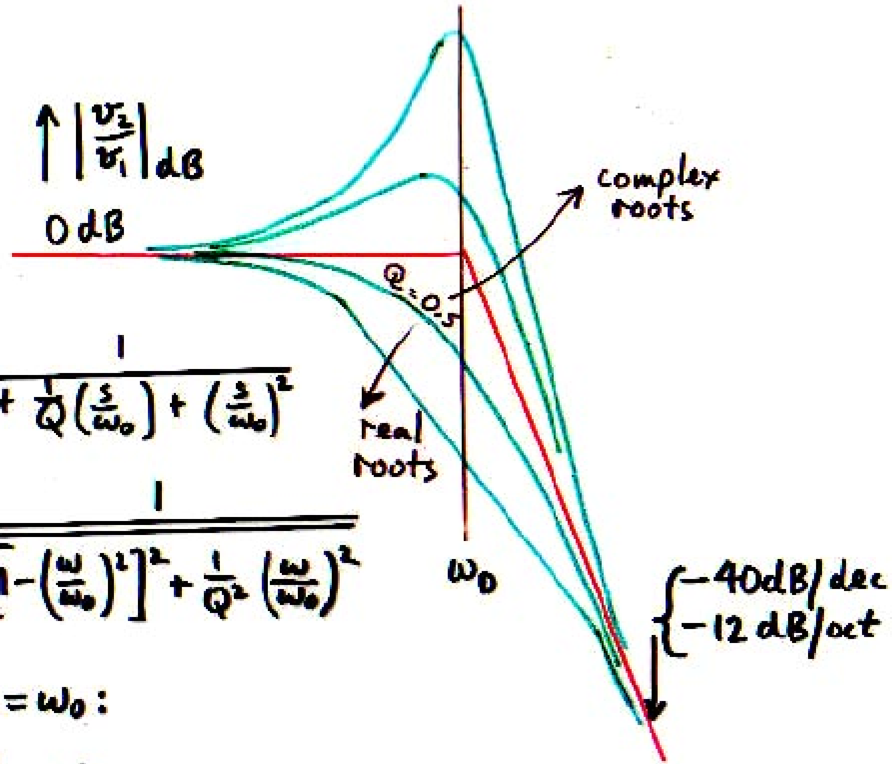
$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

At $\omega = \omega_0$:

$\left|\frac{v_2}{v_1}\right| = Q$ which is not the maximum; the maximum moves to the left for lower Q.



$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

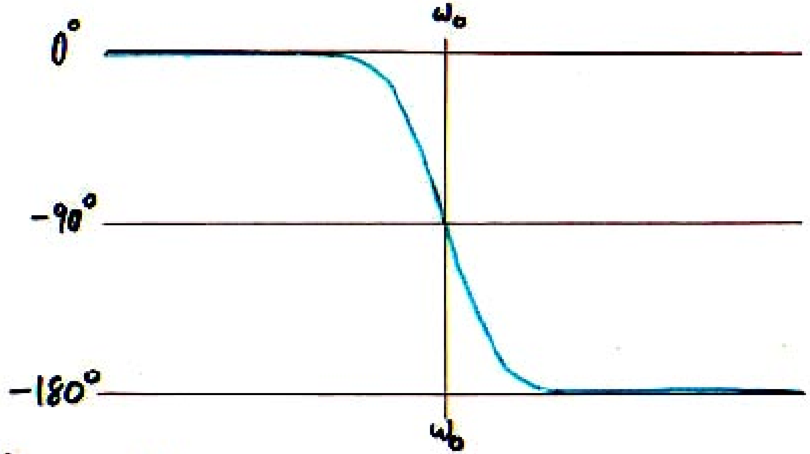
At $\omega = \omega_0$:

$$\left|\frac{v_2}{v_1}\right| = Q$$

Phase shape:

$$\angle\frac{v_2}{v_1} = -\tan^{-1}\left[\frac{\frac{1}{Q}\left(\frac{\omega}{\omega_o}\right)}{1-\left(\frac{\omega}{\omega_o}\right)^2}\right]$$

$$\left.\begin{array}{l} \xrightarrow{\omega\to 0} 0^\circ \\ \xrightarrow{\omega=\omega_o} -90^\circ \\ \xrightarrow{\omega=\infty} -180^\circ \end{array}\right\} \text{independent of } Q$$



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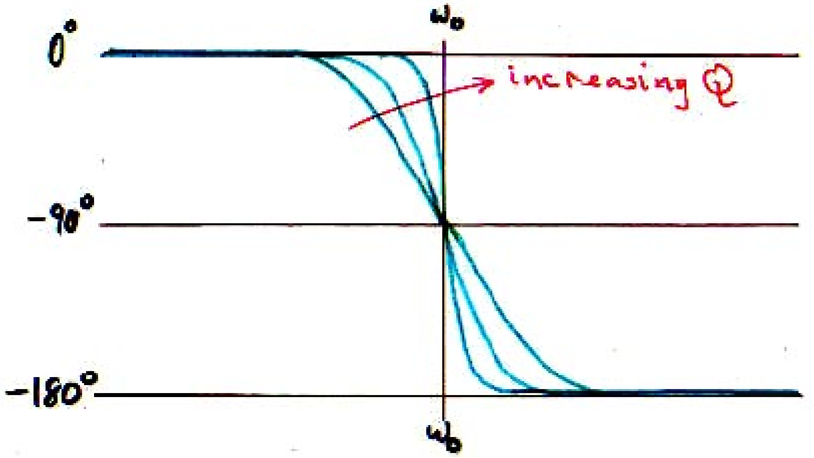
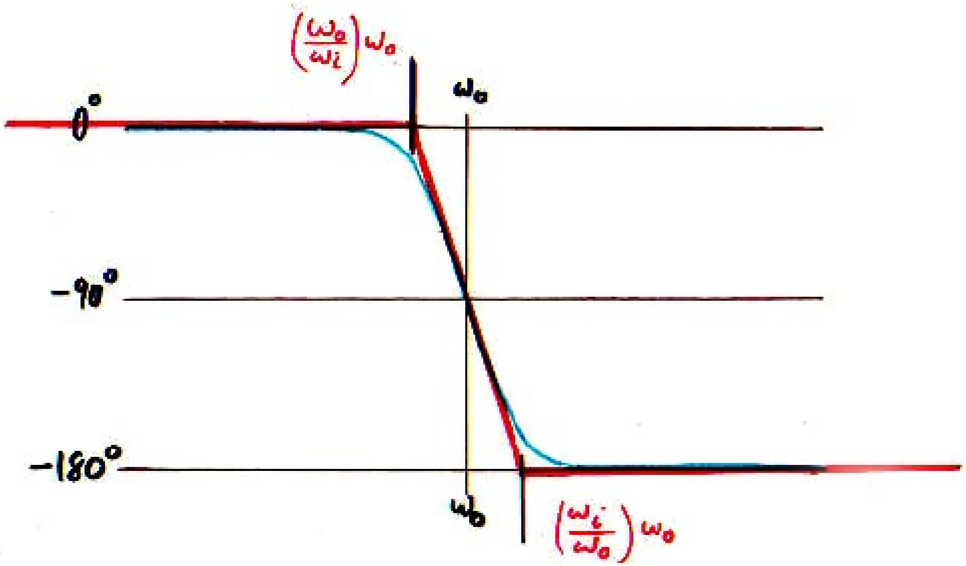


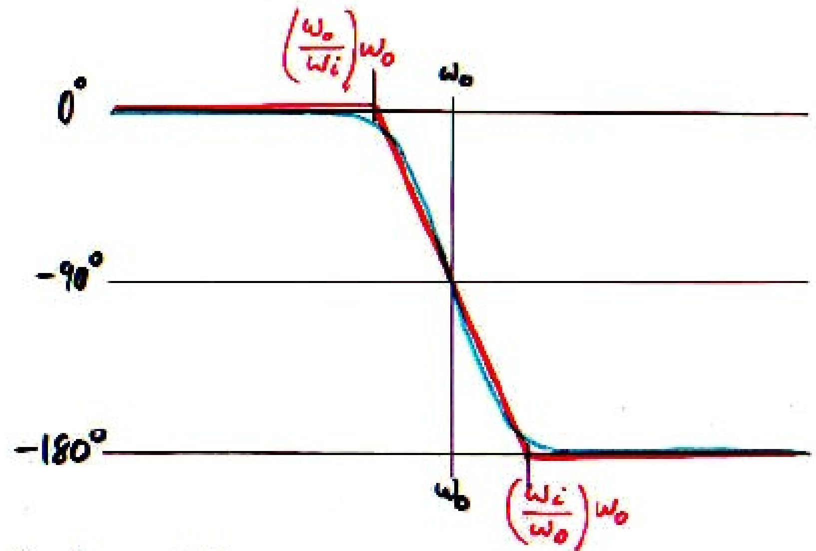
Increased Q causes sharper phase change between the 0° and −180° asymptotes.

Need: a straight-line approximation.



Choose same slope at $\omega = \omega_o$:

$$\frac{\omega_i}{\omega_o} = \left(e^{\frac{\pi}{2}}\right)^{\frac{1}{2Q}} = (4.81)^{\frac{1}{2Q}}$$



Better choice:

$$\frac{\omega_i}{\omega_o} \approx 5^{\frac{1}{2Q}}$$

An even better choice is

$$\frac{\omega_i}{\omega_o} = 10^{\frac{1}{2Q}}$$

because for Q = 0.5 (two equal real roots)

$$\frac{\omega_i}{\omega_o} = 10$$

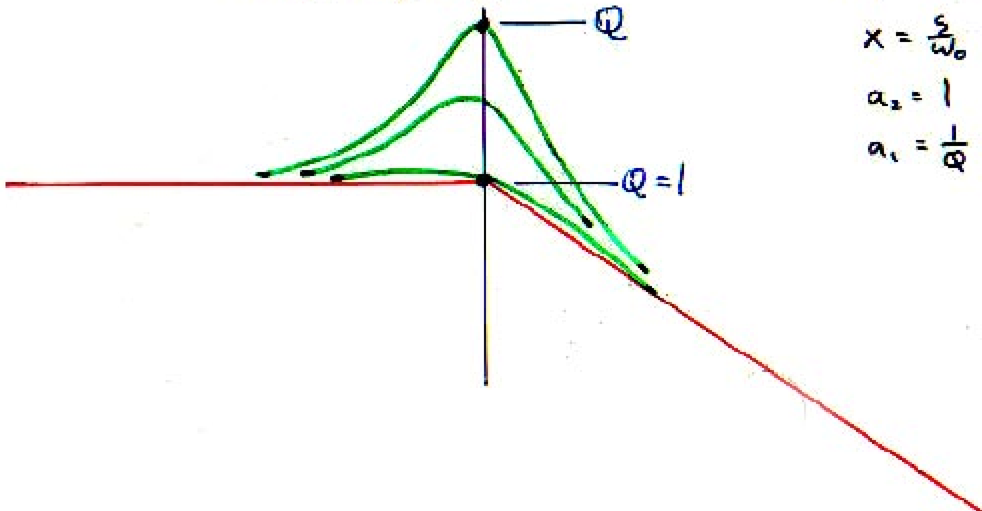
and the slope is $-90° / \text{dec}$, the same as twice the $-45° / \text{dec}$ slope for a single pole.

Second-order response:

$$A = A_1 \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$A = A_1 \frac{1}{1 + a_1(x) + a_2(x)^2} = A_1 \frac{1}{1 + \frac{a_1}{\sqrt{a_2}}\left(\sqrt{a_2}\,x\right) + \left(\sqrt{a_2}\,x\right)^2}$$

$x = \frac{s}{\omega_0}$	$x = s$
$a_2 = 1$	$a_2 = \frac{1}{\omega_0^2}$
$a_1 = \frac{1}{Q}$	$a_1 = \frac{1}{\omega_0 Q}$



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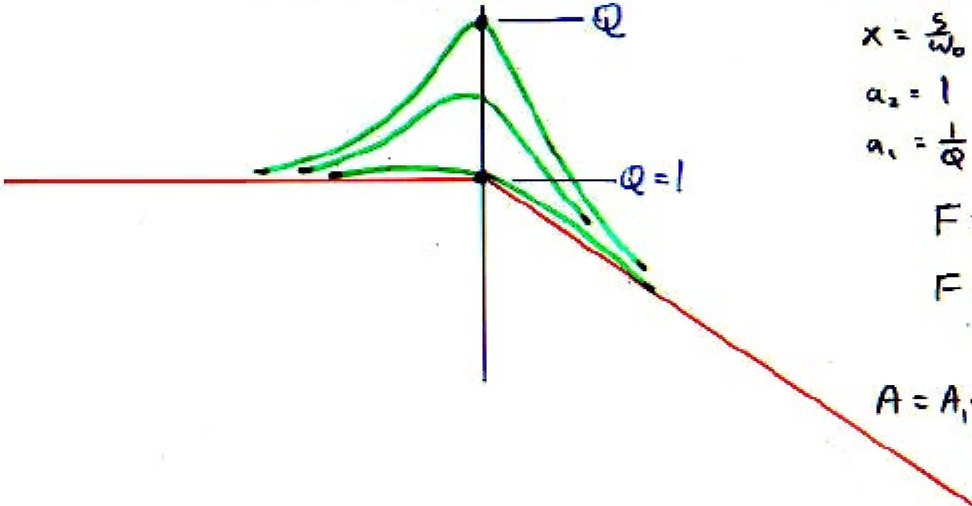

