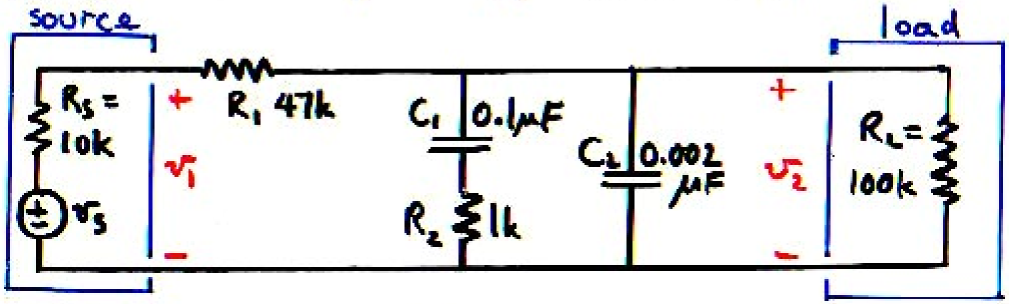
4. AN IMPROVED FORMULA FOR QUADRATIC ROOTS

The Conventional Formula suffers from two congenital defects

Example

Analyze the following circuit for the gain response v_2/v_1, using the given values to justify appropriate analytic approximations:



Express the result in the factored pole-zero form

$$\frac{v_2}{v_1} \equiv A = A_0 \frac{\prod (1 + s/\omega_x)}{\prod (1 + s/\omega_y)}$$

Sketch $|A|$ and $\angle A$ showing the straight-line asymptotes, and label salient features with both analytic expressions and numerical values.

$$A = \frac{\dfrac{\left(\dfrac{R_L}{1+sC_2R_L}\right)\left(R_2+\dfrac{1}{sC_1}\right)}{\dfrac{R_L}{1+sC_2R_L}+R_2+\dfrac{1}{sC_1}}}{\dfrac{\left(\dfrac{R_L}{1+sC_2R_L}\right)\left(R_2+\dfrac{1}{sC_1}\right)}{\dfrac{R_L}{1+sC_2R_L}+R_2+\dfrac{1}{sC_1}}+R_1}$$

a lot of algebra

$$= \frac{R_L + sC_1R_2R_L}{[R_1+R_L]+s[C_1(R_1R_2+R_LR_2+R_1R_L)+C_2R_1R_L]+s^2[C_1C_2R_1R_2R_L]}$$

This is a high-entropy expression. To lower the entropy, write the polynomials in s with a leading term of unity:

$$A = \frac{R_L}{R_1 + R_L} \cdot \frac{1 + sC_1R_2}{1 + s\left[C_1\left(\frac{R_1R_2 + R_LR_2 + R_1R_L}{R_1 + R_L}\right) + C_2\left(\frac{R_1R_L}{R_1 + R_L}\right)\right] + s^2\left[C_1C_2\left(\frac{R_1R_2R_L}{R_1 + R_L}\right)\right]}$$

Now, recognize series/parallel resistance combinations:

$$(R_2 + R_1 \| R_L) \qquad (R_1 \| R_L) \qquad R_2(R_1 \| R_L)$$

$$A = \frac{R_L}{R_1 + R_L} \frac{1 + sC_1R_2}{1 + s\left[C_1\left(\frac{R_1R_2 + R_LR_2 + R_1R_L}{R_1 + R_L}\right) + C_2\left(\frac{R_1R_L}{R_1 + R_L}\right)\right] + s^2\left[C_1C_2\left(\frac{R_1R_2R_L}{R_1+R_L}\right)\right]}$$

Now, recognize series/parallel resistance combinations:

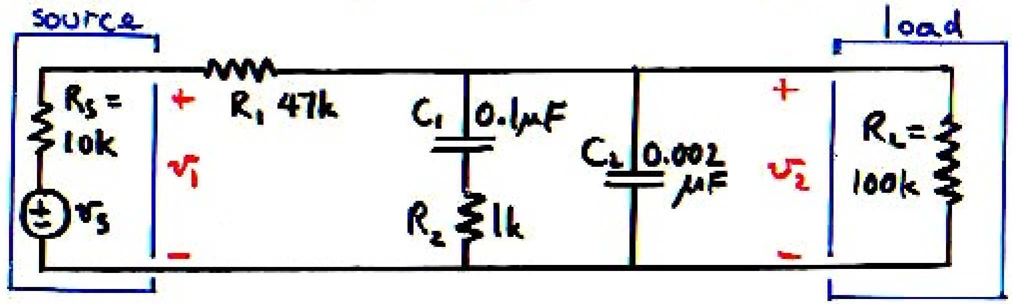
$$(R_2 + R_1 \| R_L) \qquad (R_1 \| R_L) \qquad R_2(R_1 \| R_L)$$

The same result, including the series/parallel resistance grouping, could have been obtained with less algebra by elimination, first, of one of the loops of the original circuit.

