ON THE CONVERGENCE OF CERTAIN MULTIPLE SERIES

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1. The most important of the few known tests for the conditional convergence of simple series are derived from an elementary theorem generally known as Abel's lemma. In this paper I propose to extend this theorem in such a way as to derive similar tests for the conditional convergence of multiple series. So far as I am aware, no one has yet proved the convergence of any general class of multiple series whose terms are not all positive, though the general theory of such series has been worked out in considerable detail by Pringsheim.*

2. Abel's lemma may be written in the form

$$\sum_{i=1}^{p} a_i u_i = \sum_{i=1}^{p-1} (a_i - a_{i+1}) \sum_{k=1}^{i} u_k + a_p \sum_{k=1}^{p} u_k.$$

This is an almost obvious identity. Hence

$$\sum_{i=1}^{p} \sum_{j=1}^{q} a_{i,j} u_{i,j} = \sum_{j=1}^{q} \left[\sum_{i=1}^{p-1} \beta_{i,j} v_{i,j} + a_{p,j} v_{p,j} \right], \tag{1}$$

where
$$\beta_{i,j} = a_{i,j} - a_{i+1,j},$$
$$v_{i,j} = \sum_{k=1}^{i} u_{k,j}.$$

But
$$\sum_{j=1}^{q} \beta_{i,j} v_{i,j} = \sum_{j=1}^{q-1} (\beta_{i,j} - \beta_{i,j+1}) \sum_{l=1}^{j} v_{i,l} + \beta_{i,q} \sum_{l=1}^{q} v_{i,l} \tag{2}$$

and
$$\sum_{j=1}^{q} a_{p,j} v_{p,j} = \sum_{j=1}^{q-1} \gamma_{p,j} \sum_{l=1}^{j} v_{p,l} + a_{p,q} \sum_{l=1}^{q} v_{p,l}, \tag{3}$$

where
$$\gamma_{i,j} = a_{i,j} - a_{i,j+1}.$$

It is convenient to put $\Delta = a_{p,q}$,

$$\Delta_i = \beta_{i,q} = a_{i,q} - a_{i+1,q}, \qquad \Delta_j = \gamma_{p,j} = a_{p,j} - a_{p,j+1},$$

$$\Delta_{i,j} = \beta_{i,j} - \beta_{i,j+1} = \gamma_{i,j} - \gamma_{i+1,j} = a_{i,j} - a_{i+1,j} - a_{i,j+1} + a_{i+1,j+1}.$$

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* *Sitzungsberichte d. Ak. d. Wiss. zu München*, Vol. XXVII. See also later papers by Pringsheim and London in the *Math. Annalen*.

Then, from (1), (2), and (3),

$$\sum_{i=1}^{p}\sum_{j=1}^{q} a_{i,j}u_{i,j} = \sum_{i=1}^{p-1}\sum_{j=1}^{q-1}\Delta_{i,j}\sum_{k=1}^{i}\sum_{l=1}^{j}u_{k,l}+\sum_{i=1}^{p-1}\Delta_i\sum_{k=1}^{i}\sum_{l=1}^{q}u_{k,l}$$
$$+\sum_{j=1}^{q-1}\Delta_j\sum_{k=1}^{p}\sum_{l=1}^{j}u_{k,l}+\Delta\sum_{k=1}^{p}\sum_{l=1}^{q}u_{k,l}. \quad \text{(A)}$$

3. The corresponding equation for n-ple series may be written in the form

$$\sum_{i_1=1}^{p_1}\sum_{i_2=1}^{p_2}\dots\sum_{i_n=1}^{p_n} a_{i_1,i_2,\dots,i_n}u_{i_1,i_2,\dots,i_n} = \Sigma[\Sigma\Delta(\Sigma\Sigma\dots\Sigma u_{k_1,k_2,\dots,k_n})]. \quad \text{(A')}$$

To form the right-hand side we proceed as follows. We take any selection of the suffixes $i_1, i_2, \dots, i_n$ as a suffix for Δ. If i_1 does not occur in this selection, we put $i_1 = p_1$ in $a_{i_1,\dots,i_n}$. If it does, we substitute

$$a_{i_1,i_2,\dots,i_n}-a_{i_1+1,i_2,\dots,i_n} \quad \text{for} \quad a_{i_1,i_2,\dots,i_n}.$$

We repeat this for each suffix, and the result is the corresponding Δ; thus, *e.g.*,

$$\Delta_{i_{n-1},i_n} = a_{p_1,p_2,\dots,i_{n-1},i_n}-a_{p_1,p_2,\dots,i_{n-1}+1,i_n}-a_{p_1,p_2,\dots,i_{n-1},i_n+1}$$
$$-a_{p_1,p_2,\dots,i_{n-1}+1,i_n+1}.$$

In the summation in round brackets the limits for k_ν are 1 and i_ν if i_ν is a suffix of Δ; 1 and p_ν otherwise. The summation within the square brackets applies to every i_ν which is a suffix of Δ, and the limits are 1 and $p_\nu-1$. The outside summation applies to all selections of the suffixes, including that in which no suffix is selected. In this case $\Delta = a_{p_1,p_2,\dots,p_\nu}$.