$x = \frac{s}{\omega_0}$	$x = s$
$a_2 = 1$	$a_2 = \frac{1}{\omega_0^2}$
$a_1 = \frac{1}{Q}$	$a_1 = \frac{1}{\omega_0 Q}$

$$F \approx \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4a_2/a_1^2}$$

$$F \approx \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}$$

$$A = A_1 \frac{1}{(1+a_1 F x)\left(1+\frac{a_2}{a_1 F}x\right)}$$

Second-order response:

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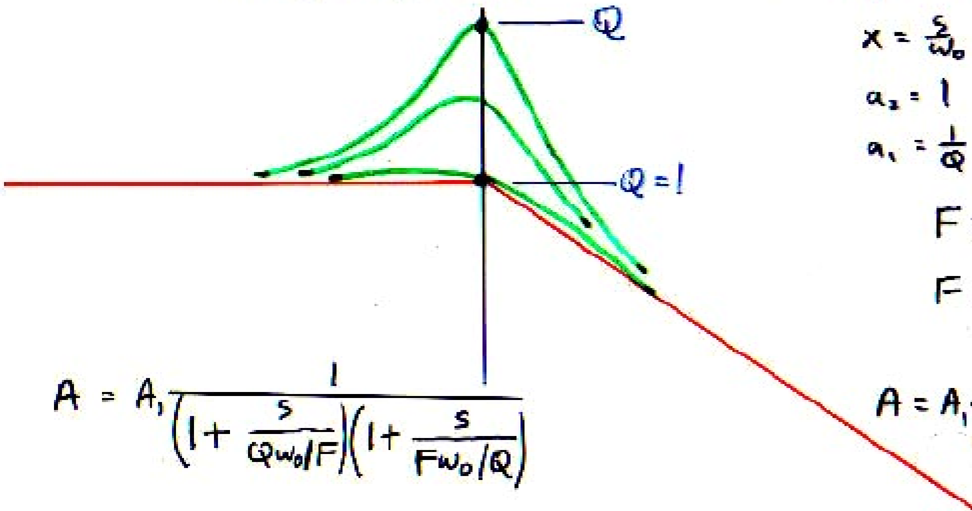

$x = \frac{s}{\omega_0}$	$x = s$
$a_2 = 1$	$a_2 = \frac{1}{\omega_0^2}$
$a_1 = \frac{1}{Q}$	$a_1 = \frac{1}{\omega_0 Q}$

$$F \approx \frac{1}{2}+\frac{1}{2}\sqrt{1-4a_2/a_1^2}$$

$$F \approx \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$A = A_1\frac{1}{\left(1+\frac{s}{Q\omega_0/F}\right)\left(1+\frac{s}{F\omega_0/Q}\right)}$$

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$x = \frac{s}{\omega_0}$	$x = s$
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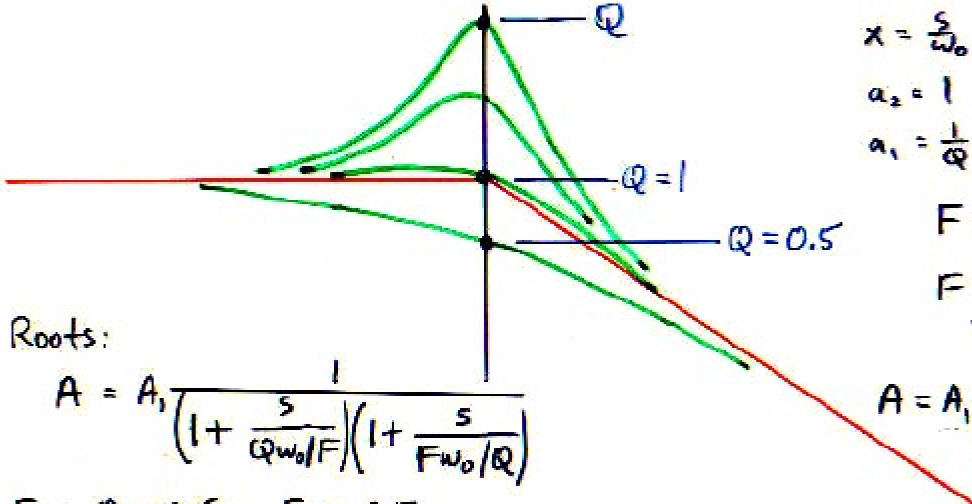

Roots:

$$A = A_1 \frac{1}{\left(1 + \frac{s}{Q\omega_0/F}\right)\left(1 + \frac{s}{F\omega_0/Q}\right)}$$

$$A = A_1 \frac{1}{(1 + a_1 F x)\left(1 + \frac{a_2}{a_1 F}x\right)}$$

For $Q = 0.5$: $F = 0.5$

$$A = A_1 \frac{1}{\left(1 + \frac{s}{\omega_0}\right)^2}$$

Second-order response:

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$a_1 = \frac{1}{Q}$ $\qquad$ $a_1 = \frac{1}{\omega_0 Q}$

$$F = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4a_2/a_1^2}$$

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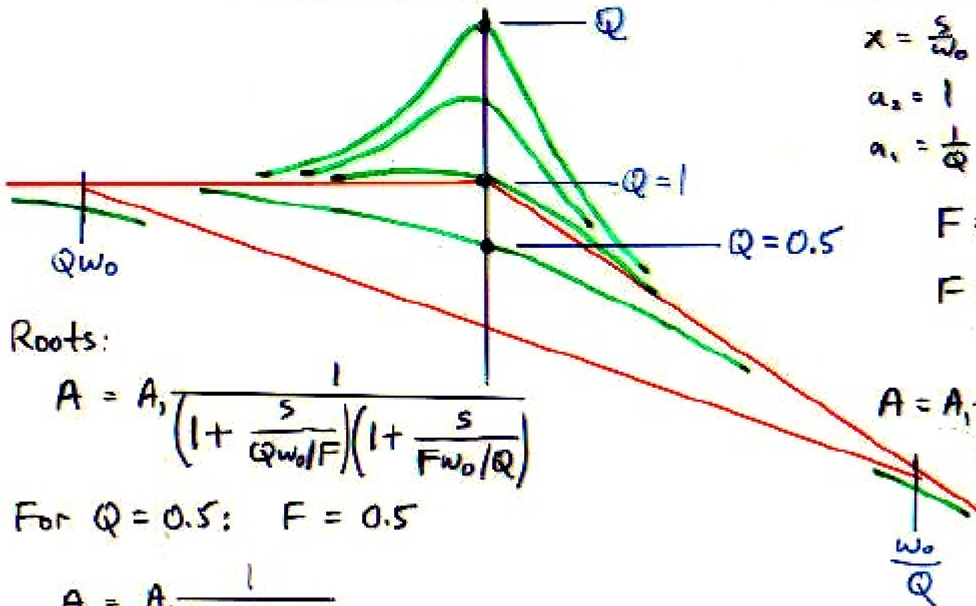


Roots:

$$A = A_1 \frac{1}{\left(1 + \frac{s}{Q\omega_0/F}\right)\left(1 + \frac{s}{F\omega_0/Q}\right)}$$

$$A = A_1 \frac{1}{(1 + a_1 F x)\left(1 + \frac{a_2}{a_1 F} x\right)}$$

For $Q = 0.5$: $\quad F = 0.5$

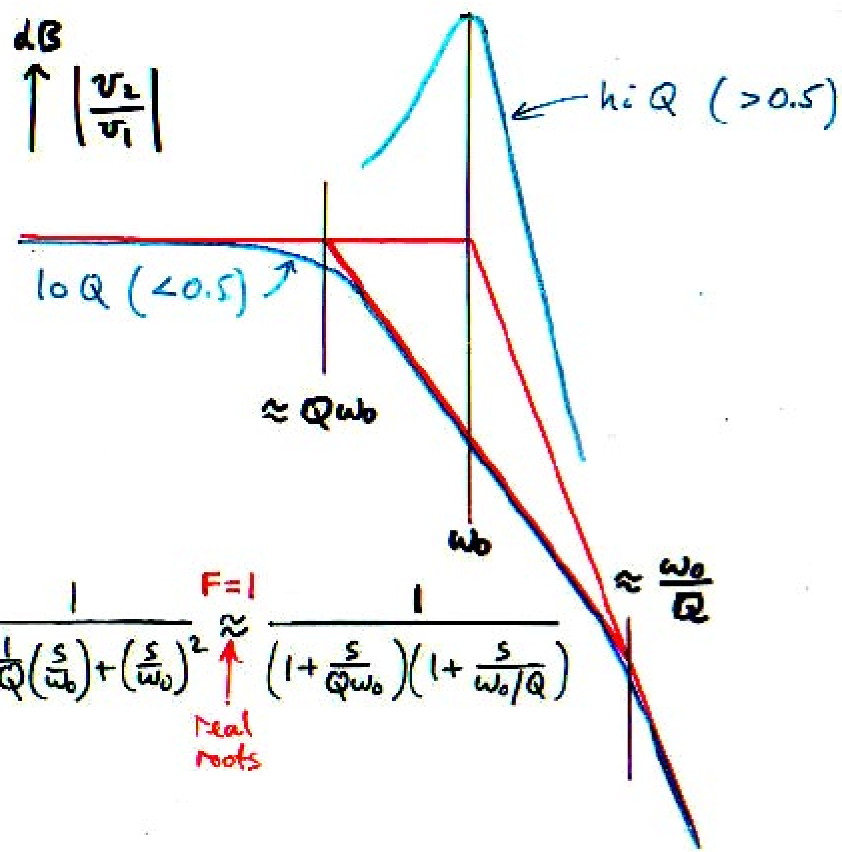
$$A = A_1 \frac{1}{\left(1 + \frac{s}{\omega_0}\right)^2}$$

For $Q \ll 0.5$: $\quad F \approx 1$

$$A = A_1 \frac{1}{\left(1 + \frac{s}{Q\omega_0}\right)\left(1 + \frac{s}{\omega_0/Q}\right)}$$

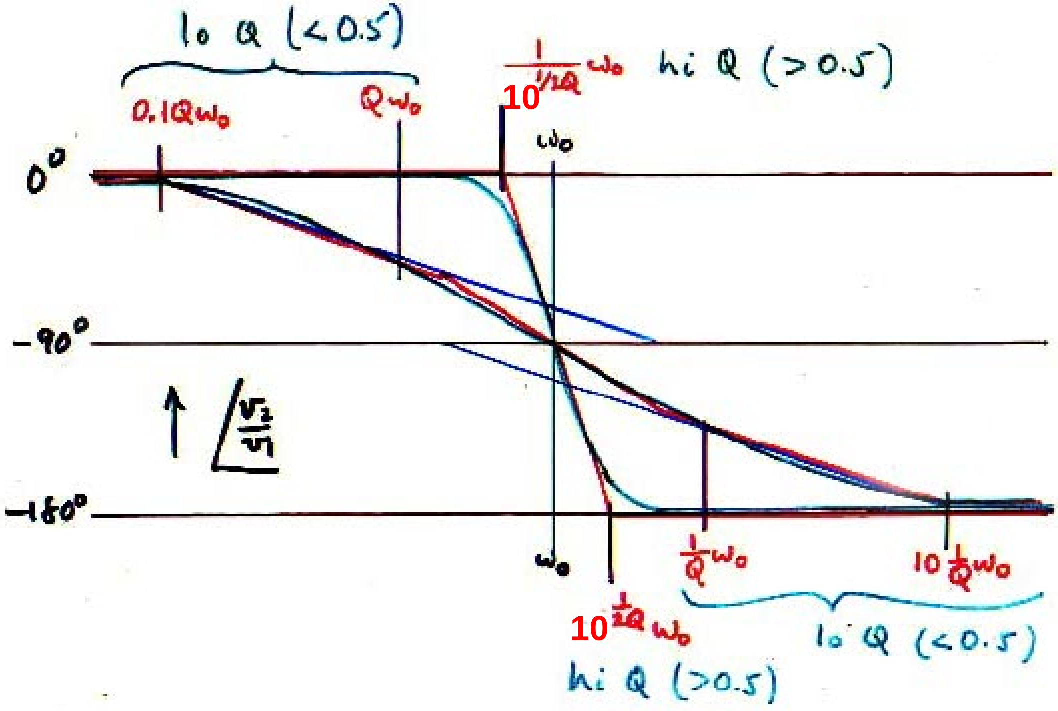
$$A = A_1 \frac{1}{(1 + a_1 x)\left(1 + \frac{a_2}{a_1} x\right)}$$

Low-pass 2-pole characteristic:



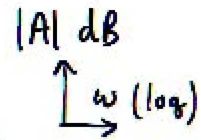
$$\frac{v_2}{v_1} = \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+\left(\frac{s}{\omega_0}\right)^2} \overset{F=1}{\approx} \frac{1}{\left(1+\frac{s}{Q\omega_0}\right)\left(1+\frac{s}{\omega_0/Q}\right)}$$