Circuit with R_1 and R_L absorbed into a Thevenin equivalent:

Example

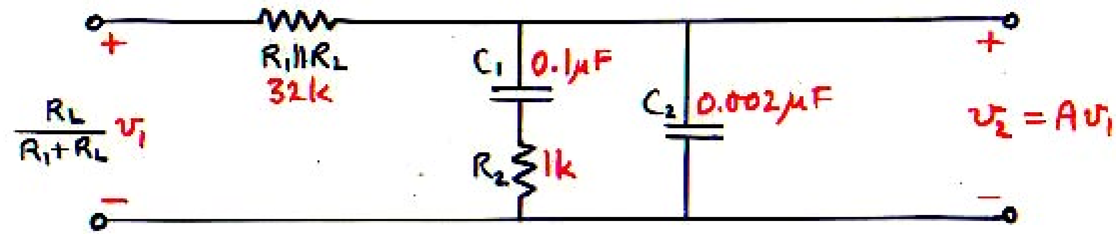
Analyze the following circuit for the gain response v_2/v_1, using the given values to justify appropriate analytic approximations:



Express the result in the factored pole-zero form

$$\frac{v_2}{v_1} \equiv A = A_0 \frac{\prod (1 + s/\omega_x)}{\prod (1 + s/\omega_y)}$$

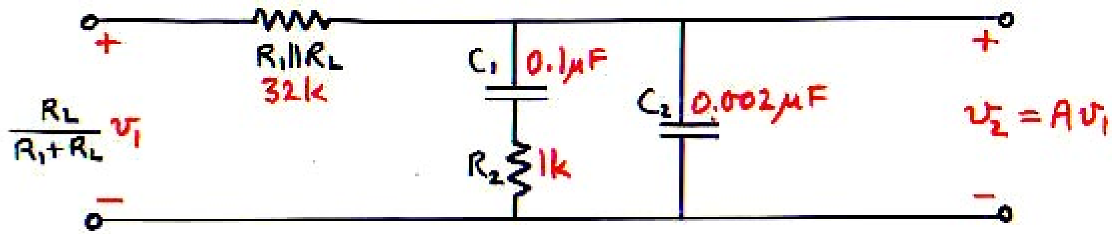
Sketch $|A|$ and $\angle A$ showing the straight-line asymptotes, and label salient features with both analytic expressions and numerical values.



$$A = \frac{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)}}{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)} + R_1\|R_L} \cdot \frac{R_L}{R_1+R_L}$$

less algebra

$$= \frac{R_L}{R_1+R_L} \cdot \frac{1+sC_1R_2}{1+s\left[C_1(R_2+R_1\|R_L)+C_2(R_1\|R_L)\right]+s^2\left[C_1C_2R_2(R_1\|R_L)\right]}$$



$$A = \frac{\dfrac{\frac{1}{sC_2}\left(R_2 + \frac{1}{sC_1}\right)}{R_2 + \frac{1}{s}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)}}{\dfrac{\frac{1}{sC_2}\left(R_2 + \frac{1}{sC_1}\right)}{R_2 + \frac{1}{s}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)} + R_1 \| R_L} \cdot \frac{R_L}{R_1 + R_L}$$

less $\Big\downarrow$ algebra

$$= \frac{R_L}{R_1 + R_L} \cdot \frac{1 + sC_1R_2}{1 + s\left[C_1(R_2 + R_1\|R_L) + \cancel{C_2(R_1\|R_L)}\right] + s^2\left[C_1C_2R_2(R_1\|R_L)\right]}$$

Use of numerical values to justify analytic approximation

Generalization: Use of Numerical Values to Justify Analytic Approximations

Use numbers to justify leaving out a term, but continue the analysis with the symbols.

This way, the analysis result can be used for design, because the numbers can be changed so that the answer has the desired value. (The approximation must be checked to ensure that it is not invalidated by the new numbers.)

You can't lose by trying!

$$A = \frac{R_L}{R_1+R_L} \frac{1+sC_1R_2}{1+s[C_1(R_2+(R_1 \| R_L))]+s^2[C_1C_2R_2(R_1 \| R_L)]}$$

$$= A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)}$$

where

$$\frac{1}{\omega_{1,3}} = \frac{\overset{3.3\times10^{-3}}{C_1(R_2+R_1\|R_L)} \pm \sqrt{\overset{10\times10^{-6}}{C_1^2(R_2+R_1\|R_L)^2} - \overset{0.026\times10^{-6}}{4C_1C_2R_2(R_1\|R_L)}}}{2}$$

This is useless for <u>design</u>, and in any case is inaccurate numerically.

Improved formulas for quadratic roots

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a(x - x_1)(x - x_2)$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

High entropy

Disadvantages of the conventional form

1. Complicated algebraic expressions in terms of element values:

$$\frac{1}{\omega_{1,3}} = \frac{C_1(R_2 + R_1 \| R_L) \pm \sqrt{C_1^2(R_2 + R_1\|R_L)^2 - 4C_1C_2R_2(R_1\|R_L)}}{2}$$

2. Computationally inaccurate when $4ac \ll b^2$:

$$\frac{1}{\omega_{1,3}} = 10^{-3}\,\frac{3.3 \pm \sqrt{3.3^2 - 0.026}}{2}$$

small difference of large numbers, for one root

These are congenital defects!

