THE EQUIVALENCE OF EXPANSIONS IN TERMS OF ORTHOGONAL FUNCTIONS[1]

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Introduction. Suppose we are considering the nth partial sum of the formal expansion of an arbitrary function $f(x)$ in terms of the set of functions $\{\phi_n(x)\}$ normal and orthogonal on the interval $0 \leqq x \leqq 1$. This sum is

$$\sum_{i=1}^{n} \phi_i(x) \int_0^1 f(y)\phi_i(y)dy = \int_0^1 f(y) \left\{ \sum_{i=1}^{n} \phi_i(x)\phi_i(y) \right\} dy. \tag{1}$$

If we are considering the nth partial sum of the expansion of $f(x)$ in terms of the functions $\phi_n(x)$ which together with the functions $\psi_n(x)$ form a biorthogonal set normalized over the interval (0,1), we find that this sum is

$$\sum_{i=1}^{n} \phi_i(x) \int_0^1 f(y)\psi_i(y)dy = \int_0^1 f(y) \left\{ \sum_{i=1}^{n} \phi_i(x)\psi_i(y) \right\} dy. \tag{2}$$

If we replace the partial sums (1) and (2) by the analogous Cesàro sums of order 1, we get

$$\int_0^1 f(y) \left\{ \sum_{i=1}^{n} \frac{n-i+1}{n} \phi_i(x)\phi_i(y) \right\} dy \tag{3}$$

$$\text{and } \int_0^1 f(y) \left\{ \sum_{i=1}^{n} \frac{n-i+1}{n} \phi_i(x)\psi_i(y) \right\} dy,$$

respectively. Similar expressions may easily be obtained for Cesàro sums of higher orders. Again, the Fourier integral representation for $f(x)$ is

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(y) \frac{\sin n(y-x)}{y-x} dy. \tag{4}$$

All of the expressions (1)–(4) have at their nth stage, as their

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nth approximation to the arbitrary function $f(x)$, an expression of the form

$$\int_a^b f(y)\, K_n(x,y)dy. \tag{5}$$

In the expansions which have been mentioned, this represents $f(x)$ in some sense or other, either by convergence (as $n \to \infty$), or convergence in the mean, or perhaps purely formally. For certain well-known sequences $\{ K_n(x,y) \}$ the nature of this mode of representation has been ascertained with more or less completeness. The question naturally arises whether for other sequences $\{ K_n \}$ the problem of the mode of representation of f may not be reducible to that of better-known sequences $\{ K_n \}$. This is the chief problem attacked in the present paper; the principal result is stated in Theorem I. We shall consider also the representation of a function by a given sequence $\{ K_n \}$ at different points of the interval considered; the results found are stated in Theorems II and III.

It is clear that the formal expansions of a function $f(x)$ in terms of the two sequences $\{ K_n \}$ and $\{ K^1{}_n \}$ will have the same properties in all that concerns convergence, convergence in the mean, uniform convergence, Gibbs's Phenomenon, etc., if merely

$$\lim_{n \to \infty} \int_a^b f(y) \left\{ K_n(x,y) - K^1{}_n(x,y) \right\} dy = 0 \tag{6}$$

uniformly in x. If this is true we shall say that the formal expansions of $f(x)$ *have the same convergence properties* or are *equivalent*. If this is true for all functions $f(x)$ of summable square[2] we shall say that not only the expansions of $f(x)$ are equivalent but that the two sequences $\{ K_n \}$ and $\{ K_n^1 \}$ are equivalent. These definitions presuppose a range of values for x which is frequently $a \leqq x \leqq b$ and will be considered to be this entire interval unless otherwise specified. The range may be specified, however, as any sub-set of points on this interval.

[2] Here and throughout this paper, when we postulate the summability of the square of a function we tacitly assume that the function is measurable; it follows that the function itself is also summable.

The notion of equivalence has been employed by Haar[3] and by Young[4] in specific cases. A similar notion involving the absoluteness of the convergence of the series

$$\sum \int_a^b f(y) \left\{ K_{n+1}(x,y) - K_{n+1}^1(x,y) - K_n(x,y) + K_n^1(x,y) \right\} dy, \quad (7)$$

has been discussed in some detail by Walsh,[5] and sufficient conditions have been found for its applicability. To the knowledge of the authors, however, no necessary and sufficient conditions for equivalence have been found. The first part of the present paper is devoted to a proof of **Theorem I**, which can be expressed in a simplified form as follows:

A necessary and sufficient condition that the sequences $\{ K_n \}$ *and* $\{ K_n^1 \}$ *be equivalent is that*

(a) *the expansions of all functions of a closed set*[6] *in terms of* K_n *and* K_n^1 *have the same convergence properties and that*

(b) *there exist a finite* A *such that for all* n *and* x

$$\int_a^b \left\{ K_n(x,y) - K_n^1(x,y) \right\}^2 dy < A. \quad (8)$$

Haar, in the second article cited, makes use of the sufficiency of this condition in the particular case of the equivalence of the Fourier and Legendre developments, and in a footnote mentions a mode of proving its sufficiency that is of perfectly general application, employing Hilbert's theory of bilinear forms in infinitely many variables.

Theorem I is applied in the first part of the present paper to the demonstration of the equivalence of the sine and cosine series.

[3] Mathematische Annalen, Vol. 69 (1910), pp. 331–371; Vol. 78 (1917–1918), pp. 121–136. The former of these papers refers to the equivalence of the Sturm-Liouville and Fourier cosine developments, the latter refers to the equivalence of the Legendre polynomial and Fourier cosine developments.