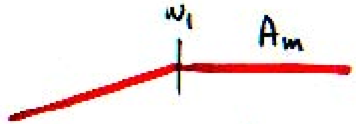
real roots



lo Q (<0.5)
0.1Qwo
Qwo
$\frac{1}{10^{1/2Q}}$wo
hi Q (>0.5)
wo
0°
-90°
-180°
wo
$\frac{1}{Q}$wo
10$\frac{1}{Q}$wo
$10^{\frac{1}{2Q}}$wo
lo Q (<0.5)
hi Q (>0.5)

Normal and inverted poles and zeros:



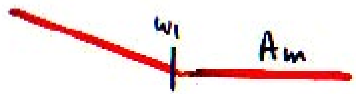


$$A = A_m \frac{1}{\left(1 + \frac{\omega_1}{s}\right)}$$

inverted pole

$$A = A_m \frac{1}{\left(1 + \frac{s}{\omega_1}\right)}$$

normal pole

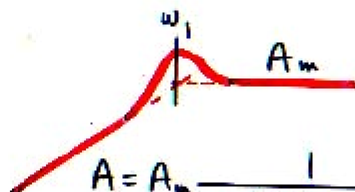


$$A = A_m \left(1 + \frac{\omega_1}{s}\right)$$

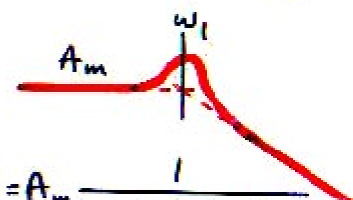
inverted zero

$$A = A_m \left(1 + \frac{s}{\omega_1}\right)$$

normal zero



$$A = A_m \frac{1}{1 + \frac{1}{Q}\left(\frac{\omega_1}{s}\right) + \left(\frac{\omega_1}{s}\right)^2}$$



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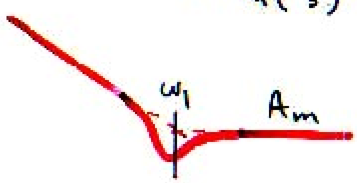
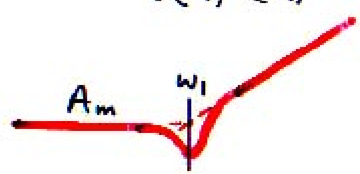
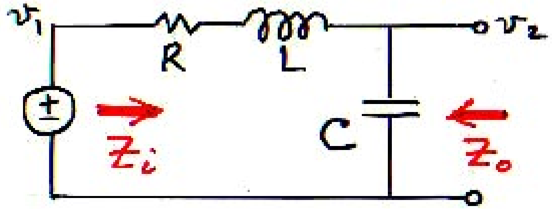


$$A = A_m \left[1 + \frac{1}{Q}\left(\frac{\omega_1}{s}\right) + \left(\frac{\omega_1}{s}\right)^2\right]$$



$$A = A_m \left[1 + \frac{1}{Q}\left(\frac{s}{\omega_1}\right) + \left(\frac{s}{\omega_1}\right)^2\right]$$

Input and Output Impedances of low-pass filter



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}} \qquad Q = \frac{R_o}{R} = \frac{1}{\omega_o CR} = \frac{\omega_o L}{R}$$

$$Z_i = \frac{1}{sC} + R + sL$$
$$\frac{1 + sCR + s^2LC}{sC}$$

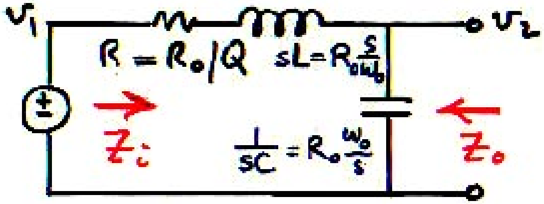
$$Z_o = \frac{\frac{1}{sC}(R+sL)}{\frac{1}{sC} + R + sL} = \frac{R+sL}{1+sCR+s^2LC}$$

Express in terms of ω_o, Q, R_o:

$$Z_i = \frac{1}{\omega_o C}\,\frac{1 + \omega_o CR\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)} = R_o\,\frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)}$$

$$Z_o = \omega_o L\,\frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{R}{sL}\right)}{1 + \omega_o CR\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2} = R_o\,\frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

Input and Output Impedances of low-pass filter



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad Q = \frac{R_0}{R}$$

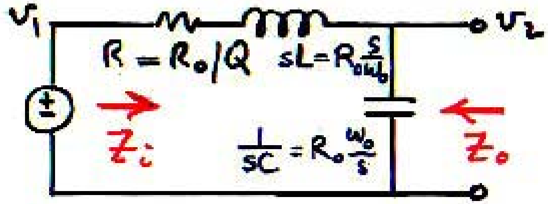
$$R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Z_i = \frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s} \qquad Z_o = \frac{\left(\frac{R_0}{Q} + R_0\frac{s}{\omega_0}\right)R_0\frac{\omega_0}{s}}{\frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}}$$

$$= R_0\,\frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)} \qquad = R_0\,\frac{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

Note how the algebra is shortened when the analysis starts with the normalized element values.

Input and Output Impedances of low-pass filter



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad Q = \frac{R_o}{R}$$

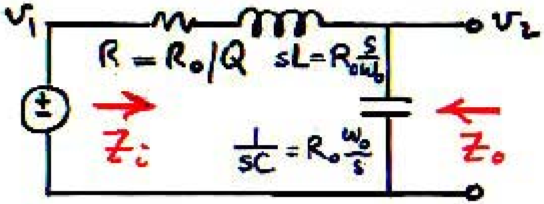
$$R_o \equiv \sqrt{\frac{L}{C}}$$

$$Z_i = \frac{R_o}{Q} + R_o \frac{s}{\omega_o} + R_o \frac{\omega_o}{s} \qquad Z_o = \frac{\left(\frac{R_o}{Q} + R_o \frac{s}{\omega_o}\right) R_o \frac{\omega_o}{s}}{\frac{R_o}{Q} + R_o \frac{s}{\omega_o} + R_o \frac{\omega_o}{s}}$$

$$= R_o \frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)} \qquad = R_o \frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

Put the quantities you know you want in the answer into the statement of the problem as soon as possible, even into the circuit diagram.

Input and Output Impedances of low-pass filter



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}} \qquad Q = \frac{R_0}{R}$$

$$Z_i = \frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}$$

$$\boxed{= R_0\frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)}}$$

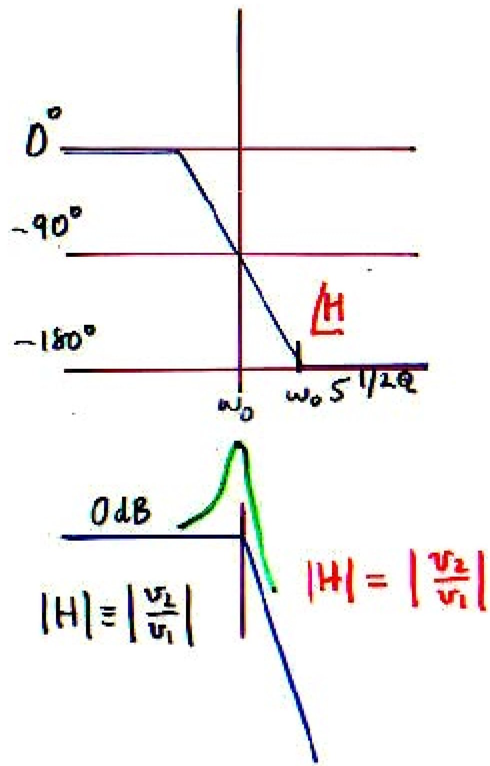
$$Z_o = \frac{\left(\frac{R_0}{Q} + R_0\frac{s}{\omega_0}\right)R_0\frac{\omega_0}{s}}{\frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}}$$

$$= R_0\frac{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

Put the quantities you know you want in the answer into the statement of the problem as soon as possible, even into the circuit diagram.

$$Z_i = \underbrace{R_o}_{\text{ref. value}} \times \underbrace{\left[1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2\right]}_{\frac{1}{H}} \times \underbrace{\left[\frac{1}{\frac{s}{\omega_o}}\right]}_{\text{single slope}}$$

R_o

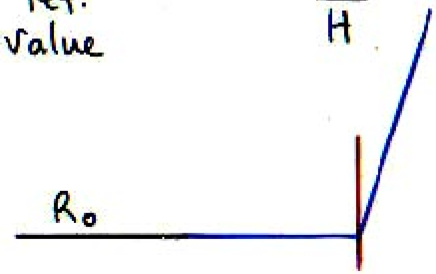

0°
-90°
-180°
∠H
ω₀
ω₀ 5^{1/2Q}
0dB
|H| ≡ |v₂/v₁|
|H| = |v₂/v₁|

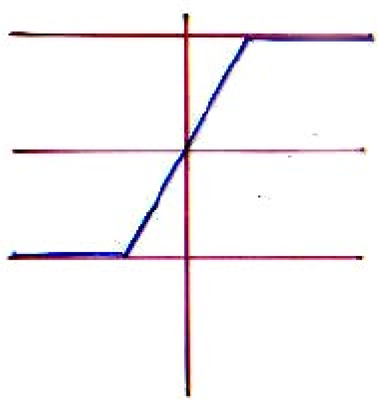
$$Z_i = R_o \times \left[1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2\right] \times \left[\frac{1}{\frac{s}{\omega_o}}\right]$$

ref. value

$\frac{1}{H}$

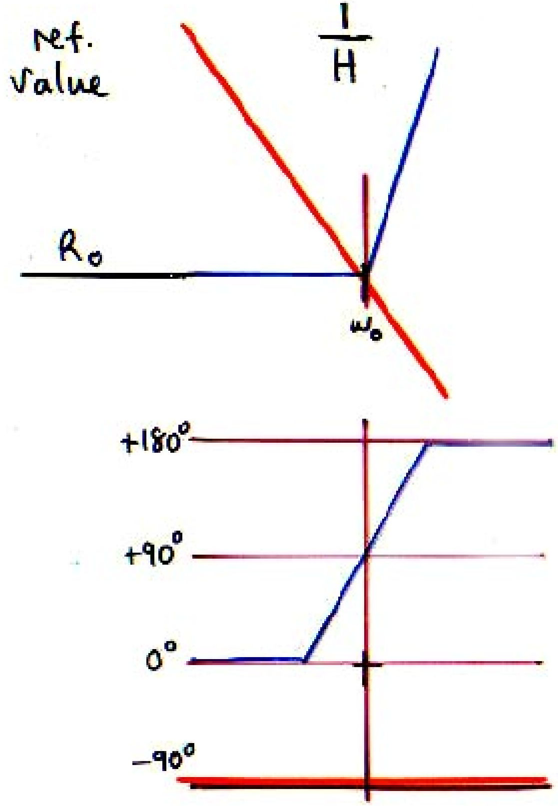
single slope





$$Z_i = R_o \times \left[1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2\right] \times \left[\frac{1}{\frac{s}{\omega_o}}\right]$$

single slope

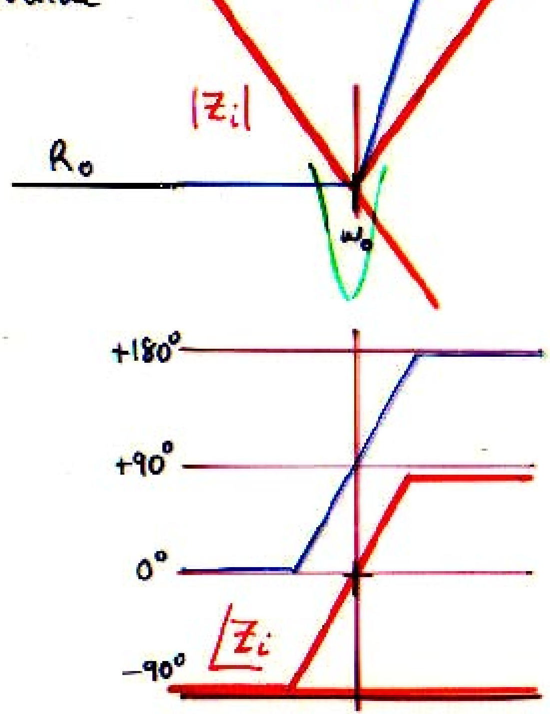


$$Z_i = R_o \times \left[1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2\right] \times \left[\frac{1}{\frac{s}{\omega_o}}\right]$$