Better method:

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right] = -\frac{b}{a}F$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}, \qquad Q^2 \equiv \frac{ac}{b^2}$$

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-4Q^2}\right]$$

$$= -\frac{b}{a}\frac{\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-4Q^2}\right]\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right]}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right]} = -\frac{b}{a}\frac{\frac{1}{4}-\frac{1}{4}(1-4Q^2)}{F}$$

$$= -\frac{b}{a}\frac{Q^2}{F} = -\frac{c}{b}\frac{1}{F}$$

Better method:

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}\right] = -\frac{b}{a}F$$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}, \qquad Q^2 \equiv \frac{ac}{b^2}$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4Q^2}\right]$$

$$= -\frac{b}{a}\,\frac{\left[\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4Q^2}\right]\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}\right]}{\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}\right]} = -\frac{b}{a}\,\frac{\frac{1}{4} - \frac{1}{4}(1 - 4Q^2)}{F}$$

$$= -\frac{b}{a}\,\frac{Q^2}{F} = -\frac{c}{b}\,\frac{1}{F}$$

Crucial step: Large numbers are subtracted exactly, leaving the small difference in analytic form.

Hence, both roots can be computed with equal accuracy:

$$x_1 = -\frac{c}{b}\,\frac{1}{F} \qquad x_2 = -\frac{b}{a}F$$

Rewrite the two roots:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

x_2 is acceptable for all values; x_1 is unacceptable for $4ac \ll b^2$.

Rewrite x_2:

$$x_2 = -\frac{b}{a}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}\right]$$

Now, instead of using the formula for x_1 directly, use the property of the quadratic that $x_1 x_2 = \frac{c}{a}$:

$$x_1 = \frac{c}{a}\frac{1}{x_2} = -\frac{c}{a}\frac{a}{b}\frac{1}{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}}$$

Thus, the improved formulas for the quadratic roots are:

$$x_1 = -\frac{c}{b}\frac{1}{\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}\right]} \qquad x_2 = -\frac{b}{a}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}\right]$$

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$$x_1 = \frac{c}{a}\frac{1}{x_2} = -\frac{c}{a}\frac{a}{b}\frac{1}{\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}}$$

Thus, the improved formulas for the quadratic roots are:

$$x_1 = -\frac{c}{b}\frac{1}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]} \qquad x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

$$\boxed{\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \\ &= a(x-x_1)(x-x_2) \end{aligned}}$$

$x_1 x_2 = \frac{c}{a}$

Rewrite the two roots:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

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$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a(x - x_1)(x - x_2)$$

More elegant form:

$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad \frac{x_1}{x_2} = \frac{Q^2}{F^2} \qquad x_2 = -\frac{b}{a}F$$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2} \quad \text{in which} \quad Q^2 \equiv \frac{ac}{b^2}$$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
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root ratio

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simple ratios of the original quadratic coefficients

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simple ratios of the original quadratic coefficients

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2} \quad \text{in which} \quad Q^2 \equiv \frac{ac}{b^2}$$

This is exact for all values.

If $Q > 0.5$, F is complex $\Rightarrow$ complex roots

If $Q < 0.5$, F is real $\Rightarrow$ real roots

If $Q \ll 0.5$, $F \approx 1$

Note how simple the analytic roots, and therefore the quadratic factorization, become if $F \approx 1$.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a(x - x_1)(x - x_2)$$

root ratio

More elegant form:

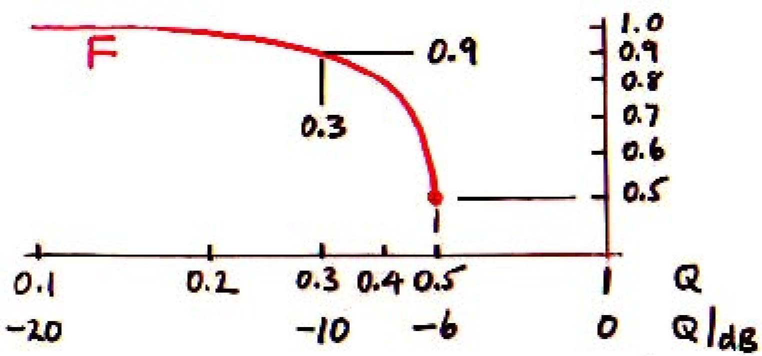
$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad \frac{x_1}{x_2} = \frac{Q^2}{F^2} \qquad x_2 = -\frac{b}{a}F$$

simple ratios of the original quadratic coefficients

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2} \quad \text{in which} \quad Q^2 \equiv \frac{ac}{b^2}$$

$F \rightarrow 1$ very rapidly as Q drops below 0.5:



$F \approx 1$ with 10% error for $Q \leq 0.3$

Remember this graph!

One more time!

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a(x - x_1)(x - x_2)$$

Bad!

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Good!

$$x_1 = -\frac{c}{b}\frac{1}{\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}\right]} \qquad x_2 = -\frac{b}{a}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}\right]$$

One more time!