[4] Proceedings of the London Mathematical Society, (2) Vol. XVIII, pp. 141–162, 163–200, especially pp. 156, 150. These papers consider the equivalence of the Legendre polynomial and Bessel developments to the better-known Fourier developments.

[5] Transactions of the American Mathematical Society, Vol. 22 (1921), pp. 230–239. See also a forthcoming paper in the Annals of Mathematics.

[6] A set of functions is said to be *closed* or *complete* if each function of the set is of summable square and if there is no non-null function of summable square orthogonal to every function of the set.

In the second part of this paper we discuss certain problems of the interval equivalence of expansions. Thus the Fourier sine development of an arbitrary function has in the interior of a period the following important properties:

(*a*) The convergence of the development of $f(x)$ for $x=x_1$ and the nature of the approach of the approximating functions (the partial sums) to the limit function in the neighborhood of x_1 depend merely on the values of $f(x)$ for arguments in the neighborhood of x_1;

(*b*) The nature of the approach of the approximating functions to the limit function in the neighborhood of x_1 depends in no way on the actual position of x_1 in the entire interval; in other words, the difference between the sine series for $f(x)$ in the neighborhood of x_1 and the sine series for $f(x+h)$ in the neighborhood of x_1-h converges uniformly to zero in x and h.

In order to make these properties more tangible, we formulate the following definitions:

(*a*) the sequence $\{K_n\}$ is said to be *regular* if, whenever $f(x)\equiv g(x)$ over the interval (c,d) $[a<c<d<b]$, we have

$$\lim_{n\to\infty}\int_a^b [f(y)-g(y)]\,K_n(x,y)dy = 0 \tag{9}$$

uniformly over any closed interval entirely interior to (c,d);

(*b*) the sequence $\{K_n\}$ is said to be *uniform* if it is regular and if whenever x and $x+h$ lie in the interval $(a+\epsilon,\ b-\epsilon)$, where ϵ is an arbitrary positive quantity, we have

$$\lim_{n\to\infty}\int_{x-\epsilon}^{x+\epsilon} f(y)\left\{K_n(x,y)-K_n(x+h,y+h)\right\}dy=0 \tag{10}$$

uniformly in x and h for every $f(y)$ of summable square; this is the condition already referred to, but it will be convenient to require also that

$$\lim_{n\to\infty}\int_a^b\left\{g(x)-\int_a^b K_n(x,y)g(y)dy\right\}^2 dx=0 \tag{11}$$

shall be significant and valid for every $g(x)$ of summable square.

That condition (11) is equivalent to property (*b*) follows from

Theorem II. *If the sequence* $\{K_n\}$ *is regular, then for every* ϕ *of summable square*

$$\lim_{n\to\infty}\int_a^{x-\epsilon}\phi(y)\,K_n(x,y)dy=\lim_{n\to\infty}\int_{x+\epsilon}^{b}\phi(y)\,K_n(x,y)dy=0 \qquad (12)$$

uniformly in x *over the interval* $(a+\epsilon,\ b-\epsilon)$.

We shall prove that if $\{K_n\}$ is uniform, it is equivalent over $(a+\epsilon,\ b-\epsilon)$ to a sequence of the form $K_n(y-x)$. From this there will follow

Theorem III. *If the sequence* $\{K_n\}$ *is uniform and if* $F(x)$ *can be written in the form,*

$$\int_a^x \Phi(u)du, \qquad (13)$$

where $\Phi(u)$ *is of summable square in* (a,b), *then the formal expansion of* $F(x)$ *in terms of the sequence* $\{K_n\}$ *converges uniformly over* $(a+\epsilon,\ b-\epsilon)$ *to the value* $F(x)$.

I. **The equivalence of expansions.**

Let $\{f_n(x)\}$ and $\{\phi_n(x)\}$ be two closed sets of functions, normal and orthogonal, on the interval $0\leqq x\leqq 1$. A condition that the sequences $\{K_n\}$ corresponding be equivalent is that the sequence of functions

$$\sum_{j=1}^{n}\{f_j(x)f_j(y)-\phi_j(x)\phi_j(y)\}=Q_n(x,y) \qquad (14)$$

be equivalent to zero. If this is true we shall say that the sets $\{f_n\}$ and $\{\phi_n\}$ are equivalent; the condition for this equivalence thus reduces to the condition that for every $F(x)$ of summable square we have

$$\lim_{n\to\infty}\int_0^1 F(y)Q_n(x,y)dy=0 \qquad (15)$$

uniformly in x.

We shall now prove

Theorem I. *A necessary and sufficient condition that a sequence of functions* $\{Q_n(x,y)\}$ *of summable square in* y *should have the* *property that for every F of summable square* (15) *should hold uniformly in x over a point set S contained in* (0.1) *is that there exist a number A such that for all n and for all x in S,*

$$\int_0^1 [Q_n(x,y)]^2 dy < A, \tag{16}$$

and that for all the functions $\{\psi_m\}$ *of a set closed on* (0, 1),

$$\lim_{n\to\infty} \int_0^1 \psi_m(y) Q_n(x,y) dy = 0 \tag{17}$$

uniformly in x over S.