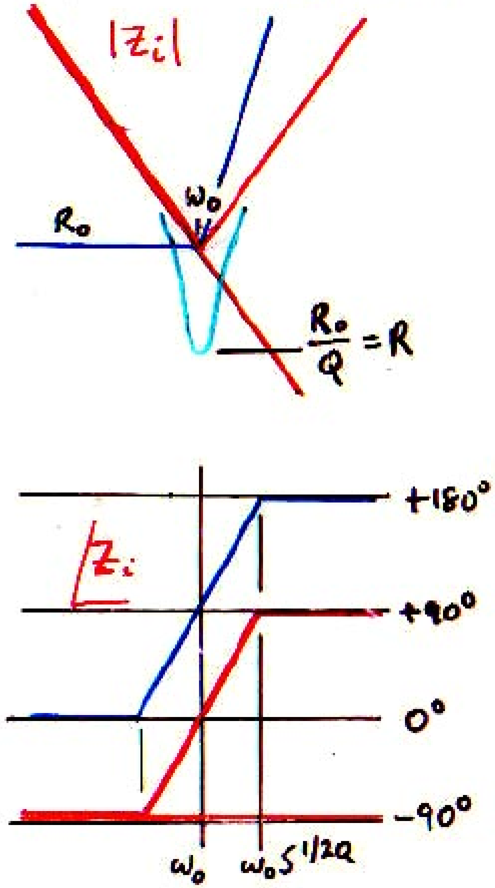
ref. value

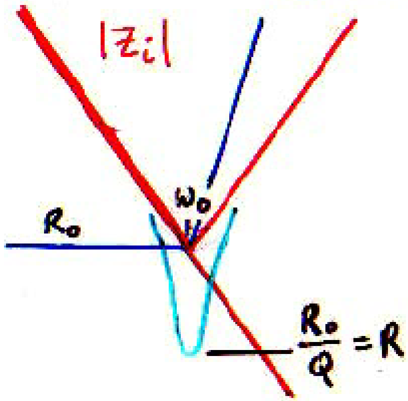
$\frac{1}{H}$

single slope

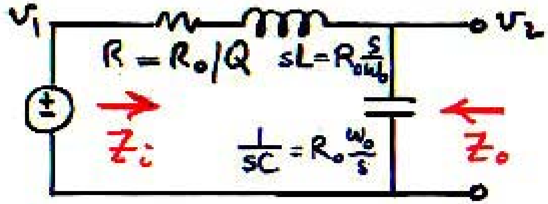


Asymptote sketches for high Q (>>0.5)




|Zi|
ω0
R0
R0/Q = R

Input and Output Impedances of low-pass filter



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}} \qquad Q = \frac{R_o}{R}$$

$$Z_i = \frac{R_o}{Q} + R_o\frac{s}{\omega_o} + R_o\frac{\omega_o}{s} = R_o\frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)}$$

$$Z_o = \frac{\left(\frac{R_o}{Q} + R_o\frac{s}{\omega_o}\right)R_o\frac{\omega_o}{s}}{\frac{R_o}{Q} + R_o\frac{s}{\omega_o} + R_o\frac{\omega_o}{s}} \qquad \boxed{= R_o\frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}}$$

Put the quantities you know you want in the answer into the statement of the problem as soon as possible, even into the circuit diagram.

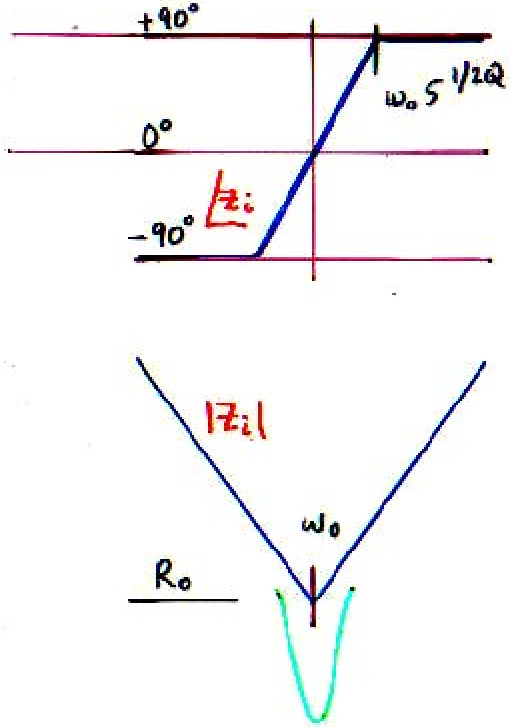
$$Z_o = R_o \times \left[\frac{\frac{s}{\omega_o}}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}\right] \times \left(1 + \frac{\omega_o/Q}{s}\right)$$

ref. value

$\frac{1}{Z_i}$

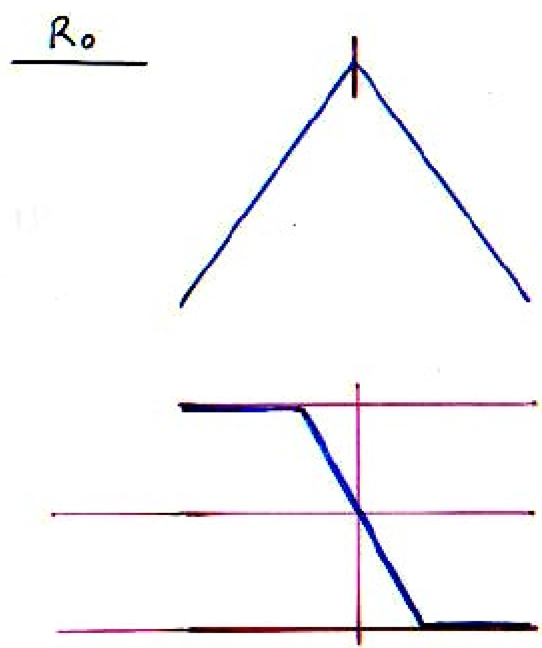
inverted zero

$\underline{R_o}$


+90°
$\omega_o 5^{1/2Q}$
0°
$\angle Z_i$
−90°
$|Z_i|$
ω_o
R_o

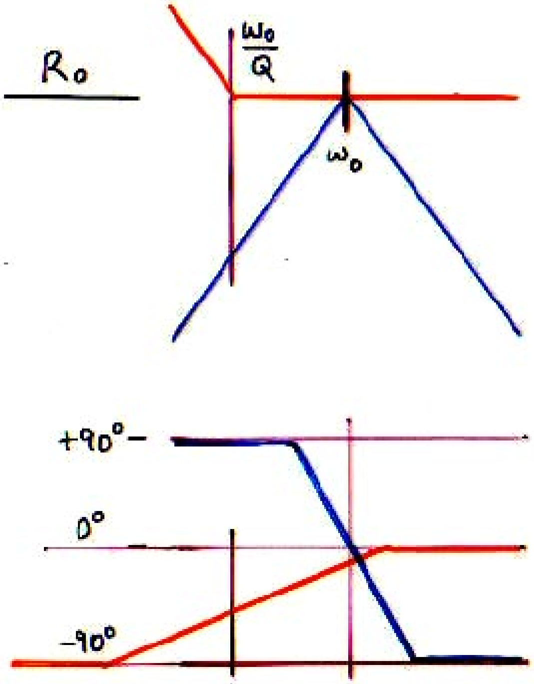
$$Z_o = R_o \times \left[\frac{\frac{s}{\omega_o}}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}\right] \times \left(1 + \frac{\omega_o/Q}{s}\right)$$

ref. value — $\frac{1}{Z_i}$ — inverted zero



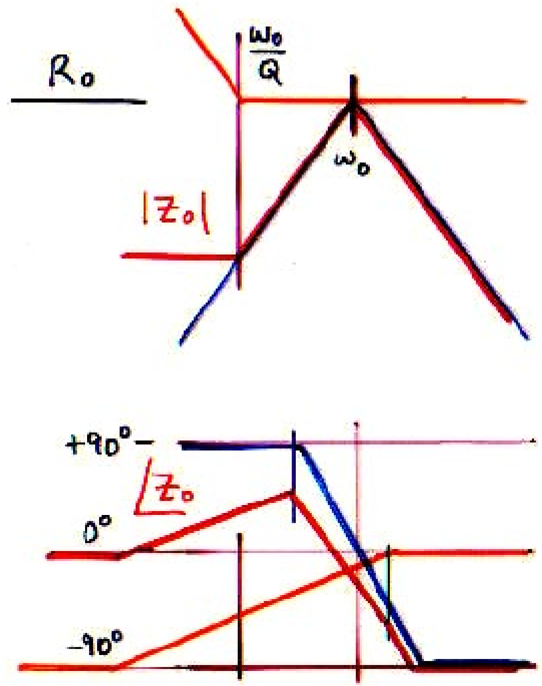
$$Z_o = R_o \times \left[\frac{\frac{s}{\omega_0}}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}\right] \times \left(1 + \frac{\omega_0/Q}{s}\right)$$

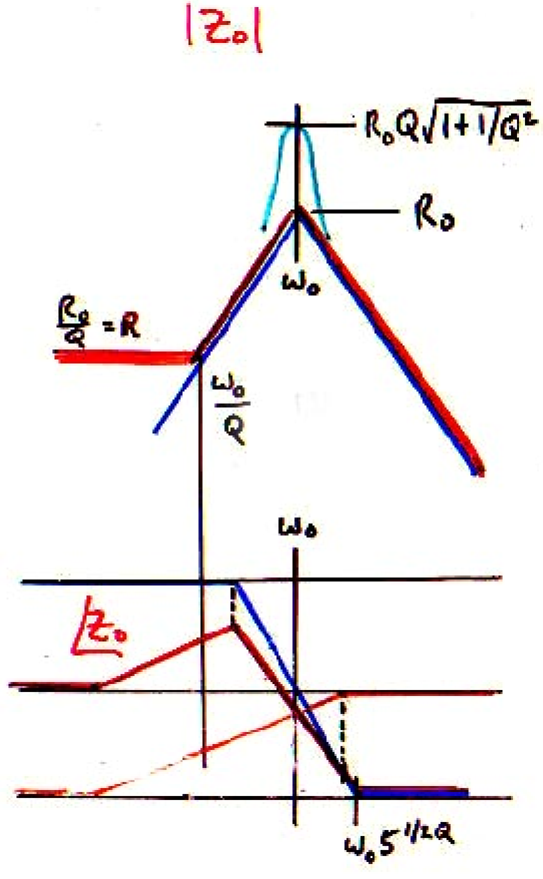
ref. value — $\frac{1}{Z_i}$ — inverted zero



$$Z_o = R_o \times \left[\frac{\frac{s}{\omega_o}}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}\right] \times \left(1 + \frac{\omega_o/Q}{s}\right)$$

ref. value — $\frac{1}{Z_i}$ — inverted zero




|Zo|
$R_oQ\sqrt{1+1/Q^2}$
R_o
ω_o
$\frac{R_o}{Q} = R$
$\frac{\omega_o}{Q}$
ω_o
$\angle Z_o$
$\omega_o 5^{1/2Q}$

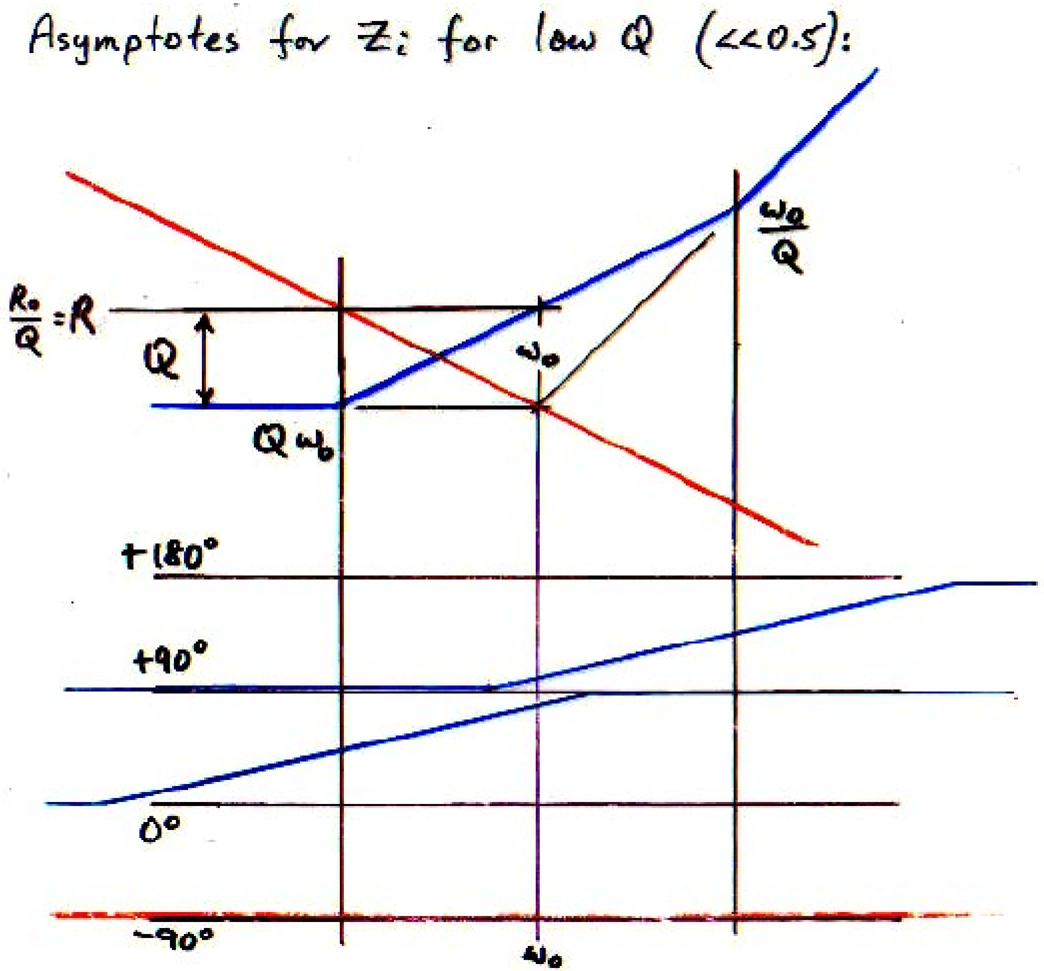
Exercise 5.1

Sketch asymptotes for Z_i and Z_o for low Q.

