"On the Value of the Constant in Legendre's Formula for the number of Primes inferior to a given number:" by J. W. L. Glaisher, F.R.S. (Proceedings of the Cambridge Philosophical Society, Vol. iii., Part vii.)

"Note on the Enumerations of Primes of the forms $4n+1$ and $4n+3$" (British Association Report, 1879): by J. W. L. Glaisher, F.R.S.

"Separate Enumerations of Primes of the form $4n+1$ and of the form $4n+3$" (Proceedings of the Royal Society, No. 197, 1879): by J. W. L. Glaisher, F.R.S.

"On Definite Integrals involving Elliptic Functions" (Proceedings of Royal Society, No. 198, 1879): by J. W. L. Glaisher, F.R.S.

"Values of the Theta and Zeta Functions for Certain Values of the Argument" (Proceedings of Royal Society, No. 198, 1879): by J. W. L. Glaisher, F.R.S.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome iii., Decembre 1879; Paris, 1879.

"Beiblätter zu den Annalen der Physik und Chemie," by G. and E. Wiedemann, Bd. iv., Stücke 4 & 5.

"Proceedings of Royal Society," Vol. xxx., No. 202.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. v., Part 1; Newhaven, 1880.

"On the Extra-meridian Determination of Time by means of a portable Transit-instrument," by Osmond Stone, A.M., Astronomer at the Cincinnati Observatory.

"American Journal of Mathematics, Pure and Applied," Vol. ii., No. 4, Dec. 1879; Baltimore, 1880.

On Functions analogous to Laplace's Functions.

By E. J. Routh, F.R.S.

[*Read April 8th*, 1880.]

Contents.

ARTS. 56, 57. The function X, which corresponds to the least value of p, keeps one sign throughout.

ARTS. 58—64. The roots of the equation $X = 0$, which corresponds to any value of p, separates the roots of the equation $X = 0$ which corresponds to a lesser value of p.

---

47. In a paper which was published in the last volume of the Society's Transactions, I discussed a method of forming functions X, Y, &c., possessing the double property, that (1) $\int XY dx = 0$, and (2) that any function could be expanded in a series of such functions. This method was purely analytical, and depended on certain elementary properties of quadrics. The method, in the first instance, led us to certain equations of differences, and thence, by taking the limit, to certain differential equations. These when solved gave the functions X, Y, &c., all the constants, except one, being determined by given conditions at the limits. These equations might have any number of independent variables, and were all of an even order.

48. To save reference to the former paper, we may state that, when there is but one independent variable, the general form of the differential

equation is $$aX - \frac{d}{dx}\left(b\frac{dX}{dx}\right) + \frac{d^2}{dx^2}\left(c\frac{d^2X}{dx^2}\right) - \&c. = pA^2X,$$

where a, b, c, &c., and A^2 are all functions of x, to be chosen at our pleasure, but independent of p. The values of p and the constants of integration, except one, are determined by the conditions

$$\left.\begin{aligned} b\frac{dX}{dx} - \frac{d}{dx}\left(c\frac{d^2X}{dx^2}\right) + \&c. &= \lambda X \\ c\frac{d^2X}{dx^2} - \&c. \qquad\qquad &= \lambda'\frac{dX}{dx} \\ \&c. &= \&c. \end{aligned}\right\},$$

which hold at one limit, say $x = \alpha$, and

$$\left.\begin{aligned} b\frac{dX}{dx} - \frac{d}{dx}\left(c\frac{d^2X}{dx^2}\right) + \&c. &= -\mu X \\ \&c. &= \&c. \end{aligned}\right\},$$

which hold at the other limit, say $x = \beta$. Here $\lambda\lambda'$, &c., $\mu\mu'$, &c., are constants at our disposal, but they must be independent of p. In this

way, we find $$X = L\psi(x, p),$$

where L is an undetermined constant and p is given by an equation which we may write $$f(p) = 0.$$

If p, q be two different roots of this equation, then

$$\int_{\alpha}^{\beta} A\psi(x, p)\, \psi(x, q)\, dx = 0.$$

49. The simplest form of the equation will be

$$aX - \frac{d}{dx}\left(b\frac{dX}{dx}\right) = pA^2X,$$

where we restrict ourselves to the second order. The conditions at the limits are

$$\left.\begin{aligned} b\frac{dX}{dx} &= \lambda X, \quad \text{when } x = \alpha \\ b\frac{dX}{dx} &= -\mu X, \text{ when } x = \beta \end{aligned}\right\}.$$

In this form the equation represents the motion of heat in a bar. The functions a and b depend on the coefficients of cooling and conduction and on the area and perimeter of the section. With this interpretation, it is essential that the functions a, b, and A^2 should be positive from one end of the bar to the other. The constants λ and μ must also be positive.