The sufficiency of (16) and (17) is readily demonstrated.[7] Let $\{\psi_m\}$ be taken, as it may without essential restriction, as a closed normal orthogonal set. Then by the Riesz-Fischer theorem, a necessary and sufficient condition that a function $F(x)$ be of summable square is that we may write

$$F(x) \sim \sum_{j=1}^{\infty} a_j \psi_j(x),\ a_j = \int_0^1 F(x)\psi_j(x)dx, \tag{18}$$

where the symbol $\sim$ is to be interpreted as convergence in the mean. Moreover, by the Riesz-Fischer theory,[8] we have

$$\int_0^1 F(y)Q_n(x,y)dy = \sum_{j=1}^{\infty} a_j \int_0^1 \psi_j(y) Q_n(x,y) dy$$

$$= \sum_{j=1}^{m} a_j \int_0^1 \psi_j(y)Q_n(x,y)dy + \sum_{j=m+1}^{\infty} a_j \int_0^1 \psi_j(y) Q_n(x,y)dy. \tag{19}$$

Then we have from the general inequality,

$$\sum_{j=1}^{k} \alpha_j \beta_j \leqq \left\{ \sum_{j=1}^{k} \alpha_j^2 \sum_{j=1}^{k} \beta_j^2 \right\}^{\frac{1}{2}} \tag{20}$$

which holds whatever may be the quantities α_j and β_j, we have

[7] Cf. Hilbert, Integralgleichungen, p. 148.

[8] This follows immediately from the Schwarz inequality.

$$\left|\sum_{j=m+1}^{\infty} a_j \int_0^1 \psi_j(y) Q_n(x,y)dy\right|$$
$$\leqq \left\{ \sum_{i=m+1}^{\infty} a_j^2 \sum_{j=m+1}^{\infty} \left[\int_0^1 \psi_j(y) Q_n(x,y)dy\right]^2 \right\}^{\frac{1}{2}}$$
$$\leqq \left\{ \sum_{j=m+1}^{\infty} a_j^2 \sum_{j=1}^{\infty} \left[\int_0^1 \psi_j(y) Q_n(x,y)dy\right]^2 \right\}^{\frac{1}{2}} \tag{21}$$
$$\leqq \left\{ \sum_{j=m+1}^{\infty} a_j^2 \int_0^1 \left[Q_n(x,y)\right]^2 dy \right\}^{\frac{1}{2}} \leqq \left\{ \sum_{j=m+1}^{\infty} a_j^2 \right\}^{\frac{1}{2}} A^{\frac{1}{2}}.$$

Let the function $F(x)$ be given. We may choose m so that the last term in (21) is less than $\epsilon/2$, by the convergence of Σa_j^2. We choose N, so that for $n > N$, $1 \leqq j \leqq m$ we have

$$\left| a_j \sum_{i=1}^{m} a_i^2 \int_0^1 \psi_j(y) Q_n(x,y)dy \right| \leqq \frac{\epsilon a_j^2}{2}, \tag{22}$$

for all x in S. Then it follows from (19) that

$$\left| \int_0^1 F(y)\, Q_n(x,y)dy \right| < \epsilon \tag{23}$$

uniformly for all x in S. This completes the proof of the sufficiency of (16) and (17).

It remains to prove the necessity of (16) and (17); in fact, merely the necessity of (16), for that of (17) is obvious. Suppose (16) not to be true. Then there are two infinite sequences of numbers n_k and x_k, such that

$$\int_0^1 \left[Q_{n_k}(x_k,y)\right]^2 dy > 2^{2k}. \tag{24}$$

Set $C_1(y) = Q_{n_1}(x_1,y)$. Let $C_2(y)$ be the first $Q_{n_k}(x_k,y)$, such that

$$\left| \int_0^1 C_2(y)\, Q_{n_1}(x,y)\, dy \right| < 1 \tag{25}$$

for all x in S; such a $Q_{n_k}(x_k,y)$ will exist because of (15). Similarly, let $C_k(y)$ be the first $Q_{n_k}(x_k,y)$, such that we have

$$\left| \int_0^1 C_k(y)\, C_j(y)\, dy \right| < 1 \tag{26}$$

for all x in S and for every $j < k$.

We now consider the series

$$\sum_{n=1}^{\infty} \frac{C_n(y)}{2^n \left\{ \int_0^1 [C_n(y)]^2 dy \right\}^{\frac{1}{2}}}, \tag{27}$$

and shall prove that this series converges in the mean. That is, we are to prove

$$\lim_{n\to\infty} \int_0^1 \left\{ \sum_{j=n}^{n+p} \frac{C_j(y)}{2^j \left\{ \int_0^1 [C_j(y)]^2 dy \right\}^{\frac{1}{2}}} \right\}^2 dy = 0; \tag{28}$$

or, what is the same thing,

$$\lim_{n\to\infty} \sum_{k=n}^{n+p} \sum_{j=n}^{n+p} \frac{\int_0^1 C_j(y)\, C_k(y)\, dy}{2^{j+k} \left\{ \int_0^1 [C_j(y)]^2 dy \int_0^1 [C_k(y)]^2 dy \right\}^{\frac{1}{2}}} = 0. \tag{29}$$

We find at once from the Schwarz inequality that

$$\int_0^1 C_j(y)\, C_k(y) dy \leq \left\{ \int_0^1 [C_j(y)]^2 dy \int_0^1 [C_k(y)]^2 dy \right\}^{\frac{1}{2}}, \tag{30}$$

so that (29) will follow if merely

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} \sum_{j=n}^{\infty} 2^{-(j+k)} = 0\,. \tag{31}$$

This is, however, immediately true, since

$$\sum_{k=n}^{\infty} \sum_{j=n}^{\infty} 2^{-(j+k)} = \left\{ \sum_{j=n}^{\infty} 2^{-j} \right\}^2 = 2^{2-2n}. \tag{32}$$

Hence, there is a function $\Phi(x)$ to which the series (27) converges in the mean.[9] The limit function $\Phi(x)$ is summable and of summable square, and (27) can be integrated term by term.