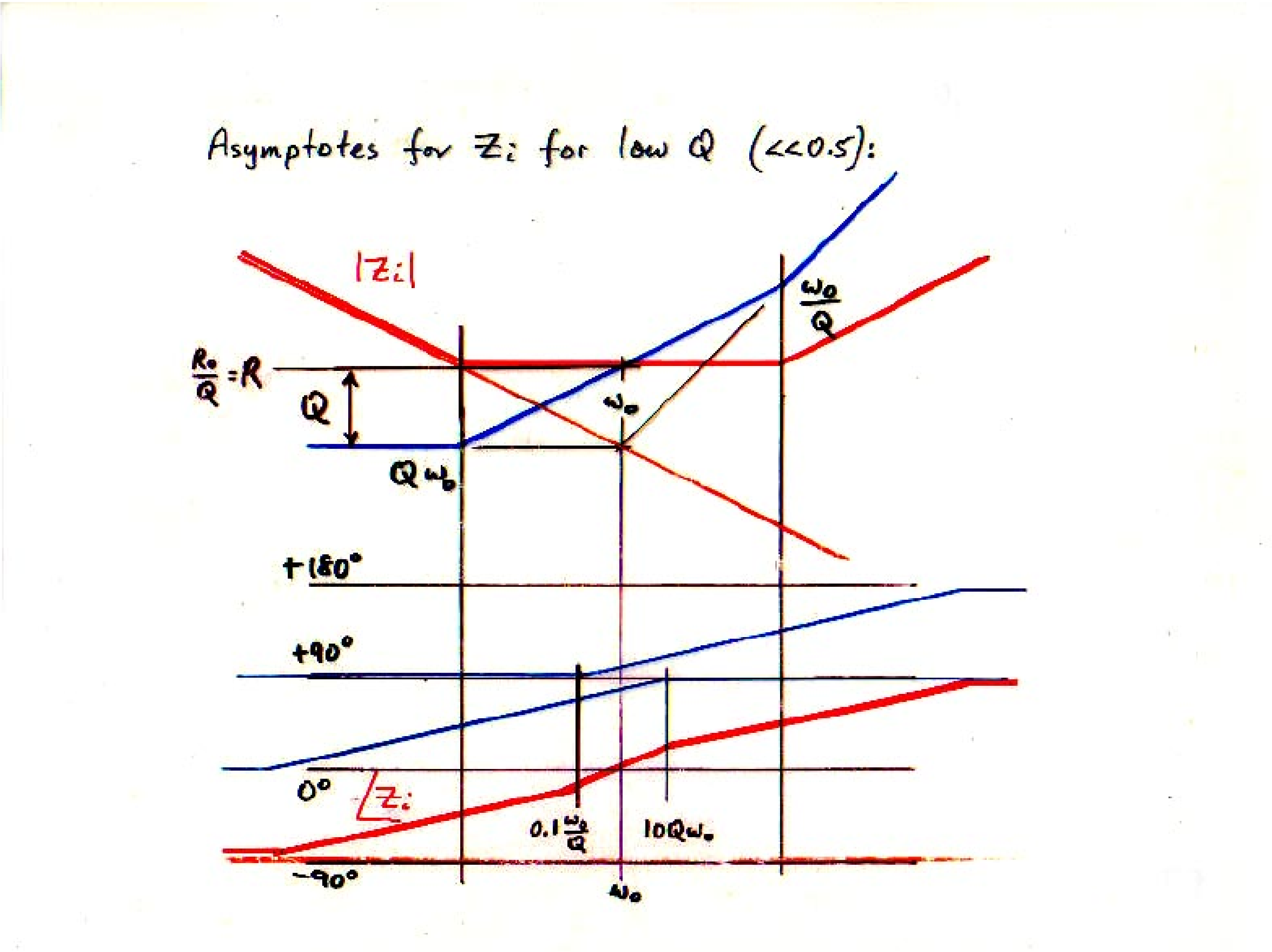
Exercise

For the two-pole low-pass LC filter, sketch the magnitude and phase asymptotes of Z_i and Z_o for low Q ($<<0.5$).
(But take $Q > 0.1$)

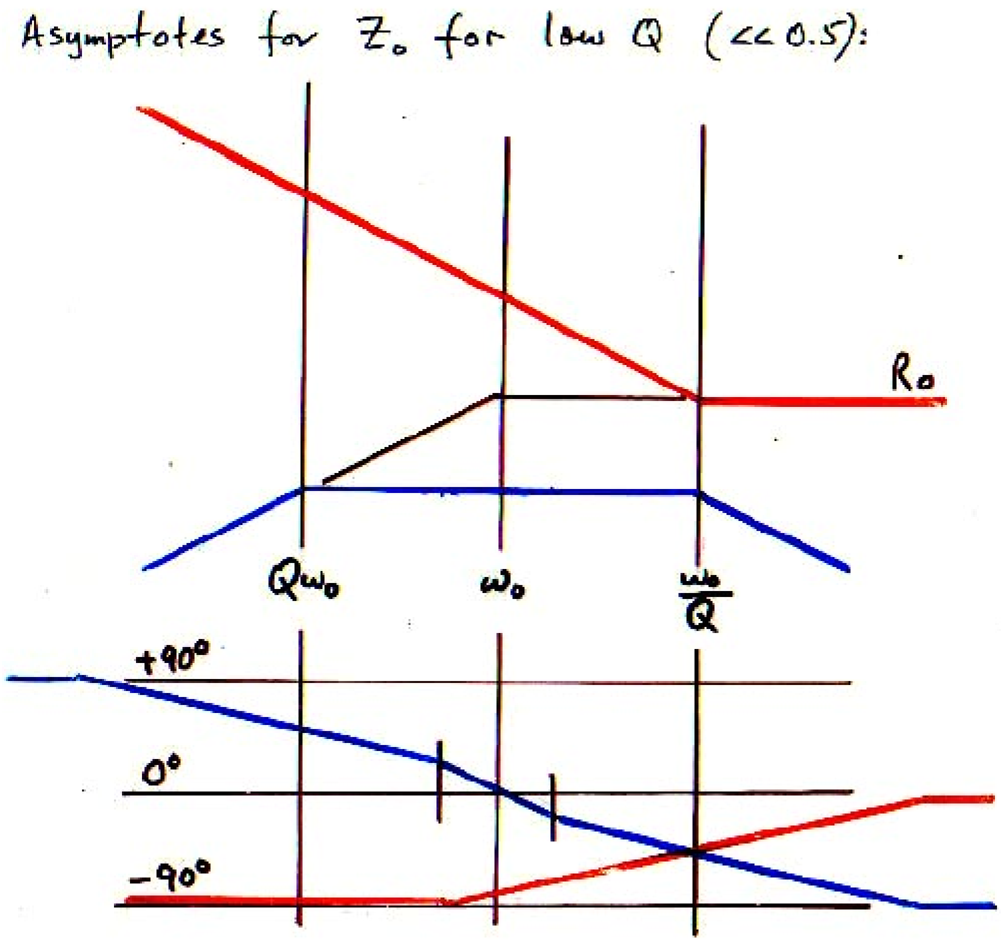
Exercise 5.1 - Solution



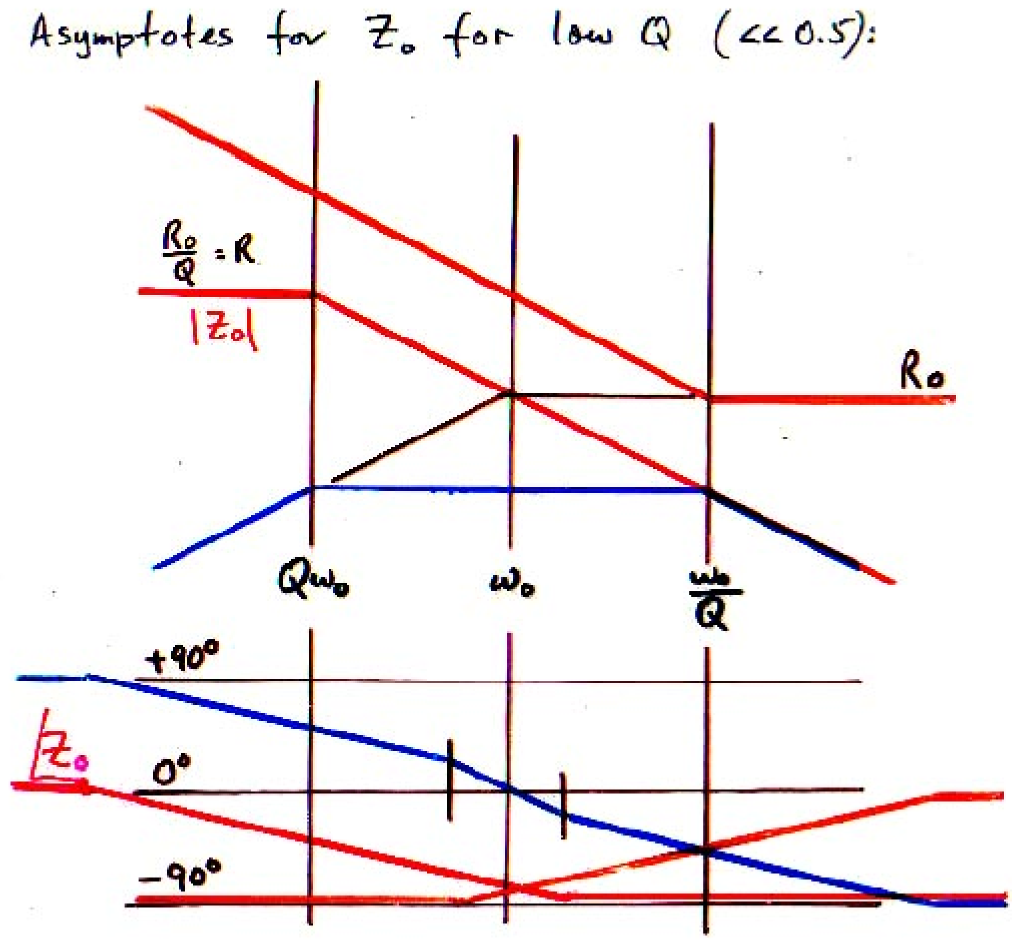
Exercise 5.1 - Solution

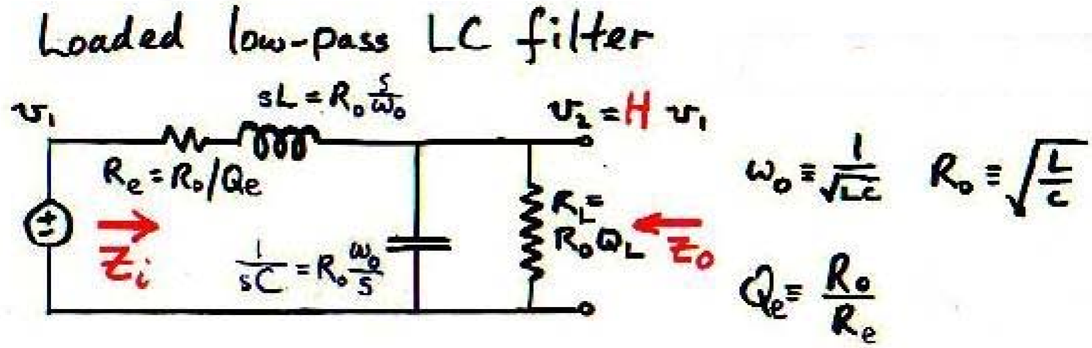


Exercise 5.1 - Solution



Exercise 5.1 - Solution

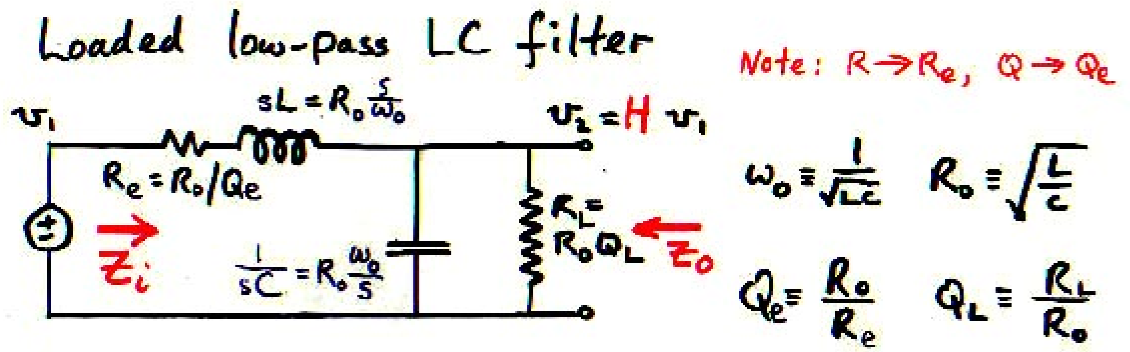




Since there are now two resistances, re-name

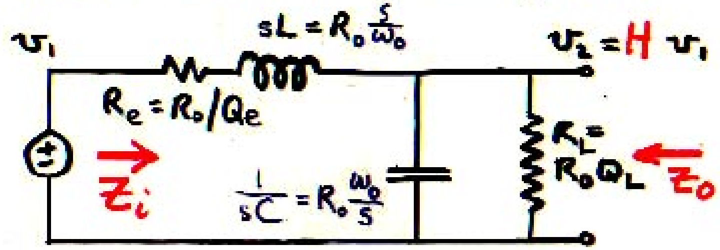
$$\mathbf{R \rightarrow R_e, \quad Q \rightarrow Q_e}$$

By analogy, define $\mathbf{Q_L \equiv \frac{R_L}{R_o}}$



Why is Q_L defined "upside down" relative to Q_e?