This simple form of the equation has been discussed by C. Sturm in the first volume of Liouville's "Journal." I have not yet examined the whole paper, but he gives a very full introduction, in which he sums up his more remarkable results. These are of two kinds,—first, those which relate to the properties of the equation to find p, and secondly, those which relate to the properties of the function X.

(1). He remarks that Poisson has shown that the equation to find p has no imaginary roots. He now shows that the equation has no negative or equal roots.

(2). None of the functions X, Y, &c., can vanish without changing sign.

The first of these, *i.e.*, the function which corresponds to the least root of the equation to find p, preserves one sign from one end of the bar to the other. The second, *i.e.*, the function which corresponds to the next smallest root, changes sign once at a point somewhere between the two given extremities of the bar. The third changes sign twice, the fourth three times, and so on.

Two functions corresponding to two consecutive roots change sign alternately, one after the other, that which corresponds to the greater vanishing first, as we proceed from end of the bar to the other.

Analogous theorems are then proved relative to the maxima and minima values of these functions.

I propose to show in this paper that some of these results follow at once from elementary properties of the quadrics made use of in my former paper. Many of the properties are true in the general case, when we do not restrict the differential equation to be of the second order. They are also true when the functions are given by an equation of differences.

This, however, is not the communication alluded to in the preface to my former paper.

50. *To find the geometrical meaning of the symbol p.*

Consider the two quadrics

$$A_1^2 X_1^2 + A_2^2 X_2^2 + \ldots = H^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$2U = a_{11} X_1^2 + 2a_{12} X_1 X_2 + \ldots = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

We seek (as in Art. 6 of the former paper) the common conjugate diameters of the two quadrics. Let $X_1 X_2 \ldots X_n$ be the coordinates of the extremity of one such diameter, then*

$$\left.\begin{aligned} \frac{dU}{dX_1} &= pA_1^2 X_1 \\ \frac{dU}{dX_2} &= pA_2^2 X_2 \\ \&c. &= \&c. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Let this diameter cut the first quadric in P and the second in Q. This is, of course, a short mode of stating that, if $X_1 X_2 \ldots X_n$ be obtained from (1) and (3), then

$$OP^2 = X_1^2 + X_2^2 + \ldots + X_n^2;$$

but if $X_1 X_2 \ldots$ be obtained from (2) and (3), then

$$OQ^2 = X_1^2 + X_2^2 + \ldots + X_n^2.$$

If we now multiply equations (3) by $X_1 X_2 \ldots X_n$, and add the results, we find, if $X_1 X_2 \ldots X_n$ be the coordinates of Q,

$$1 = p\,(A_1^2 X_1^2 + \ldots + A_n^2 X_n^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

* The property $\Sigma A^2 XY = 0$ follows at once from these equations without assuming the reality of the values of p, or referring to any geometrical property. Let q be another value of p, and let $Y_1 Y_2 \ldots$ be the corresponding values of $X_1 X_2 \ldots$,

then we easily find $\quad X_1 \dfrac{dU}{dY_1} - Y_1 \dfrac{dU}{dX_1} = (p-q)\, A_1^2 X_1 Y_1.$

Substitute for U and sum for all the suffixes; we have

$$0 = (p-q)\, \Sigma A^2 XY.$$

This gives at once $$p = \frac{1}{H^2}\left(\frac{OP}{OQ}\right)^2 \quad \text{.....................................(5).}$$

If we eliminate $X_1 X_2 \dots X_n$ from the equations (3), we of course get the determinant

$$\begin{vmatrix} a_{11}-pA_1^2, & a_{12}, & \&c. \\ a_{12}, & a_{22}-pA_2^2, & \\ \&c. & & \end{vmatrix} = 0 \quad \text{..........................(6)}$$

already given in the article referred to in the former paper.

51. *The values of p given by the determinant are all real.*

Three different proofs of this proposition have been given by Dr. Salmon in his "Lessons on Higher Algebra." The proof given by Poisson, in Art. 90 of his "Théorie de la Chaleur," is applied to the case of a differential equation of the second order, but it may easily be extended to the general case of a differential equation or an equation of differences of any order.