Let us now consider $\int_0^1 \Phi(x) C_n(x) dx.$

We have

$$\int_0^1 \Phi(x) C_n(x) dx = \sum_{m=1}^{\infty} \frac{\int_0^1 C_n(x)\, C_m(x) dx}{2^m \left\{ \int_0^1 [C_m(x)]^2 dx \right\}^{\frac{1}{2}}}. \tag{33}$$

[9] See, for example, Plancherel, Rendiconti di Palermo, Vol. 30 (1910), p. 292.

We find from (24)

$$\frac{\int_0^1 [C_n(x)]^2 dx}{2^n \left\{ \int_0^1 [C_n(x)]^2 dx \right\}^{\frac{1}{2}}} > 1; \tag{34}$$

and we find from (24) and (26)

$$\frac{\left| \int_0^1 C_n(x)\, C_m(x) dx \right|}{2^m \left\{ \int_0^1 [C_m(x)]^2 dx \right\}^{\frac{1}{2}}} < 2^{-2m}. \tag{35}$$

Then we have finally,

$$\int_0^1 \Phi(x) C_n(x) dx > 1 - \sum_{m=1}^{\infty} 2^{-2m} = \frac{2}{3}, \tag{36}$$

which is inconsistent with (15). This completes the proof of the necessity of (16) and the proof of Theorem I.

In particular, we have proved that a necessary and sufficient condition that the normal orthogonal sets $\{f_n\}$ and $\{\phi_n\}$ be equivalent is that [in the notation of (14)]

$$\int_0^1 [Q_n(x,y)]^2 dy < A \tag{37}$$

for all n and for all x on the interval (0, 1), while [$\{f_n\}$ being assumed closed]

$$\lim_{n \to \infty} \int_0^1 f_j(y)\, Q_n(x,y) dy = 0 \tag{38}$$

for every j uniformly in x.

Formulæ (37) and (38) are susceptible of a good deal of transformation. We may substitute for (37)

$$\sum_{j=1}^{n} [f_j(x)]^2 + \sum_{j=1}^{n} [\phi_j(x)]^2 - 2 \sum_{j=1}^{n} \sum_{k=1}^{n} f_j(x) \phi_k(x) \int_0^1 f_j(y) \phi_k(y)\, dy < A, \tag{39}$$

and for (38)

$$\lim_{n \to \infty} \left\{ f_j(x) - \sum_{k=1}^{n} \phi_k(x) \int_0^1 \phi_k(y) f_j(y) dy \right\} = 0, \tag{40}$$

uniformly in x for every j. Formula (40) simply states that the ϕ series for each f_j converges uniformly to the value f_j.

Equivalence as defined for these sets $\{f_n\}$ and $\{\phi_n\}$ refers to the identity of the convergence properties of the two series formed as in (18) from the two sets of functions. This equivalence includes equivalence of the two new series found by summing those two former series by the Cesàro mean of the first order. There is, however, an equivalence for the new series without any reference to the old series, or, what is the same thing, which involves the identity of the convergence properties of the original series so far as concerns summability or uniform summability of the first or of higher orders. The definition of this equivalence is that for every function F of summable square we must have

$$\lim_{n\to\infty}\frac{1}{n}\int_0^1 F(y)\sum_{j=1}^{n} Q_j(x,y)dy = 0 \tag{41}$$

uniformly in x. For this equivalence it is necessary and sufficient that there be an A such that for all n and x

$$A > \frac{1}{n^2}\int_0^1\left\{\sum_{j=1}^{n} Q_j(x,y)\right\}^2 dy = \frac{1}{n^2}\int_0^1\left\{\sum_{j=1}^{n}(n+1-j)[f_j(x)f_j(y) - \phi_j(x)\phi_j(y)]\right\}^2 dy$$

$$= \frac{1}{n^2}\left\{\sum_{j=1}^{n}(n+1-j)^2\left\{[f_j(x)]^2+[\phi_j(x)]^2\right\} \right. \tag{42}$$

$$\left. -2\sum_{j=1}^{n}\sum_{k=1}^{n}(n+1-j)(n+1-k)\left\{f_j(x)\phi_k(x)\int_0^1 f_j(y)\phi_k(y)dy\right\}\right\},$$

while

$$\lim_{n\to\infty}\frac{1}{n}\int_0^1 f_j(y)\sum_{k=1}^{n} Q_k(x,y)dy = 0 \tag{43}$$

uniformly in x for all j. Condition (43) may be written

$$\lim_{n\to\infty}\left[\frac{n+1-j}{n}f_j(x) - \sum_{1}^{n}\frac{(n+1-k)\phi_k(x)\int_0^1\phi_k(y)f_j(y)dy}{n}\right] = 0, \tag{44}$$

which merely asserts that every f_j is the uniform Cesàro sum of its ϕ series. It is of course assumed here that the set $\{f_j\}$ is closed.

Every type of summation has a type of equivalence appropriate to it, and every such type of equivalence admits of a treatment similar to the one just given for Cesàro summability of the first kind. We shall have a boundedness condition analogous to (42) and a set of summability conditions not unlike (44).