Loaded low-pass LC filter



Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$

$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_L \equiv \frac{R_L}{R_o}$$

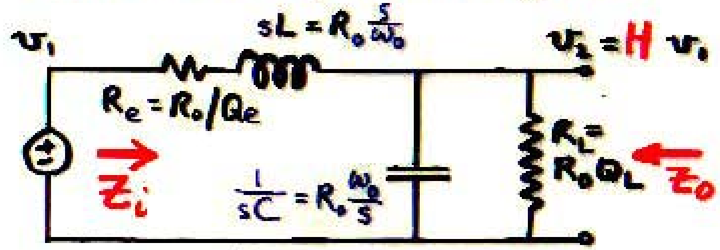
$$H = \frac{\dfrac{R_o Q_L}{1+Q_L \frac{s}{\omega_o}}}{\dfrac{R_o Q_L}{1+Q_L \frac{s}{\omega_o}} + \dfrac{R_o}{Q_e} + R_o \dfrac{s}{\omega_o}} = \frac{Q_L}{Q_L + \frac{1}{Q_e} + \left(\frac{Q_L}{Q_e}+1\right)\left(\frac{s}{\omega_o}\right) + Q_L\left(\frac{s}{\omega_o}\right)^2}$$

$$= \frac{1}{1+1/Q_eQ_L} \frac{1}{1 + \dfrac{\left(\frac{1}{Q_e}+\frac{1}{Q_L}\right)}{1+1/Q_eQ_L}\left(\frac{s}{\omega_o}\right) + \dfrac{1}{1+1/Q_eQ_L}\left(\frac{s}{\omega_o}\right)^2}$$

Conventional result:

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



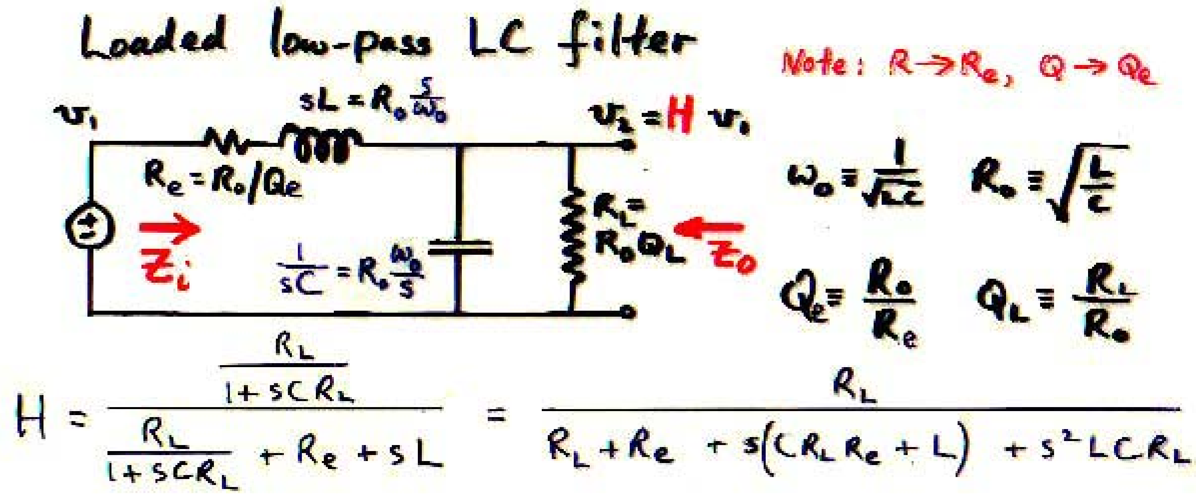
$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

$$H = \frac{\dfrac{R_L}{1+sCR_L}}{\dfrac{R_L}{1+sCR_L} + R_e + sL} = \frac{R_L}{R_L + R_e + s(CR_LR_e + L) + s^2LCR_L}$$

Reveals no insight

Conventional result:



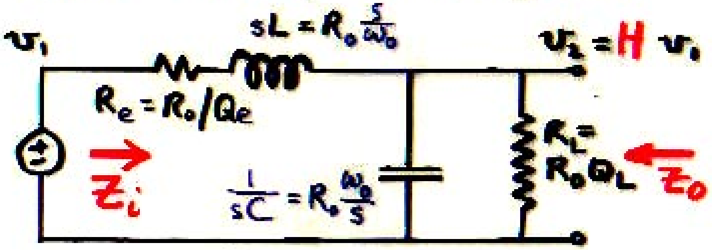
Reveals no insight

This high entropy result can be converted into the desired low entropy version by application of mental energy, but it takes quite an effort, and you have to know where you're going!

Conventional result:

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

$$H = \frac{\dfrac{R_L}{1+sCR_L}}{\dfrac{R_L}{1+sCR_L} + R_e + sL} = \frac{R_L}{R_L + R_e + s(CR_LR_e + L) + s^2LCR_L}$$

$$= \frac{R_L}{R_L + R_e} \frac{1}{1 + s\left(CR_e + \dfrac{L}{R_L}\right)\dfrac{R_L}{R_L + R_e} + s^2LC\dfrac{R_L}{R_L + R_e}}$$

$$= \frac{R_L}{R_L + R_e} \frac{1}{1 + s\sqrt{LC}\left(R_e\sqrt{\dfrac{C}{L}} + \dfrac{1}{R_L}\sqrt{\dfrac{L}{C}}\right)\dfrac{R_L}{R_L + R_e} + s^2LC\dfrac{R_L}{R_L + R_e}}$$

$$= \frac{1}{1 + 1/Q_eQ_L} \frac{1}{1 + \dfrac{\left(\dfrac{1}{Q_e} + \dfrac{1}{Q_L}\right)}{1 + 1/Q_eQ_L}\left(\dfrac{s}{\omega_0}\right) + \dfrac{1}{1 + 1/Q_eQ_L}\left(\dfrac{s}{\omega_0}\right)^2}$$

$$H=\frac{1}{1+1/Q_eQ_L}\frac{1}{1+\frac{\frac{1}{Q_e}+\frac{1}{Q_L}}{\sqrt{1+1/Q_eQ_L}}\left(\frac{s}{\omega_o\sqrt{1+1/Q_eQ_L}}\right)+\left(\frac{s}{\omega_o\sqrt{1+1/Q_eQ_L}}\right)^2}$$

Result, compared with unloaded case:

1. Low-freq. asymptote is $\frac{1}{1+1/Q_eQ_L}=\frac{R_L}{R_L+R_e}$
(resistive divider)

2. The corner frequency is changed to
$\sqrt{1+1/Q_eQ_L}\,\omega_o$

3. The damping coefficient is changed to
$\frac{\frac{1}{Q_e}+\frac{1}{Q_L}}{\sqrt{1+1/Q_eQ_L}}$

This is a good example of how a low entropy format can allow one equation to disclose more than one useful piece of information.

$$H=\frac{1}{1+1/Q_eQ_L}\frac{1}{1+\frac{\frac{1}{Q_e}+\frac{1}{Q_L}}{\sqrt{1+1/Q_eQ_L}}\left(\frac{s}{\omega_o\sqrt{1+1/Q_eQ_L}}\right)+\left(\frac{s}{\omega_o\sqrt{1+1/Q_eQ_L}}\right)^2}$$

Result, compared with unloaded case:

second order → 1. Low-freq. asymptote is $\frac{1}{1+1/Q_eQ_L}=\frac{R_L}{R_L+R_e}$ (resistive divider)

second order → 2. The corner frequency is changed to $\sqrt{1+1/Q_eQ_L}\,\omega_o$

first order → 3. The damping coefficient is changed to

$$\frac{\frac{1}{Q_e}+\frac{1}{Q_L}}{\sqrt{1+1/Q_eQ_L}}$$

For the high-Q case, $Q_e, Q_L >> 0.5$, $Q_eQ_L >> 1$ and the first two effects are negligible, and the damping coefficient becomes

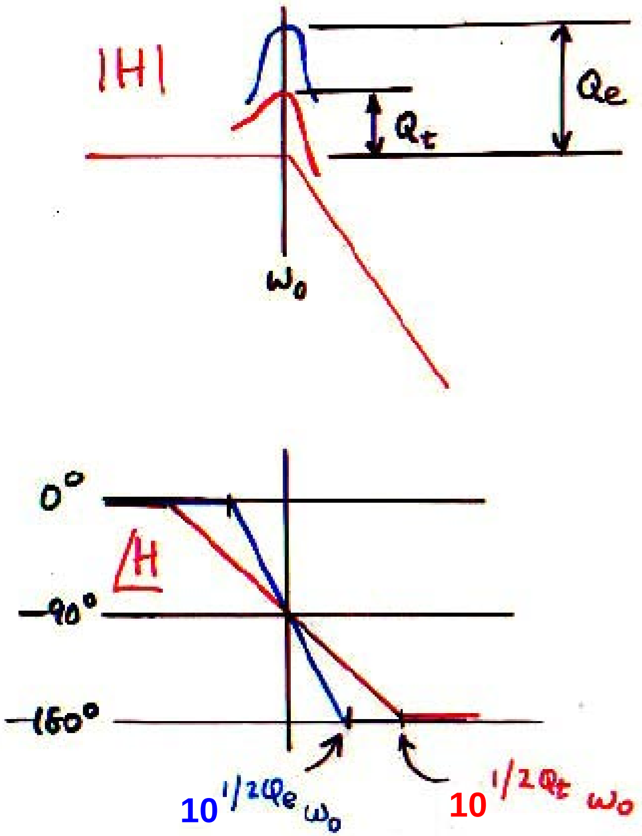
$$\frac{1}{Q_e}+\frac{1}{Q_L}$$

Hence, for the high-Q case,

$$H \approx \frac{1}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

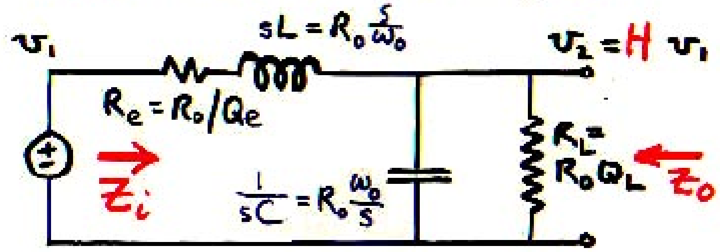
where Q_t is a "total" Q-factor given by the "parallel combination"

$$\frac{1}{Q_t} \equiv \frac{1}{Q_e} + \frac{1}{Q_L}$$


|H|
Q_e
Q_t
ω_0
0°
∠H
-90°
-180°
$10^{1/2Q_e}\omega_0$
$10^{1/2Q_t}\omega_0$

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

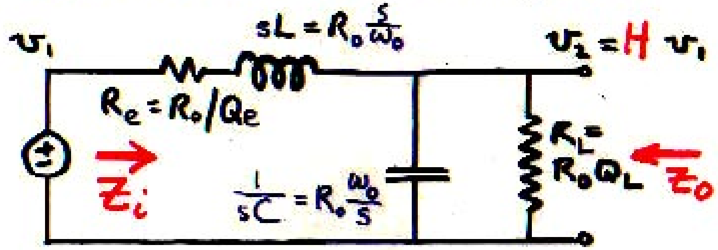
$$Z_i = \frac{R_0 Q_L}{1 + Q_L\left(\frac{s}{\omega_0}\right)} + \frac{R_0}{Q_e} + R_0 \frac{s}{\omega_0} = R_0 \frac{Q_L + \frac{1}{Q_e} + \left(\frac{Q_L}{Q_e} + 1\right)\left(\frac{s}{\omega_0}\right) + Q_L\left(\frac{s}{\omega_0}\right)^2}{1 + Q_L\left(\frac{s}{\omega_0}\right)}$$

$$= R_0\left(1 + 1/Q_eQ_L\right) \frac{1 + \frac{\frac{1}{Q_e} + \frac{1}{Q_L}}{1 + 1/Q_eQ_L}\left(\frac{s}{\omega_0}\right) + \frac{1}{1 + 1/Q_eQ_L}\left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q_L}{s}\right)}$$

Same three effects as for H, but with addition of an inverted pole at ω_0/Q_L.