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
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Further information:

TT DVD Ch 3

Paper

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IMPORTANT: Dated Materials Enclosed

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

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$$x_1 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

$$x_2 = -\frac{c}{b}\Bigg/\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

IMPORTANT: Dated Materials Enclosed

General result:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a(x - x_1)(x - x_2)$$

$$= a\left(x + \frac{c}{b}\frac{1}{F}\right)\left(x + \frac{b}{a}F\right)$$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}$$

$$x_1 = -\frac{c}{b}\frac{1}{F}$$

$$x_2 = -\frac{b}{a}F$$

General result:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a(x - x_1)(x - x_2)$$

$$= a\left(x + \frac{c}{b}\frac{1}{F}\right)\left(x + \frac{b}{a}F\right) \overset{F=1}{\approx} a\left(x + \frac{c}{b}\right)\left(x + \frac{b}{a}\right)$$

Good approximation for real roots, $Q \equiv \sqrt{\frac{ac}{b^2}} \leq 0.5$: $F \approx 1$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}$$

$$x_1 = -\frac{c}{b}\frac{1}{F} \approx -\frac{c}{b}$$

$$x_2 = -\frac{b}{a}F \approx -\frac{b}{a}$$

Alternative format:

$$ax^2 + bx + c = c\left(1 + \frac{b}{c}x + \frac{a}{c}x^2\right)$$

$$= c\left(1 - \frac{x}{x_1}\right)\left(1 - \frac{x}{x_2}\right)$$

$$= c\left(1 + \frac{b}{c}Fx\right)\left(1 + \frac{a}{b}\frac{1}{F}x\right)$$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2}$$

Alternative format:

$$ax^2+bx+c = c\left(1+\overset{\overset{a_1}{\downarrow}}{\frac{b}{c}}x+\overset{\overset{a_2}{\downarrow}}{\frac{a}{c}}x^2\right)$$

$$= c\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= c\left(1+\frac{b}{c}Fx\right)\left(1+\frac{a}{b}\frac{1}{F}x\right)$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2}$$

Redefine coefficients:

$$1+a_1x+a_2x^2 = \left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= \left(1+a_1Fx\right)\left(1+\frac{a_2}{a_1}\frac{1}{F}x\right)$$

where
$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{a_2}{a_1^2}$$

Alternative format:

$$ax^2+bx+c = c\left(1+\overset{a_1}{\frac{b}{c}}x+\overset{a_2}{\frac{a}{c}}x^2\right)$$

$$= c\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

Good approximation for real roots, $Q \le 0.5$: $F \approx 1$

$$= c\left(1+\frac{b}{c}Fx\right)\left(1+\frac{a}{b}\frac{1}{F}x\right) \overset{F=1}{\approx} c\left(1+\frac{b}{c}x\right)\left(1+\frac{a}{b}x\right)$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2}$$

Redefine coefficients:

$$1+a_1x+a_2x^2 = \left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= \left(1+a_1Fx\right)\left(1+\frac{a_2}{a_1}\frac{1}{F}x\right) \overset{F=1}{\approx} \left(1+a_1x\right)\left(1+\frac{a_2}{a_1}x\right)$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{a_2}{a_1^2}$$

Generalization: Improved Formulas for Roots of a Quadratic

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2} \qquad\qquad Q^2 \equiv \frac{a_2}{a_1^2}$$

$$ax^2 + bx + c = a(x-x_1)(x-x_2) \qquad\qquad 1 + a_1x + a_2x^2 = \left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad\qquad x_1 = -\frac{1}{a_1F}$$

$$\frac{x_1}{x_2} = \frac{Q^2}{F^2}$$

$$x_2 = -\frac{b}{a}F \qquad\qquad x_2 = -\frac{a_1}{a_2}F$$

$$ax^2 + bx + c = a\left(x+\frac{c}{b}\frac{1}{F}\right)\left(x+\frac{b}{a}F\right) \qquad\qquad 1 + a_1x + a_2x^2 = (1+a_1Fx)\left(1+\frac{a_2}{a_1}\frac{1}{F}x\right)$$

For real roots, $Q \le 0.5$ and $F \approx 1$:

$$x_1 \approx -\frac{c}{b} \qquad\qquad x_1 \approx -\frac{1}{a_1}$$

$$\frac{x_1}{x_2} \approx Q^2$$

$$x_2 \approx -\frac{b}{a} \qquad\qquad x_2 \approx -\frac{a_1}{a_2}$$

$$\underset{①}{ax^2} + \underset{②}{bx} + \underset{③}{c} \approx a\left(x+\underset{③/②}{\frac{c}{b}}\right)\left(x+\underset{②/①}{\frac{b}{a}}\right) \qquad\qquad \underset{①}{1} + \underset{②}{a_1x} + \underset{③}{a_2x^2} \approx (1+\underset{②/①}{a_1}x)\left(1+\underset{③/②}{\frac{a_2}{a_1}}x\right)$$

Advantages over the conventional formulas

1. Both roots can be computed with equal accuracy (avoids small difference of large numbers).

2. For real roots, to a very good approximation, there is no $\sqrt{}$ anywhere in the results, and each root is a simple ratio of coefficients of the original quadratic.