In many cases of equivalence of the sets $\{f_n\}$ and $\{\phi_n\}$, the two sets of functions are so related that we have, for all n and x,

$$\sum_{j=1}^{n} \left\{ f_j(x) - \phi_j(x) \right\}^2 < A. \tag{45}$$

In this case it is possible to replace (39) by an equivalent condition to the effect that,

$$\left| \sum_{j=1}^{n} \sum_{k=1}^{n} A_{jk} f_j(x) \phi_k(x) \right| < B, \tag{46}$$

where

$$A_{jk} = \int_0^1 f_j(x) \phi_k(x) dx - \delta_{jk}, \tag{47}$$

δ_{jk} being the Kronecker symbol that is zero or unity according as $j \neq k$ or $j = k$.

The special case of Theorem I, where the range S is a single point, deserves some detailed consideration. Thus we have the theorem concerning a sequence $\{Q_n\}$ of functions of summable square:

In order that for all functions $F(y)$ of summable square we have

$$\lim_{n \to \infty} \int_0^1 F(y) Q_n(y) dy = 0, \tag{48}$$

it is necessary and sufficient that (48) *be true for a closed set of F's and that there be an A such that for all n*

$$\int_0^1 [Q_n(y)]^2 dy < A. \tag{49}$$

We shall now prove the following theorem:

Let there be a sequence $\{Q_n(y)\}$ such that (49) *obtains. Then a* *necessary and sufficient condition that there exist a function $Q(y)$ such that*

$$\lim_{n \to \infty} \int_0^1 F(y) Q_n(y) dy = \int_0^1 F(y) Q(y) dy \tag{50}$$

for all functions $F(y)$ of summable square is that, for all functions of a closed set $\{F_m(y)\}$,

$$\lim_{n \to \infty} \int_0^1 F_m(y) Q_n(y) dy$$

exist.

The necessity of the condition is obvious; we proceed to prove its sufficiency. We clearly have the following relation, if the closed set $\{F_m(y)\}$ is chosen (as may be done with no loss of generality) as a normal orthogonal set:

$$\sum_{m=1}^{\infty} \left\{ \int_0^1 F_m(y) Q_n(y) dy \right\}^2 < A. \tag{51}$$

It therefore follows that,

$$\sum_{m=1}^{\infty} \left\{ \lim_{n \to \infty} \int_0^1 F_m(y) Q_n(y) dy \right\}^2 \leqq A. \tag{52}$$

Then, by the Riesz-Fischer Theorem, there exists a function $Q(y)$ of summable square such that,

$$Q(y) \frown\!\!\smile \sum_{m=1}^{\infty} F_m(y) \left\{ \lim_{n \to \infty} \int_0^1 F_m(y) Q_n(y) dy \right\}, \tag{53}$$

where the sign $\frown\!\!\smile$ has the same significance as in (18).

Then we have

$$\int_0^1 Q(y) F_m(y) dy = \lim_{n \to \infty} \int_0^1 F_m(y) Q_n(y) dy, \tag{54}$$

$$\lim_{n \to \infty} \int_0^1 F_m(y) \left\{ Q_n(y) - Q(y) \right\} dy = 0. \tag{55}$$

We have also

$$\int_0^1 \left\{ Q_n(y) - Q(y)^2 \right\} dy \leqq 2 \int_0^1 [Q_n(y)]^2 dy + 2 \int_0^1 [Q(y)]^2 dy < 4A, \tag{56}$$

so that as in (48) we have

$$\lim_{n\to\infty}\int_0^1 F(y)\left\{Q_n(y)-Q(y)\right\}dy=0, \tag{57}$$

and our proof is complete.

It may also be remarked that if (50) is assumed, we may prove successively (55), (54), (53), (52), (49), under the assumption that the integrals that occur in (49) all exist.

The relation between $Q_n(x)$ and $Q(x)$ in (50) is analogous to convergence in the mean, and is perhaps sufficiently important to deserve a special name. The authors suggest, as a possible verbal equivalent of (50), the statement that $\{Q_n\}$ is *quasi-convergent* to Q.

The equivalence of the sine and cosine series. The nth partial sum of the sine series for $f(x)$ over the interval $(0,\pi)$ is

$$\frac{1}{2\pi}\int_0^\pi\left\{f(y)\frac{\sin(2n+1)\frac{y-x}{2}}{\sin\frac{y-x}{2}}-f(y)\frac{\sin(2n+1)\frac{y+x}{2}}{\sin\frac{y+x}{2}}\right\}dy. \tag{58}$$

The corresponding nth partial sum of the cosine series is

$$\frac{1}{2\pi}\int_0^\pi\left\{f(y)\frac{\sin(2n+1)\frac{y-x}{2}}{\sin\frac{y-x}{2}}+f(y)\frac{\sin(2n+1)\frac{y+x}{2}}{\sin\frac{y+x}{2}}\right\}dy. \tag{59}$$

The difference between these nth partial sums is

$$\frac{1}{\pi}\int_0^\pi f(y)\frac{\sin(2n+1)\frac{x+y}{2}}{\sin\frac{y+x}{2}}dy. \tag{60}$$

Accordingly, the condition analogous to (16) is the uniform boundedness of

$$\int_0^\pi \frac{\sin^2(2n+1)\frac{y+x}{2}}{\sin^2\frac{y+x}{2}}\,dy < \int_0^\pi \csc^2\frac{y+x}{2}\,dy$$
$$= 2\int_{\frac{x}{2}}^{\frac{\pi+x}{2}} \csc^2 u\,du = 2\cot\frac{x}{2} - 2\cot\frac{\pi+x}{2}. \tag{61}$$

But the extreme right-hand member of (61) is uniformly bounded over any interval $(\epsilon, \pi-\epsilon)$ if $\epsilon > 0$. Furthermore, the cosine expansions of the sine functions converge uniformly over $(0, \pi)$. Hence, by Theorem I, the sine and cosine expansions of an arbitrary function (summable and of summable square) have the same convergence properties over the interval $(\epsilon, \pi-\epsilon)$.