It is easy to prove (2) by induction. We assume it for n indices, and suppose each a and u affected with a new suffix i_{n+1}. We then sum from $i_{n+1}=1$ to $i_{n+1}=p_{n+1}$, and apply Abel's lemma to each of the terms on the right. Then it is almost obvious that we obtain

$$\Sigma[\Sigma\Delta(\Sigma\Sigma\dots\Sigma u_{k_1,k_2,\dots,k_n,k_{n+1}})]. \quad \text{(A'')}$$

For the term of (A′) whose characteristic suffixes are $i_\theta, i_\phi, \dots, i_\omega$ gives the two terms of (A″) whose characteristic suffixes are $i_\theta, i_\phi, \dots, i_\omega, i_{n+1}$ and $i_\theta, i_\phi, \dots, i_\omega$ respectively.

The theorem expressed by the equation (A′) is therefore true generally.

4. Now suppose that $\Delta'_{i_\theta,i_\phi,\dots,i_\omega}$ is the quantity formed in the same way as $\Delta_{i_\theta,i_\phi,\dots,i_\omega}$, except that the suffixes i_κ which are not suffixes of Δ'

are *not* put equal to p_κ. And suppose that the quantities a, u satisfy the following conditions:—

(i.) All the quantities Δ' are positive;

(ii.) $\lim_{i_r=\infty} a_{i_1, \ldots, i_r, \ldots, i_n} = 0$ $(i = 1, 2, \ldots, n)$ *uniformly* for all values of $i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_n$;

(iii.) $\left| \sum_1^{i_1} \sum_1^{i_2} \ldots \sum_1^{i_n} u_{k_1, k_2, \ldots, k_n} \right|$ is less than a constant C for all values of $i_1, i_2, \ldots, i_n$.

Then *the series* $\sum_1^\infty \sum_1^\infty \ldots \sum_1^\infty a_{i_1, i_2, \ldots, i_n} u_{i_1, i_2, \ldots, i_n}$ *is convergent.*

It is evident that (ii.) implies that the s-ple limit obtained by keeping any $n-s$ suffixes constant and making the remaining s tend simultaneously to infinity is zero.

5. To prove the convergence of the series we have to show that however small be σ we can so choose M that

$$\left| \sum_{i_1=1}^{m_1+p_1} \sum_{i_2=1}^{m_1+p_2} \ldots \sum_{i_n=1}^{m_n+p_n} a_{i_1, i_2, \ldots, i_n} u_{i_1, i_2, \ldots, i_n} - \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \ldots \sum_{i_n=1}^{m_n} a_{i_1, i_2, \ldots, i_n} u_{i_1, i_2, \ldots, i_n} \right| < \sigma \quad (4)$$

for any values of $m_1, m_2, \ldots, m_n$ all $> M$ and all positive values of $p_1, p_2, \ldots, p_n$.

Now let us take any selection of the i's (including at least one) and form the sum $\Sigma\Sigma \ldots \Sigma a_{i_1, i_2, \ldots, i_n} u_{i_1, i_2, \ldots, i_n}$, in which the limits for i_ν are $m_\nu+1$ to $m_\nu+p_\nu$ if i_ν is selected, 1 to m_ν if not. We can form 2^n-1 such sums, and their sum is the quantity whose modulus figures in (4).

Consider, for example, the sum for which the selected i's are $i_1, i_2, \ldots, i_\mu$, and put

$$j_\nu = i_\nu - m_\nu \quad (\nu = 1, 2, \ldots, \mu),$$

$$j_\nu = i_\nu \quad (\nu = \mu+1, \ldots, n),$$

$$q_\nu = p_\nu \quad (\nu = 1, 2, \ldots, \mu),$$

$$q_\nu = m_\nu \quad (\nu = \mu+1, \ldots, n),$$

$$a'_{j_1, j_2, \ldots, j_n} = a_{j_1+m_1, j_2+m_2, \ldots, j_\mu+m_\mu, \ldots, j_n},$$