Useful format of quadratic $1 + a_1 s + a_2 s^2$

Define: $Q \equiv \frac{\sqrt{a_2}}{a_1}$

If $Q > 0.5$ (F complex), roots are complex.
Leave in quadratic form:

$$1 + a_1 s + a_2 s^2 = 1 + \underbrace{\frac{a_1}{\sqrt{a_2}}}_{\frac{1}{Q}} \left(\sqrt{a_2}\, s\right) + \underbrace{\left(\sqrt{a_2}\, s\right)}_{\text{normalized frequency}}{}^2$$

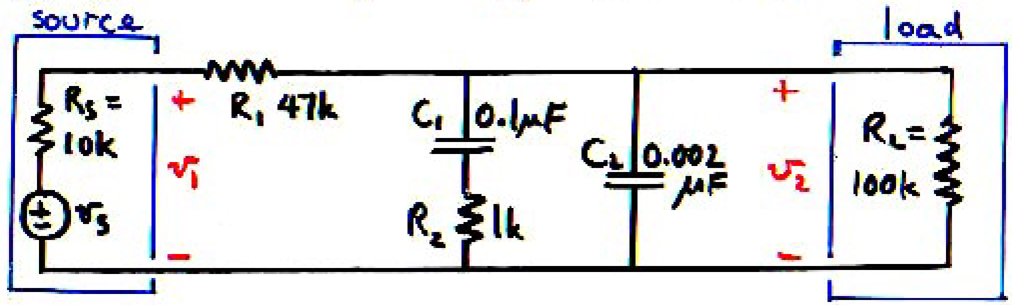
If $Q < 0.5$ ($F \approx 1$), roots are real.
Factor into two real roots: real corner frequencies

$$1 + a_1 s + a_2 s^2 \approx \left(1 + a_1 s\right)\left(1 + \frac{a_2}{a_1} s\right)$$

$$= \left[1 + \underbrace{\frac{a_1}{\sqrt{a_2}}}_{\frac{1}{Q}} \left(\sqrt{a_2}\, s\right)\right]\left[1 + \underbrace{\frac{\sqrt{a_2}}{a_1}}_{Q} \underbrace{\left(\sqrt{a_2}\, s\right)}_{\text{normalized frequency}}\right]$$

Return to the circuit example:

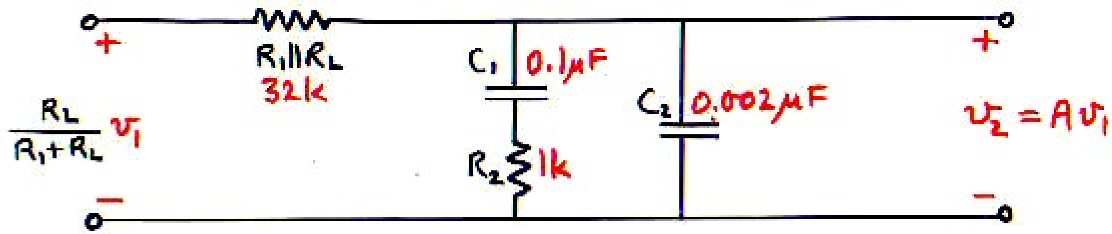
Analyze the following circuit for the gain response v_2/v_1, using the given values to justify appropriate analytic approximations:



Express the result in the factored pole-zero form

$$\frac{v_2}{v_1} \equiv A = A_0 \frac{\prod (1+s/\omega_x)}{\prod (1+s/\omega_y)}$$

Sketch $|A|$ and $\angle A$ showing the straight-line asymptotes, and label salient features with both analytic expressions and numerical values.



$$A = \frac{\dfrac{\frac{1}{sC_2}\left(R_2 + \frac{1}{sC_1}\right)}{R_2 + \frac{1}{s}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)}}{\dfrac{\frac{1}{sC_2}\left(R_2 + \frac{1}{sC_1}\right)}{R_2 + \frac{1}{s}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)} + R_1 \| R_L} \cdot \frac{R_L}{R_1 + R_L}$$

less algebra

$$= \frac{R_L}{R_1 + R_L} \cdot \frac{1 + sC_1R_2}{1 + s\left[C_1(R_2 + R_1\|R_L) + \cancel{C_2(R_1\|R_L)}\right] + s^2\left[C_1C_2R_2(R_1\|R_L)\right]}$$

Use of numerical values to justify analytic approximation

Find, both analytically and numerically, the Q and hence the roots ω_1 and ω_3 of the quadratic:

$$1 + C_1[R_2 + R_1 \| R_L]s + [C_1 C_2 R_2 (R_1 \| R_L)]s^2$$

where $C_1 = 0.1\mu F$, $C_2 = 0.002\mu F$, $R_1 = 47k$, $R_2 = 1k$, $R_L = 100k$. Express the analytic results in terms of series/parallel element combinations, and express the numerical results in Hz or kHz.