II. Regular and uniform sequences.

Let the sequence $\{K_n\}$ be regular over the interval (0,1) according to the definition already given. This is equivalent to saying that the expansion in terms of $\{K_n\}$ of a function which is of summable square and is zero throughout an interval (a,b) [which is part of the interval (0,1)] converges uniformly to zero over any closed interval (a^1,b^1) entirely interior to (a,b). That is, if ϕ is any function of summable square, we must have

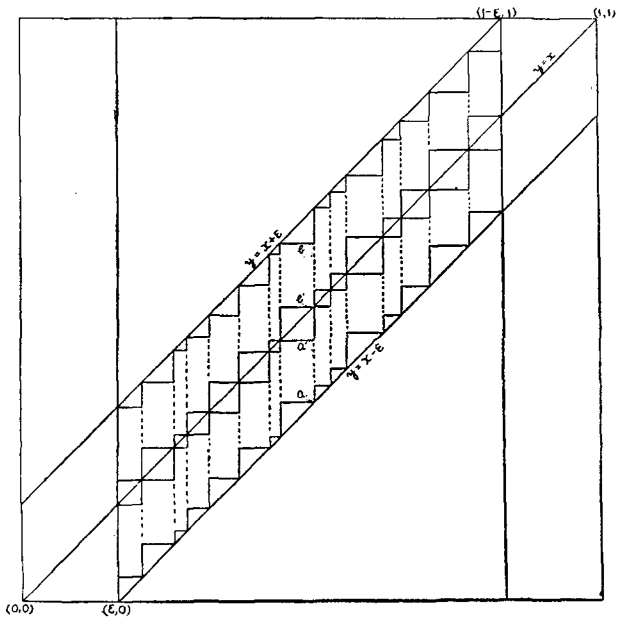
$$\lim_{n\to\infty}\int_0^a \phi(y)\,K_n(x,y)dy = 0,$$
$$\lim_{n\to\infty}\int_b^1 \phi(y)\,K_n(x,y)dy = 0, \tag{62}$$

uniformly over (a^1,b^1). It follows at once that there exists a constant A such that

$$\int_0^a [K_n(x,y)]^2\,dy < A,$$
$$\int_b^1 [K_n(x,y)]\,dy < A, \tag{63}$$

for all n and for all x in (a^1, b^1). The first of inequalities (63) must hold for every $x > a$, and the second for every $x < b$.

Consider the following diagram of the range of variation of x and y in $K_n(x,y)$:



The portion of the figure formed by the lines $x=0$, $x=\epsilon$, $x=1-\epsilon$, $x=1$, $y=0$, $y=1$, $y=x$, $y=x-\epsilon$ and $y=x+\epsilon$ needs no explanation. The parallelogram bounded by the lines $x=\epsilon$, $x=1-\epsilon$, $y=x-\epsilon$, $y=x+\epsilon$ is divided by a finite number of verticals at horizontal distances each less than $\epsilon/2$. The resulting subdivisions of this parallelogram are numbered from the left, 1, 2, . . n . . . . In the nth subdivision four horizontal lines are drawn. The line $y=b_n$ passes through the left-hand upper corner of the subdivision, that is, passes through the intersection of the left-hand bounding vertical line, and $y=x+\epsilon$. Similarly, the line $y=a_n$ passes through the right-hand lower corner, the intersection of the right-hand bounding vertical and $y=x-\epsilon$. The lines $y=a_n^1$ and $y=b_n^1$ pass through the intersection of the line $y=x$ with the left-hand and right-hand bounding verticals, respectively. Obviously we have $a_n<a_n^1<b_n^1<b_n$.

It follows immediately from (62) that we have

$$\lim_{m\to\infty}\int_0^{a_n}\phi(y)\,K_m(x,y)dy=0,$$
$$\lim_{m\to\infty}\int_{b_n}^{1}\phi(y)\,K_m(x,y)dy=0, \tag{64}$$

uniformly over (a_n^1,b_n^1). We shall use these relations to establish Theorem II.