and similarly for the u's. Then the sum is

$$\sum_{j_1=1}^{q_1} \sum_{j_2=1}^{q_2} \dots \sum_{j_n=1}^{q_n} a'_{j_1, j_2, \dots, j_n} u'_{j_1, j_2, \dots, j_n},$$

which, by (A′), $$= \Sigma[\Sigma\Delta(\Sigma\Sigma \dots \Sigma u'_{k_1, k_2, \dots, k_n})],$$

the Δ's being now formed from a' instead of a. The modulus of this $< C\Sigma[\Sigma\Delta]$. But to find $\Sigma[\Sigma\Delta]$ we have only to suppose that $u = 1$ if all its suffixes are $= 1$, and $= 0$ otherwise. This gives

$$a'_{1, 1, \dots, 1} = \Sigma[\Sigma\Delta].$$

Thus the modulus of our sum is $< Ca'_{1, 1, \dots, 1}$, *i.e.*, $< Ca_{m_1+1, \dots, m_\mu+1, 1, \dots, 1}$, and can therefore be made $< \sigma/2^n$ by choice of M.

Exactly the same argument applies to the other 2^n-2 partial sums; and (4) follows. Therefore the series is convergent.

6. The most interesting case is that in which

$$a_{i_1, i_2, \dots, i_n} = \phi\left(\sum_{\nu=1}^{n} a_\nu i_\nu\right),$$

where $a_1, \dots, a_n$ are positive, and $\phi(u)$ is a function which has 0 as its limit for $u = \infty$ and has continuous derivates $\phi'(u), \phi''(u), \dots, \phi^{(n)}(u)$, such that $\phi'(u) < 0$, $\phi''(u) > 0$, $\phi'''(u) < 0$, ..., and

$$u_{i_1, i_2, \dots, i_n} = \exp\left\{\left(\sum_{\nu=1}^{n} i_\nu \theta_\nu\right)\sqrt{(-1)}\right\},$$

where $\theta_1, \dots, \theta_n$ are any real quantities other than multiples of 2π. Then

$$(-)^s \Delta'_{i_1, i_2, \dots, i_s} = a_1 a_2 \dots a_s \int_{i_1}^{i_1+1}\int_{i_2}^{i_2+1} \dots \int_{i_s}^{i_s+1} \phi^{(s)}(a_1 x_1 + \dots + a_s x_s + a_{s+1} i_{s+1} + \dots + a_n i_n)\, dx_1\, dx_2 \dots dx_s,$$

which has the sign of $(-)^s$, so that $\Delta'_{i_1, i_2, \dots, i_s} > 0$, and similarly for each Δ'. Thus (i.) of § 4 is satisfied. Evidently (ii.) is satisfied. Finally,

$$\left|\sum_1^{i_1}\sum_1^{i_2} \dots \sum_1^{i_n} \exp\{(i_1\theta_1 + \dots + i_n\theta_n)\sqrt{(-1)}\}\right|$$
$$= \left|\Pi \frac{e^{\theta_\nu \sqrt{(-1)}}\{1 - e^{i_\nu \theta_\nu \sqrt{(-1)}}\}}{1 - e^{\theta_\nu \sqrt{(-1)}}}\right| \leqslant \operatorname{cosec}\tfrac{1}{2}\theta_1 \operatorname{cosec}\tfrac{1}{2}\theta_2 \dots \operatorname{cosec}\tfrac{1}{2}\theta_\nu;$$

so that (iii.) is satisfied.

In particular we may suppose

$$\phi(u) = \frac{1}{u^\rho} \quad (\rho > 0).$$

Hence *the series*

$$\sum_1^\infty \sum_1^\infty \dots \sum_1^\infty \frac{\begin{matrix}\cos \\ \sin\end{matrix}(i_1\theta_1 + i_2\theta_2 + \dots + i_n\theta_n)}{(a_1 i_1 + a_2 i_2 + \dots + a_n i_n)^\rho}$$

are convergent if $\rho > 0$.

It was with the object of proving the convergence of these series, which form the natural generalization of some of the simplest single series considered in the books, that I undertook the preceding investigation.

[*Note added October 4th,* 1903. — A very interesting question is whether the multiple series written above is also convergent for all complex values of ρ whose real part is positive. The argument of §§ 4–6 fails when ρ is complex, but it can be proved directly from (A′) that the series is convergent if the right values of the complex powers are taken. A further question is how far the restriction that the a's are to be real and positive is necessary.

These questions, however, belong rather to the theory of zeta and allied functions than to the elementary theory of series, and would probably be answered most easily by totally different methods depending on Cauchy's theorem.]