Find, both analytically and numerically, the Q and hence the roots ω_1 and ω_3 of the quadratic:

$$1 + \underbrace{C_1[R_2 + R_1\|R_L]}_{a_1}s + \underbrace{[C_1C_2R_2(R_1\|R_L)]}_{a_2}s^2$$

where $C_1 = 0.1\mu F$, $C_2 = 0.002\mu F$, $R_1 = 47k$, $R_2 = 1k$, $R_L = 100k$. Express the analytic results in terms of series/parallel element combinations, and express the numerical results in Hz or kHz.

$$Q^2 = \frac{a_2}{a_1^2} = \frac{C_1C_2R_2(R_1\|R_L)}{C_1^2(R_2+R_1\|R_L)^2} = \frac{C_2}{C_1}\frac{R_2\|\cancel{R_1\|R_L}}{\cancel{R_2}+R_1\|R_L} \approx \frac{C_2}{C_1}\frac{R_2}{R_1\|R_L} = \frac{1}{50}\frac{1}{47\|100} = \frac{1}{1{,}600}$$

Hence, $F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2} \approx 1$, so the roots are real: $(F = 0.9994)$ $\quad Q = \frac{1}{40} \ll 0.5$

$$\omega_1 = \frac{1}{a_1} = \frac{1}{C_1(R_2+R_1\|R_L)} \qquad f_1 = \frac{159}{0.1(1+\underbrace{47\|100}_{32})}\,\text{Hz} = 48\,\text{Hz}$$

$$\omega_3 = \frac{a_1}{a_2} = \frac{C_1(R_2+R_1\|R_L)}{C_1C_2R_2(R_1\|R_L)} = \frac{1}{C_2(R_2\|R_1\|R_L)} \qquad f_3 = \frac{159}{0.002(\underbrace{1\|32}_{0.97})}\,\text{Hz} = 82\,\text{kHz}$$

Hence

$$A \approx \frac{R_L}{R_L + R_1} \frac{1 + sC_1R_2}{[1 + C_1(R_2 + R_1 \| R_L)s][1 + C_2(R_1 \| R_2 \| R_L)s]}$$

$$= A_0 \frac{\left(1 + \frac{s}{\omega_2}\right)}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_3}\right)}$$

where

$$A_0 \equiv \frac{R_L}{R_L + R_1}$$

$$\omega_1 \equiv \frac{1}{C_1(R_2 + R_1 \| R_L)}$$

$$\omega_2 \equiv \frac{1}{C_1 R_2}$$

$$\omega_3 \equiv \frac{1}{C_2(R_1 \| R_2 \| R_L)}$$

Hence

$$A \approx \frac{R_L}{R_L+R_1} \frac{1+sC_1R_2}{[1+C_1(R_2+R_1\|R_L)s][1+C_2(R_1\|R_2\|R_L)s]}$$

$$= A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)}$$

where

$$A_0 \equiv \frac{R_L}{R_L+R_1} = \frac{100}{100+47} = 0.68 \Rightarrow -3.4\text{dB}$$

$$\omega_1 \equiv \frac{1}{C_1(R_2+R_1\|R_L)} \qquad f_1 = \frac{159}{0.1(1+\underbrace{47\|100}_{32})} = 48\text{Hz}$$

$$\omega_2 \equiv \frac{1}{C_1R_2} \qquad f_2 = \frac{159}{0.1\times 1} = 1.6\text{kHz}$$

$$\omega_3 \equiv \frac{1}{C_2(R_1\|R_2\|R_L)} \qquad f_3 = \frac{159}{0.002(47\|1\|100)} = 82\text{kHz}$$

The conventional quadratic formula for the two poles ω_1 and ω_3 is much higher entropy (gives much less useful information) than does the modified formula.

Conventional:

$$\frac{1}{\omega_{1,3}} = \frac{C_1(R_2+R_1\|R_L) \pm \sqrt{C_1^2(R_2+R_1\|R_L)^2 - 4C_1C_2R_2(R_1\|R_L)}}{2}$$

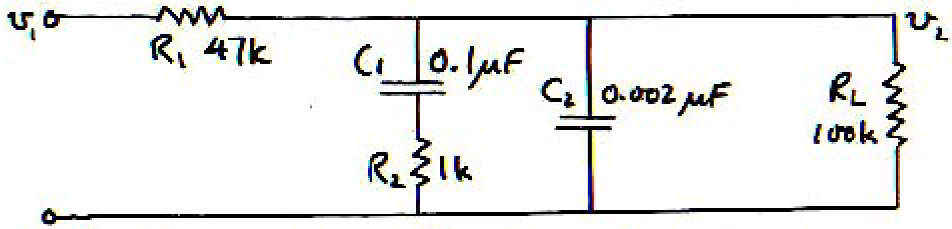
Modified:

$$\omega_1 = \frac{1}{C_1(R_2+R_1\|R_L)} \qquad \omega_3 = \frac{1}{C_2(R_1\|R_2\|R_L)}$$

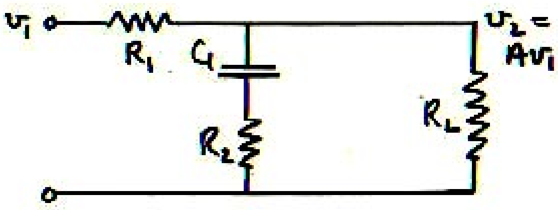
Note, in particular, (when the two roots are real and well-separated) that the modified formula is much lower entropy and not only gives both roots with equal numerical accuracy, but also exposes the fact that C_1 affects only ω_1 and C_2 affects only ω_3 — which is useful information for design purposes.