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



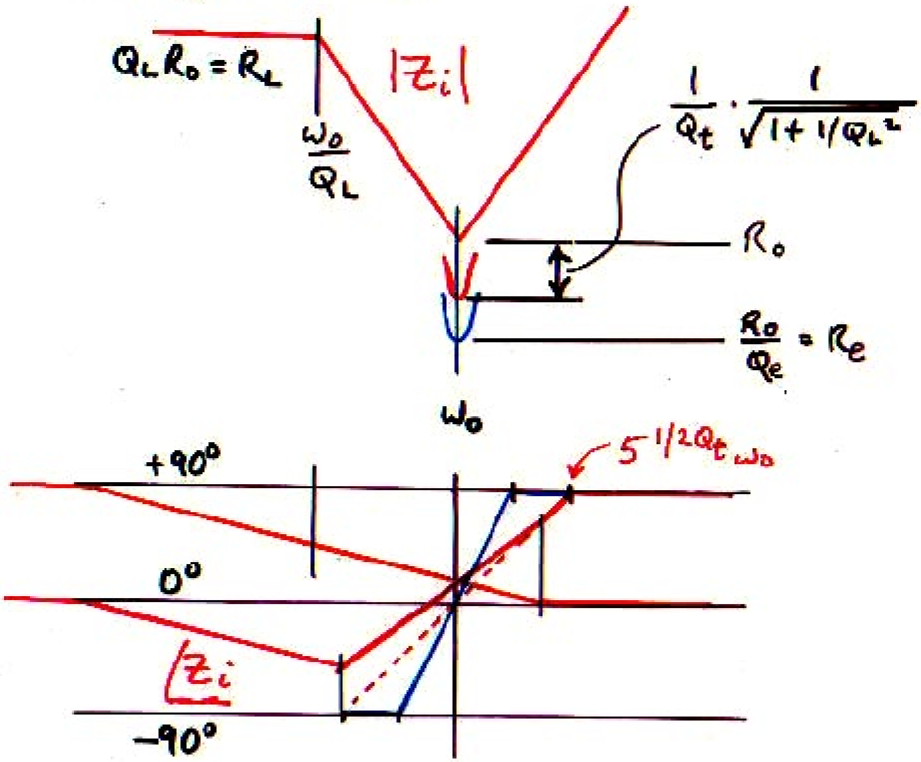
$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

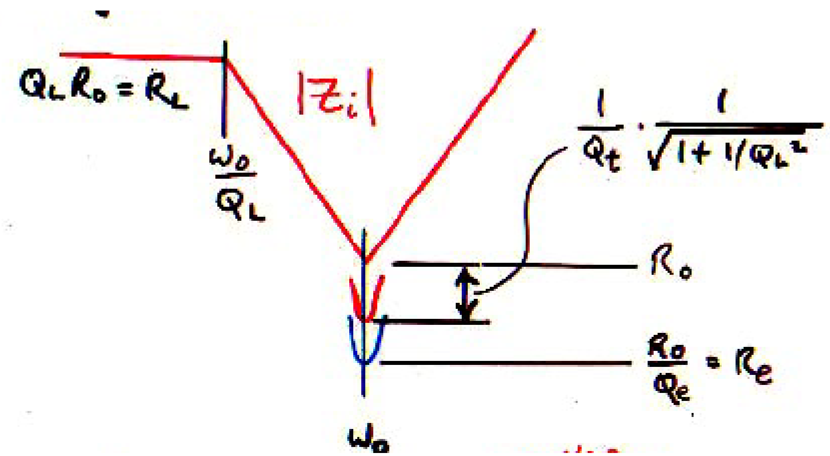
$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_L \equiv \frac{R_L}{R_o}$$

Hence, for the high-Q case,

$$Z_i \approx R_o \, \frac{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q_L}{s}\right)}$$

For high-Q case:



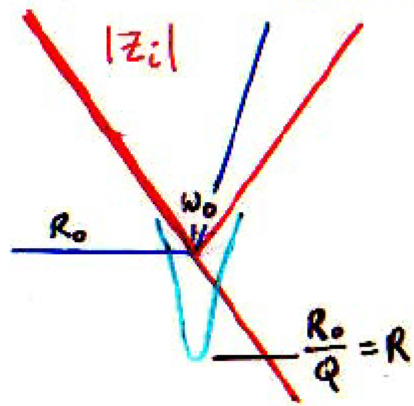


$Q_L R_o = R_L$
$|Z_i|$
$\frac{\omega_o}{Q_L}$
$\frac{1}{Q_t} \cdot \frac{1}{\sqrt{1+1/Q_L^2}}$
R_o
$\frac{R_o}{Q_e} = R_e$
ω_o

Compare the Z_i asymptotes with and without R_L:

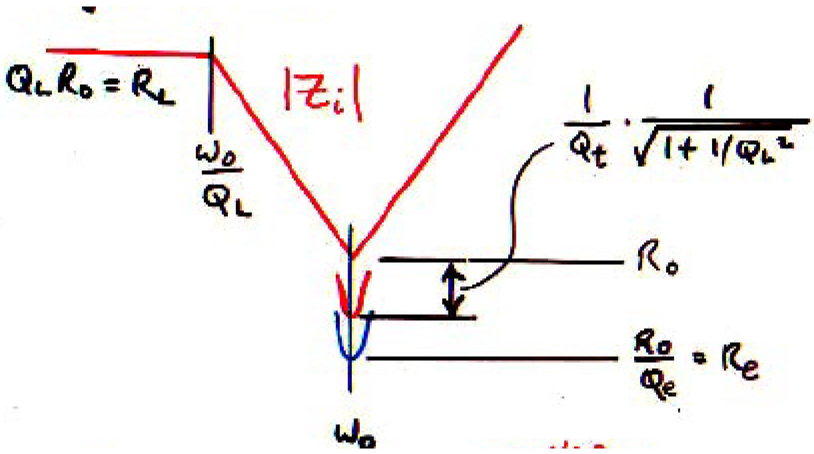
Without R_L:

$(Q_L = \infty)$



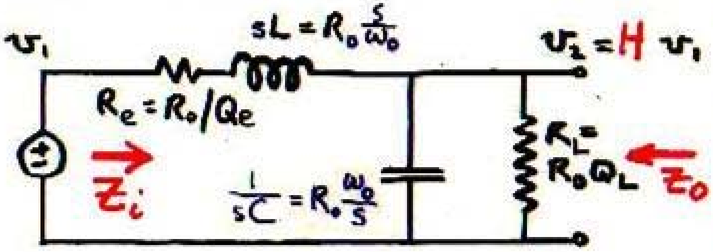
With R_L:

$(Q_L \neq \infty)$



The appearance of the new corner frequency ω_o/Q_L can be confirmed by a mental frequency sweep:

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



$\omega_o \equiv \frac{1}{\sqrt{LC}}$ $\quad R_o \equiv \sqrt{\frac{L}{C}}$

$Q_e \equiv \frac{R_o}{R_e}$ $\quad Q_L \equiv \frac{R_L}{R_o}$

Hence, for the high-Q case,

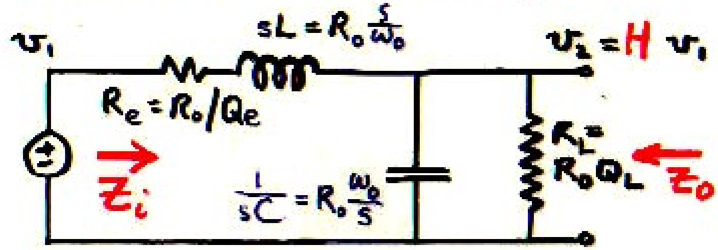
$$Z_i \approx R_o \frac{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q_L}{s}\right)}$$

Without R_L, $Z_i \rightarrow \infty$ as $\omega \rightarrow 0$ because of the capacitive reactance.

With R_L, Z_i flattens, so a concave downwards corner is introduced, which is an inverted pole.

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_L \equiv \frac{R_L}{R_o}$$

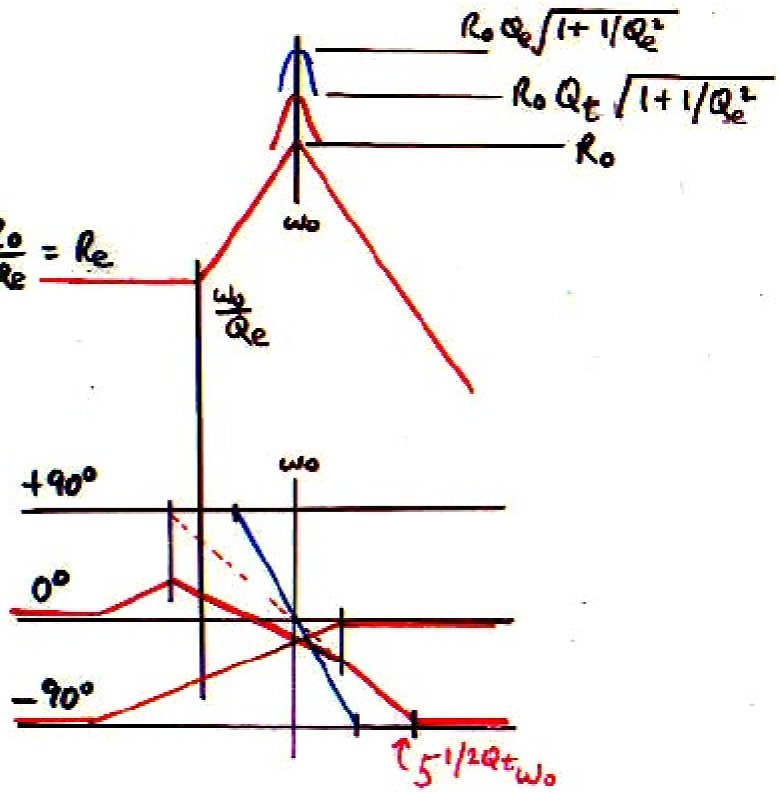
$$Z_o = \frac{\dfrac{Q_L R_o}{1+Q_L\left(\frac{s}{\omega_o}\right)}\left(\dfrac{R_o}{Q_e} + R_o\dfrac{s}{\omega_o}\right)}{\dfrac{Q_L R_o}{1+Q_L\left(\frac{s}{\omega_o}\right)} + \dfrac{R_o}{Q_e} + R_o\dfrac{s}{\omega_o}} = R_o Q_L \frac{\dfrac{1}{Q_e} + \dfrac{s}{\omega_o}}{Q_L + \dfrac{1}{Q_e} + \left(\dfrac{Q_L}{Q_e}+1\right)\left(\dfrac{s}{\omega_o}\right) + \left(\dfrac{s}{\omega_o}\right)^2}$$

$$= R_o \cdot \frac{1}{1+1/Q_eQ_L} \cdot \frac{\left(\dfrac{s}{\omega_o}\right)\left(1 + \dfrac{\omega_o/Q_e}{s}\right)}{1 + \dfrac{\left(\frac{1}{Q_e}+\frac{1}{Q_L}\right)}{1+1/Q_eQ_L}\left(\dfrac{s}{\omega_o}\right) + \dfrac{1}{1+1/Q_eQ_L}\left(\dfrac{s}{\omega_o}\right)^2}$$

Same three effects as for H, so for high-Q case

$$Z_o \approx R_o \frac{\left(\dfrac{s}{\omega_o}\right)\left(1 + \dfrac{\omega_o/Q_e}{s}\right)}{1 + \dfrac{1}{Q_L}\left(\dfrac{s}{\omega_o}\right) + \left(\dfrac{s}{\omega_o}\right)^2}$$

For high-Q case:



When an LC filter is loaded, a 4th effect needs to be accounted for:

Result, compared with unloaded case:

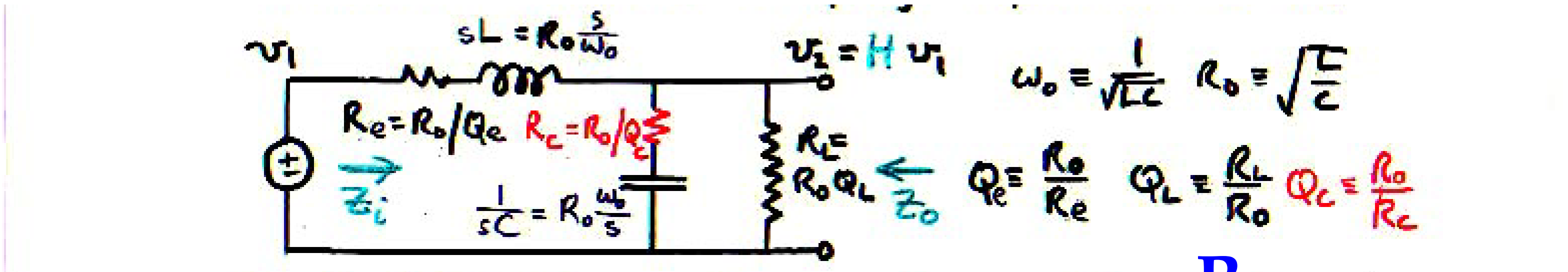
second order → 1. Low-freq. asymptote is $\frac{1}{1+1/Q_eQ_L} = \frac{R_L}{R_L+R_e}$
(resistive divider)

second order → 2. The corner frequency is changed to
$\sqrt{1+1/Q_eQ_L}\,\omega_0$

first order → 3. The damping coefficient is changed to
$$\frac{\frac{1}{Q_e}+\frac{1}{Q_L}}{\sqrt{1+1/Q_eQ_L}}$$

first order → **4. New corner frequencies may appear in some transfer functions**

A third damping resistance R_c may be present, representing the capacitor esr:



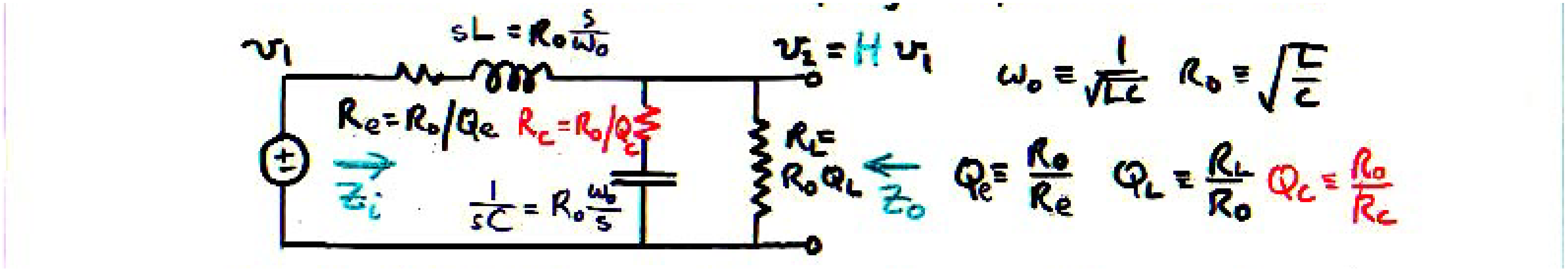
By analogy with Q_e, define $Q_c \equiv \dfrac{R_o}{R_c}$

The analysis for H, Z_i, and Z_o could be re-done in the same way.