Let us apply the Schwarz inequality to $\int_{x-\epsilon}^{\alpha}\phi\,(y)\,K_m(x,y)\,dy$, where $x-\epsilon\leqq\alpha\leqq a_n$; we have

$$\int_{x-\epsilon}^{\alpha}\phi(y)\,K_m(x,y)dy\Big|\leqq\sqrt{\int^{\alpha}\Big\{K_m(x,y)\Big\}^2dy\int_{x-\epsilon}^{\alpha}\Big\{\phi(y)\Big\}^2dy}. \tag{65}$$

Hence, so long as $a_n^1\leqq x\leqq b_n^1$ it follows from (63) that there is a constant A_n independent of x, m and α, such that

$$\left|\int_{x-\epsilon}^{\alpha}\phi(y)\,K_m(x,y)dy\right|\leqq\sqrt{A_n\int_{x-\epsilon}^{\alpha}\Big\{\phi(y)\Big\}^2dy}. \tag{66}$$

We now subdivide each region of the x-axis between $x=a_n^1$ and $x=b_n^1$ by verticals $x=c_1^{(n)}=a_n^1$, $x=c_2^{(n)}$, . . . $x=c_p^{(n)}=b_n^1$. These lines determine a succession of triangles much like the triangles shown in the figure. Each of the new triangles is bounded by the line $y=x-\epsilon$, by a vertical line $x=c_q^{(n)}$, and by a horizontal line $y=c_{q+1}^{(n)}-\epsilon$. If we increase indefinitely the divisions in such a manner that the maximum value of $c_{q+1}^{(n)}-c_q^{(n)}$ approaches zero, and if α is always chosen so as to be the $c_{q+1}^{(n)}-\epsilon$ next greater than x, it follows from the uniform continuity of $\int_0^{\zeta}\Big\{\phi(y)\Big\}^2dy$, and the fact that the A_n's are constant, that

$$\lim_{\max(c_{q+1}^{(n)}-c_q^{(n)})\to 0}\left|\int_{x-\epsilon}^{\alpha}\phi(y)\,K_n(x,y)dy\right|=0, \tag{67}$$

uniformly for all such integrals.

Let the process of subdivision just described be carried so far that the uniform upper bound of the integrals in (67) is $\eta/2$.

It is then possible to make m so large, by (64), that

$$\left| \int_0^\alpha \phi(y)\, K_m(x,y)dy \right| < \frac{\eta}{2}, \tag{68}$$

where the appropriate α is chosen for every x, uniformly over each of the intervals $a_n^1 \leqq x \leqq b_n^1$ and uniformly for all the intervals. Thus we have

$$\left| \int_0^{x-\epsilon} \phi(y)\, K_m(x,y)dy \right| \leqq \left| \int_0^\alpha \phi(y)\, K_m(x,y)dy \right| + \left| \int_{x-\epsilon}^\alpha \phi(y)\, K_m(x,y)dy \right| < \eta \tag{69}$$

uniformly for each of the intervals $a_n^1 \leqq x \leqq b_n^1$ and uniformly for all those intervals.

This means that

$$\lim_{m\to\infty} \int_0^{x-\epsilon} \phi(y)\, K_m(x,y)dy = 0 \tag{70}$$

uniformly in x for $\epsilon \leqq x \leqq 1-\epsilon$. This establishes the first part of Theorem II, and the second part may be established in precisely the same way.

Uniform expansions. If the sequence $\{K_n\}$ is uniform over (0,1), then by Theorem II and (10) $\{K_n\}$ is equivalent over the sub-interval $(\epsilon, 1-\epsilon)$ to the sequence whose nth term is $K_n(\epsilon, y-x+\epsilon)$ for $x-\epsilon \leqq y \leqq x+\epsilon$ and zero elsewhere. We shall denote this latter function by $K_n(y-x)$.

Then we see from (11) that

$$\lim_{n\to\infty} \int_\epsilon^{1-\epsilon} \left\{ g(x) - \int_\epsilon^{1-\epsilon} K_n(y-x) g(y) dy \right\}^2 dx = 0. \tag{71}$$

Hence, it is true uniformly for the whole class C of functions such that $\int_0^1 \{f(x)\}^2 dx < A$, that

$$\lim_{n\to\infty} \int_\epsilon^{1-\epsilon} f(x)dx \int_\epsilon^{1-\epsilon} K_n(y-x)\, g(y) dy = \int_\epsilon^{1-\epsilon} f(x) g(x) dx. \tag{72}$$

The definition of $K_n(u)$ and (11) assure the existence of $\int_{-1}^{1}\{K_n(u)\}^2du$. Then the Fourier series $\frac{a_0}{2}+\sum_1^\infty(a_i\cos\pi u+b_i\sin\pi u)$ of $K_n(u)$ converges in the mean to the value $K_n(u)$. Denote by $S_m(u)$ the mth partial sum of this series and by $R_m(u)$ the corresponding remainder. We clearly have

$$\int_\epsilon^{1-\epsilon}f(x)dx\int_\epsilon^{1-\epsilon}S_m(y-x)g(y)dy=\int_\epsilon^{1-\epsilon}g(y)dy\int_\epsilon^{1-\epsilon}S_m(y-x)f(x)dx,$$

for S_m is the sum of a finite number of products of functions of y by functions of x. Moreover, we have, by the Schwarz inequality,

$$\begin{aligned}&\left[\int_\epsilon^{1-\epsilon}f(x)dx\int_\epsilon^{1-\epsilon}R_m(y-x)g(y)dy\right]^2\\&\leqq\int_\epsilon^{1-\epsilon}\{f(x)\}^2dx\int_\epsilon^{1-\epsilon}\left\{\int_\epsilon^{1-\epsilon}R_m(y-x)g(y)dy\right\}^2dx\\&\leqq\int_\epsilon^{1-\epsilon}\{f(x)\}^2dx\int_\epsilon^{1-\epsilon}dx\left\{\int_\epsilon^{1-\epsilon}\{R_m(y-x)\}^2dy\int_\epsilon^{1-\epsilon}\{g(y)\}^2dy\right\}\\&\leqq\int_\epsilon^{1-\epsilon}\{f(x)\}^2dx\int_{-1}^{1}\{R_m(u)\}^2du\int_\epsilon^{1-\epsilon}\{g(y)\}^2dy.\end{aligned}$$