A still better solution:
Apply the mental frequency sweep

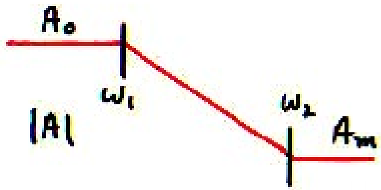
Look at the original circuit and consider response as frequency increases:



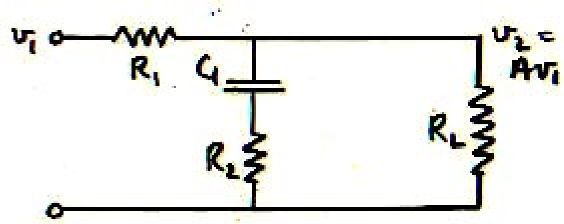
At low frequencies, both capacitances are open, so have flat response. As frequency increases, the reactance of C_1, the larger capacitance, comes down causing a pole. When the reactance of C_1 drops below R_2, the response flattens causing a zero. However, at this frequency the reactance of C_2 is still 50 times higher than R_2, so C_2 has negligible effect.

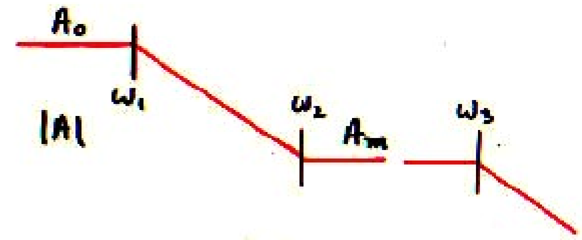


v_1
R_1
C_1
R_2
R_L
$v_2 = Av_1$

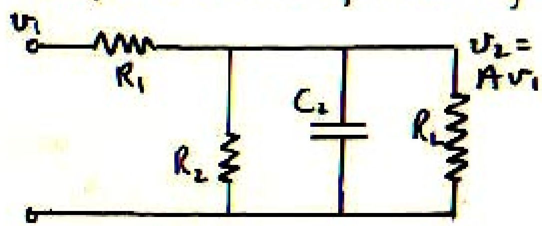


A_0
$|A|$
ω_1
ω_2
A_m



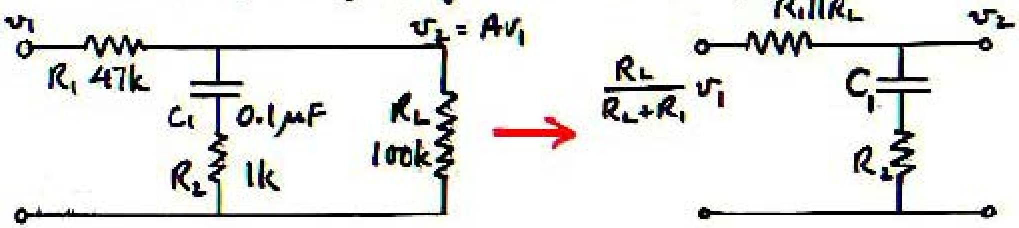


At still higher frequencies, the reactance of C_2 drops below R_2, causing a second pole



Hence, the solution can be obtained in two parts, each containing only one reactance (one pole).

Low-frequency range (C_2 open)



$$A = \frac{R_L}{R_L+R_1}\,\frac{R_2+\frac{1}{sC_1}}{R_2+\frac{1}{sC_1}+R_1\|R_L} = A_0\,\frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} = A_m\frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

where

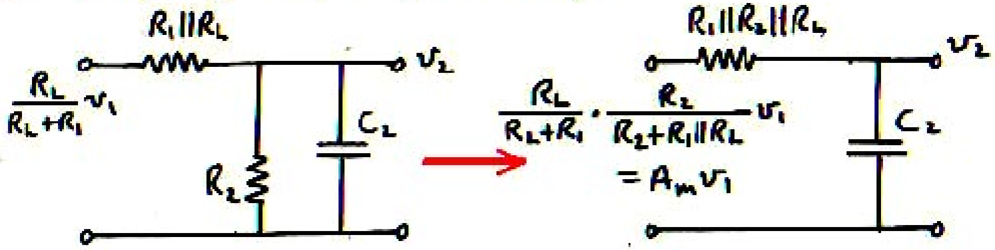
$$A_0 \equiv \frac{R_L}{R_L+R_1} = \frac{100}{100+47} = 0.68 \Rightarrow -3.4\,\mathrm{dB}$$

$$\omega_1 \equiv \frac{1}{C_1(R_2+R_1\|R_L)} \qquad f_1 = \frac{159}{0.1(1+\underbrace{47\|100}_{32})} = 48\,\mathrm{Hz}$$

$$\omega_2 \equiv \frac{1}{C_1R_2} \qquad f_2 = \frac{159}{0.1\times 1} = 1.6\,\mathrm{kHz}$$

$$A_m \equiv A_0\frac{\omega_1}{\omega_2} = \frac{R_L}{R_L+R_1}\cdot\frac{R_2}{R_2+R_1\|R_L} = 0.68\,\frac{0.048}{1.6} = 0.02 \Rightarrow -34\,\mathrm{dB}$$