Instead, let's *build* the result by applying what we already know about the two simpler cases.

The price we are willing to pay, in order to leap-frog directly to the result, is that the second-order effects will be omitted.

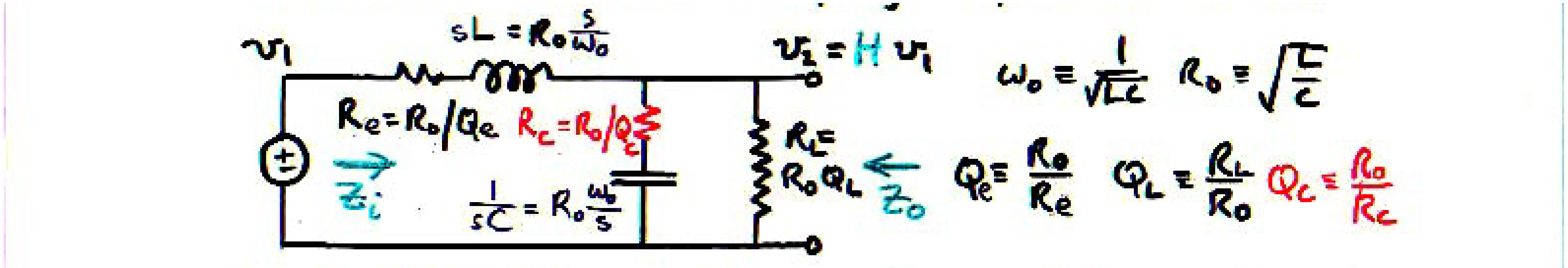
A third damping resistance R_c may be present, representing the capacitor esr:



One first-order effect of adding a second damping resistance was to lower the total Q_t to the parallel combination

$$\frac{1}{Q_t}=\frac{1}{Q_e}+\frac{1}{Q_L}$$

A third damping resistance R_c may be present, representing the capacitor esr:



A *good guess* would be that adding a third damping resistance would lower the total Q_t to the triple parallel combination

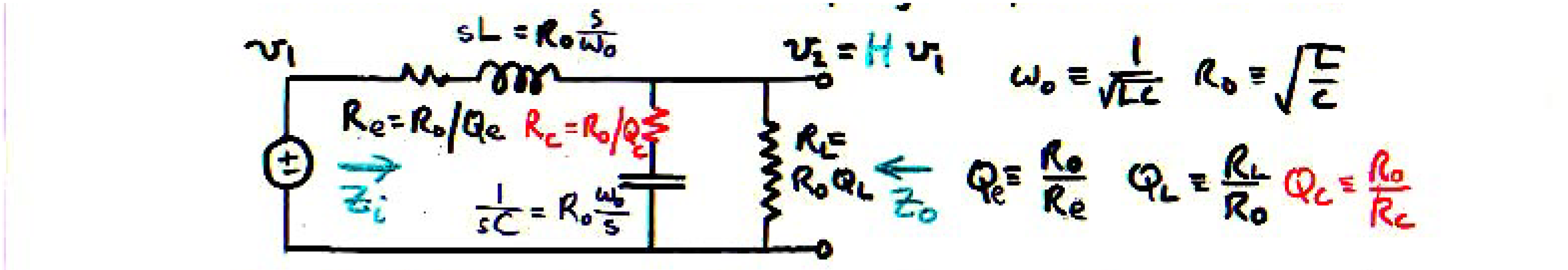
$$\frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_L} + \frac{1}{Q_c}$$

Another possible first-order effect of adding a third damping resistance is the appearance of additional corner frequencies.

A mental frequency sweep can be used to verify an analytical result, but it can also be used "in reverse" to expose new corner frequencies.

The strategy is to determine whether or not the addition of the third damping resistance changes the asymptote slope as frequency approaches either zero or infinity.

For the voltage transfer function H:



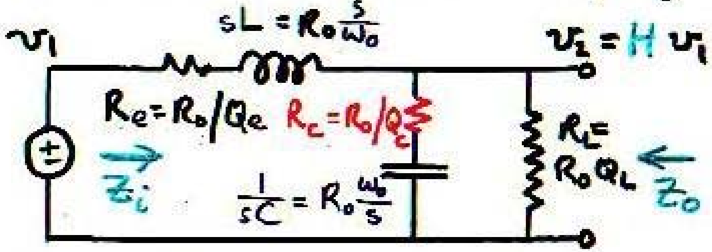
$\omega \to 0$: no change of slope, so no new inverted pole or zero;

$\omega \to \infty$: a concave upwards corner appears, so there is a new normal zero.

Further, the value of the corner is where $1/\omega C = R_c$, which is $1/RC = Q_c \omega_o$.

Assembled results:

Consider additional damping: Capacitor esr R_c



$\omega_0 \equiv \frac{1}{\sqrt{LC}}$ $\quad R_0 \equiv \sqrt{\frac{L}{C}}$

$Q_e \equiv \frac{R_0}{R_e}$ $\quad Q_L \equiv \frac{R_L}{R_0}$ $\quad Q_c \equiv \frac{R_0}{R_c}$

For the high-Q case, the previous results can be extended by inspection:

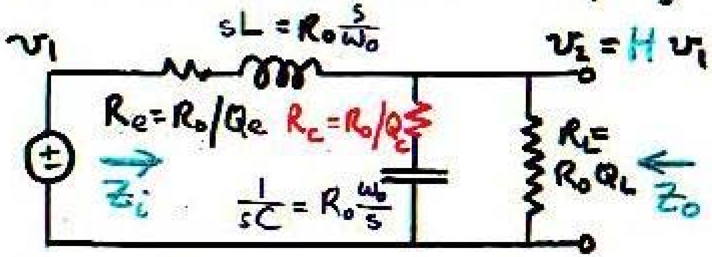
$$H = \frac{1 + \frac{1}{Q_c}\left(\frac{s}{\omega_0}\right)}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2} \quad \Leftarrow 1 + sR_cC$$

triple parallel combination:

$$\frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_L} + \frac{1}{Q_c}$$

Assembled results:

Consider additional damping: Capacitor esr R_c



$\omega_0 \equiv \frac{1}{\sqrt{LC}}$ $\quad R_0 \equiv \sqrt{\frac{L}{C}}$

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For the high-Q case, the previous results can be extended by inspection:

$$H = \frac{1 + \frac{1}{Q_c}\left(\frac{s}{\omega_0}\right)}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2} \Longleftarrow 1 + sR_cC$$

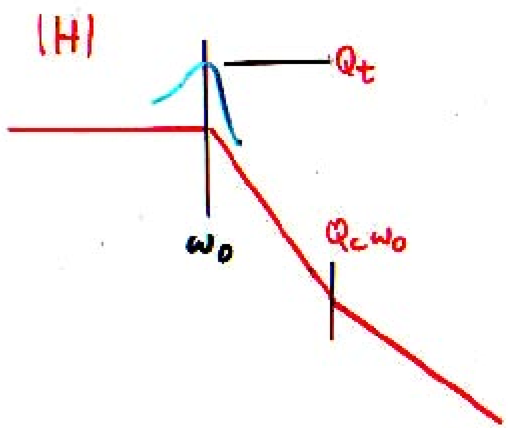
triple parallel combination:

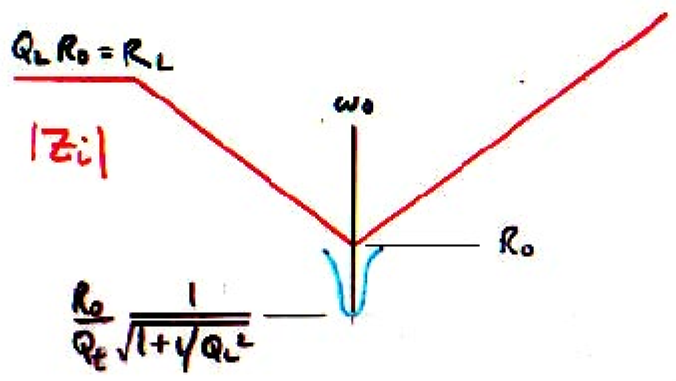
$$\frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_L} + \frac{1}{Q_c}$$

A similar process leads to the assembled result for Z_i:

$$Z_i = R_0 \frac{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q_L}{s}\right)}$$

(No new corners)


|H|
Q_t
ω_0
$Q_L \omega_0$


$Q_L R_0 = R_L$
$|Z_i|$
ω_0
R_0
$\frac{R_0}{Q_t} \frac{1}{\sqrt{1+1/Q_L^2}}$

Principle for extension of results to a more complicated case:

1. Determine the new total Q_t.

2. Add any additional pole or zero factors (Is there any change in the $\omega \to 0$ or $\omega \to \infty$ asymptotes?)

Exercise:

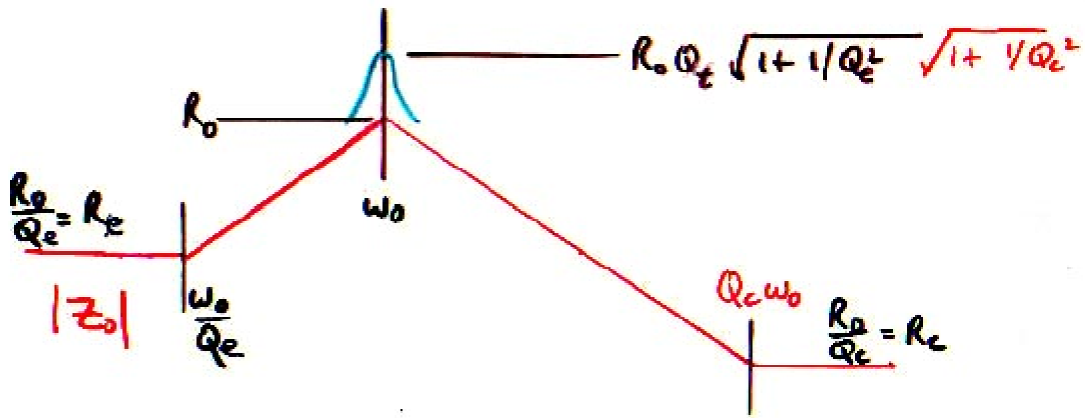
Obtain the corresponding results for Z_0.

Exercise solution:

$$Z_o = R_o \frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q_e}{s}\right)\left(1 + \frac{1}{Q_c}\frac{s}{\omega_o}\right)}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

Check high-frequency limit:

$$Z_o \xrightarrow{\omega \to \infty} \frac{R_o}{Q_c} = R_c$$



The key step is now to determine T12 and T22 from the small signal model for the condition $\hat{vi} = 0$:

$$T12 = \left[\frac{VO \times (RL - D^2 \times N^2 \times Z) \times (1 + sC \times RC)}{(D \times N \times \Delta)} \right] \qquad [15]$$

$$T22 = \left[\frac{VO \times (RL - D^2 \times N^2 \times Z) \times (1 + sC \times RL)}{(D \times N \times RL \times \Delta)} \right] \qquad [16]$$

where

$$Z = \left[\frac{\overbrace{(s^2L1 \times C1 \times RC1) + (sC1 \times RC1 \times RL1) + sL1 + RL1}^{(sL_1 + R_{L1})(sC_1 R_{C1} + 1)}}{(s^2L1 \times C1) + sC1 \times (RL1 + RC1) + 1} \right] \qquad [17]$$

$$\Delta = \left[a_1 + (D^2 \times N^2 \times Z) \times (1 + sC \times RL) \right] \qquad [18]$$

$$a_1 = \left[(s^2L \times C \times RL) + sC \times RL \times \left(RI + RC + \frac{L}{(C \times RL)}\right) + RL \right] \qquad [19]$$

At the resonant frequency of the input filter, the impedance Z will attain a very high value, limited only by the series resistances RL1 and RC1. The peaking in the value of Z will affect both the numerators and denominators of the transfer functions T12 and T22, as shown in equations 15 and 16. The net effect will be a reduction in the loop gain G_f and a corresponding phase margin reduction.