Hence, since a similar inequality holds if the integration is performed in the reverse order, we have

$$\begin{aligned}&\int_\epsilon^{1-\epsilon}f(x)dx\int_\epsilon^{1-\epsilon}K_n(y-x)g(y)dy\\&\qquad=\lim_{m\to\infty}\int_\epsilon^{1-\epsilon}f(x)dx\int_\epsilon^{1-\epsilon}S_m(y-x)g(y)dy\\&\qquad=\lim_{m\to\infty}\int_\epsilon^{1-\epsilon}g(y)dy\int_\epsilon^{1-\epsilon}S_m(y-x)f(x)dx\\&\qquad=\int_\epsilon^{1-\epsilon}g(y)dy\int_\epsilon^{1-\epsilon}K_n(y-x)f(x)dx.\end{aligned}$$

Thus we find, from (72),

$$\lim_{n\to\infty}\int_\epsilon^{1-\epsilon}g(y)dy\int_\epsilon^{1-\epsilon}K_n(y-x)f(x)dx=\int_\epsilon^{1-\epsilon}f(x)g(x)dx. \tag{73}$$

Let us set $g(x)=0$ outside of the interval $(2\epsilon, 1-2\epsilon)$, so that we have

$$\begin{aligned}\int_{2\epsilon}^{1-2\epsilon} f(x)g(x)dx &= \lim_{n\to\infty}\int_{2\epsilon}^{1-2\epsilon} g(y)dy\int_{\epsilon}^{1-\epsilon} K_n(y-x)f(x)dx \\ &= \lim_{n\to\infty}\int_{2\epsilon}^{1-2\epsilon} g(y)dy\int_{y-\epsilon}^{y+\epsilon} K_n(y-x)f(x)dx \\ &= \lim_{n\to\infty}\int_{2\epsilon}^{1-2\epsilon} g(y)dy\int_{-\epsilon}^{\epsilon} K_n(u)f(y-u)du \\ &= \lim_{n\to\infty}\int_{-\epsilon}^{\epsilon} K_n(u)du\int_{2\epsilon}^{1-2\epsilon} g(y)f(y-u)dy \\ &= \lim_{n\to\infty}\int_{-\epsilon+\zeta}^{\epsilon+\zeta} K_n(\eta-\zeta)d\eta\int_{2\epsilon}^{1-2\epsilon} g(y)\, f(y+\zeta-\eta)dy \\ &= \lim_{n\to\infty}\int_{0}^{1} K_n(\eta-\zeta)d\eta\int_{2\epsilon}^{1-2\epsilon} g(y)f(y+\zeta-\eta)dy,\end{aligned} \tag{74}$$

uniformly for all f's in C and for all values of ζ on the interval $(\epsilon,\ 1-\epsilon)$. Let us set $F(\eta)=f_\zeta(\zeta-\eta)$, so that we have $f_\zeta(x)=F(\zeta-x)$, and hence

$$\int_0^1\left\{ f_\zeta(x)\right\}^2 dx=\int_0^1\left\{F(\zeta-x)\right\}^2 dx=\int_{\zeta-1}^{\zeta}\left\{F(u)\right\}^2 du. \tag{75}$$

Then if $F(u)$ is defined over $(2\epsilon-1, 1-2\epsilon)$, and if $\int_{2\epsilon-1}^{1-2\epsilon}\{F(u)\}^2du$ exists, we have

$$\lim_{n\to\infty}\int_0^1 K_n(\eta-\zeta)d\eta\int_{2\epsilon}^{1-2\epsilon} F(\eta-y)g(y)dy=\int_{2\epsilon}^{1-2\epsilon} f_\zeta(x)g(x)dx \tag{76}$$

uniformly for $\epsilon\leqq\zeta\leqq 1-\epsilon$. That is, any function $\Phi(x)$ that can be expressed in the form

$$\int_{2\epsilon}^{1-2\epsilon} F(x-y)g(y)dy,$$

is the uniform limit of its expansion in terms of $\{K_n\}$.

Among the functions of the form

$$\int_{2\epsilon}^{1-2\epsilon} F(x-y)g(y)dy$$

are those for which $g(x) \equiv 1$. For such a function we have

$$\Phi(x) = \int_{2\epsilon}^{1-2\epsilon} F(x-y)dy = \int_{x+2\epsilon-1}^{x-2\epsilon} F(u)du = \int_{x+4\epsilon-1}^{x} F(\nu-2\epsilon)d\nu. \tag{77}$$

If we set $F(\nu-2\epsilon) = G(\nu)$ and let $G(\nu)=0$ for $\nu < 0$, we ν find

$$\Phi(x) = \int_0^x G(\nu)d\nu \tag{78}$$

for $x \leqq 1-4\epsilon$. Thus, if we use (78) as the definition of $\Phi(x)$, the $\{K_n(x,y)\}$ expansion of Φ will converge uniformly to the value Φ over the interval $(4\epsilon, 1-4\epsilon)$. This completes the proof of Theorem III.

The property expressed in Theorem III is of course familiar in the case of the Fourier sine expansion.[10]

[10] Since receiving page proof of this paper, it has come to the attention of the authors that a necessary and sufficient condition for (48) including (49) as part of its enunciation has been given by Lebesque (*Annales de la Faculté de Toulouse*, (3), Vol. I (1909), p. 55).