High-frequency range (C_1 short)

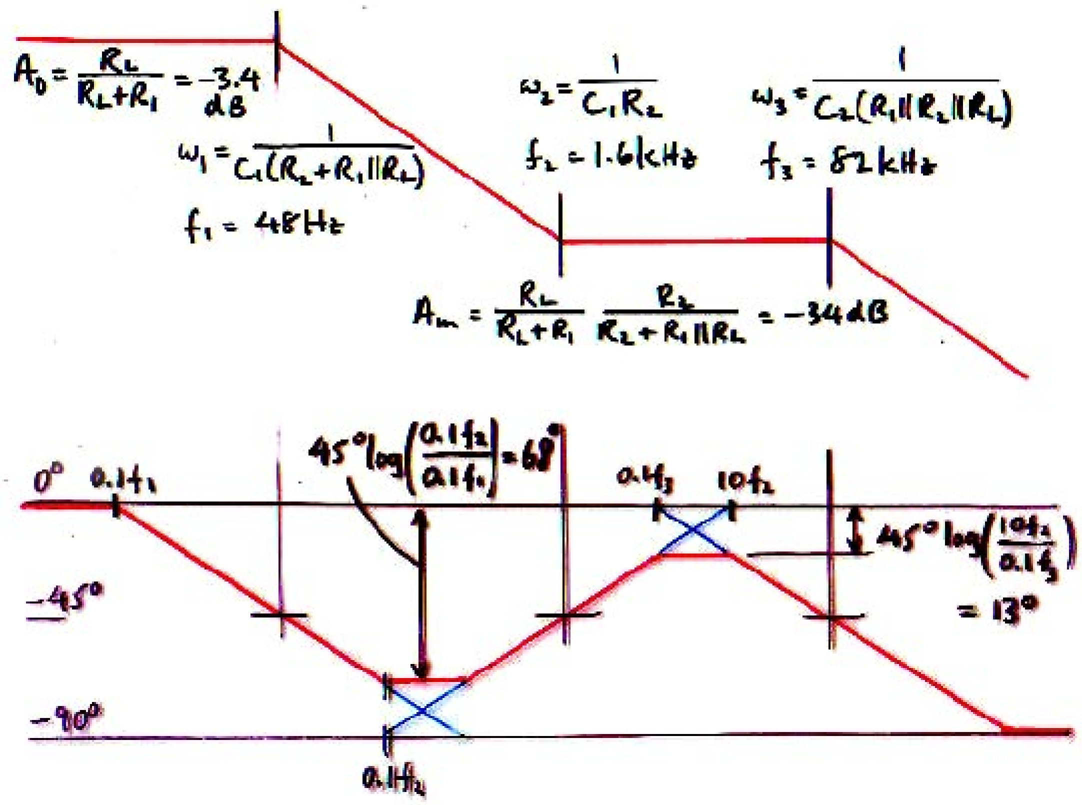


$$A = A_m \frac{1}{1+\frac{s}{\omega_3}} \quad \text{where} \quad \omega_3 \equiv \frac{1}{C_2(R_1 \| R_2 \| R_L)}$$

$$f_3 = \frac{159}{0.002(47 \| 1 \| 100)} = 82\,\text{kHz}$$

Hence, overall response is

$$A = A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)} = A_m \frac{\left(1+\frac{\omega_2}{s}\right)}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_3}\right)}$$


$A_0 = \frac{R_L}{R_L+R_1} = -3.4$ dB
$\omega_1 = \frac{1}{C_1(R_2+R_1\|R_L)}$
$f_1 = 48$ Hz
$\omega_2 = \frac{1}{C_1R_2}$
$f_2 = 1.6$ kHz
$\omega_3 = \frac{1}{C_2(R_1\|R_2\|R_L)}$
$f_3 = 82$ kHz
$A_m = \frac{R_L}{R_L+R_1}\frac{R_2}{R_2+R_1\|R_L} = -34$ dB
0°
$0.1f_1$
$45^\circ \log\left(\frac{0.1f_2}{0.1f_1}\right) = 68^\circ$
$0.1f_3$
$10f_2$
-45°
$45^\circ \log\left(\frac{10f_2}{0.1f_3}\right)$
$= 13^\circ$
-90°
$0.1f_2$

$$= A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)}$$

where

$$\frac{1}{\omega_{1,3}} = \frac{C_1(\overset{3.3\times10^{-3}}{R_2+R_1 \| R_L}) \pm \sqrt{C_1^2(\overset{10\times10^{-6}}{R_2+R_1 \| R_L})^2 - 4\overset{0.026\times10^{-6}}{C_1 C_2 R_2 (R_1 \| R_2)}}}{2}$$

This is useless for <u>design</u>, and in any case is inaccurate numerically.

Generalization: Presentation of Results

Sketch magnitude and phase by straight-line asymptotes, and label salient features (flat gains, corner frequencies, Q's, etc.) with both analytic expressions and numerical values.

This is a compact summary so that both the analytic and numerical results can be interpreted at a glance, which is especially useful for reports, design reviews, etc. so that managers can easily and quickly see and understand the results obtained by others.

For design, the element values that must be changed to give different numerical results can easily be seen.