If possible, let $p = h+k\sqrt{-1}$ be an imaginary root, then, substituting in equations (3), we have

$$\frac{X_1}{L_1+M_1\sqrt{-1}} = \frac{X_2}{L_2+M_2\sqrt{-1}} = \&c.$$

But we must also have another root, viz., $p = h-k\sqrt{-1}$. Substituting this also in the same equations and representing the new values of $X_1 X_2$, &c., by $Y_1 Y_2$, &c., we have

$$\frac{Y_1}{L_1-M_1\sqrt{-1}} = \frac{Y_2}{L_2-M_2\sqrt{-1}} = \&c.$$

But we also have $\quad A_1^2 X_1 Y_1 + A_2^2 X_2 Y_2 + \dots = 0.$

This leads to $\quad A_1^2 (L_1^2 + M_1^2) + A_2^2 (L_2^2 + M_2^2) + \dots = 0,$

which is impossible if $A_1^2 A_2^2 \dots$ be all positive or all negative.

Referring back to Art. 48 of this paper, *we infer that, if A^2 keeps one sign as x varies from one limit to the other, the equation $f(p) = 0$ has no imaginary roots.*

52. *To determine if any of the values of p can be negative.*

Since the values of p are real, it follows from equations (3) that the ratios of $X_1 X_2 \dots$ are all real.

In the quadric (1), the coefficients $A_1^2 A_2^2 \dots$ are all positive, so that, if there were but three coordinates, we should call (1) an ellipsoid. It easily follows that OP^2 is positive, and therefore OP is real.

In quadric (2), the coefficients $a_{11} a_{12} \dots$ are unrestricted, and may be either positive or negative, so that, if there were but three coordinates, we should say that (2) may be either an ellipsoid or a hyperboloid. Thus OQ^2 may be either positive or negative. In the former case p is positive, in the latter negative.

But if we wish to arrive at the differential equation

$$aX - \frac{d}{dx}\left(b\frac{dX}{dx}\right) + \frac{d^2}{dx^2}\left(\frac{d^2X}{dx^2}\right) - \&c. = pA^2X$$

in its most general form, we must, as in Art. 23 of the former paper, choose as our quadric (2)

$$2U = a_1X_1^2 + b_1(X_2 - X_1)^2 + c_1(X_3 - 2X_2 + X_1)^2 + \dots$$
$$+ \text{similar terms with increased suffixes.}$$

If now a, b, c, as well as A^2, be positive from one limiting value of x to the other, then this quadric also is of the ellipsoid class, and thus all the values of p will be also positive.

This argument may, however, be even more generally applied. If we take the case of two independent variables as given in Art. 38 of the former paper, we see that the quadric $2U = 1$ is certainly of the ellipsoidal class, if we put the e's equal to zero. *So that, when the differential equation takes the form*

$$aX - \frac{d}{dx}\left(b\frac{dX}{dx}\right) - \frac{d}{dy}\left(c\frac{dX}{dy}\right) = pA^2X,$$

where a, b, c, A are positive over the area of integration, then the values of p are all positive. We may also apply the argument to the more general case in which the differential equation is of an order higher than the second.

53. Returning to the quadric (2), we see that some of the values of p may be negative, if one of the functions b, c, &c. becomes negative between the limits of integration. We may easily understand the geometrical meaning of this. If we put $A_1X_1 = \xi_1$, $A_2X_2 = \xi_2$, &c., the process described in Art. 50 is really that of making

$$\xi_1^2 + \xi_2^2 + \dots = \text{max. or min.}$$

We are therefore finding what we may call the principal diameters of the quadric (2), when we treat $\xi_1\xi_2 \dots$ as the running coordinates. By equation (4) of Art. 50, we see that p is the reciprocal of the square of any such semi-diameter. Thus, if p, q, r ... be the values of p given by the equation (6), the quadric (2) may be written

$$pA_1^2X_1^2 + qA_2^2X_2^2 + \dots = 1.$$

Extending the phraseology of solid geometry, there will be no negative roots, one negative root, two negative roots, &c., according as the quadric (2) is an ellipsoid, a hyperboloid of one, two, &c. sheets.

54. We may also obtain some simple limits to the values of p. It is clear that the greatest semi-diameter must be greater than the greatest of the quantities found by putting all the ξ's, except one, equal to zero. Let us take for our quadric (2), the expression

$$\begin{aligned} 2U = & \, a_1 X_1^2 + b_1 (X_2 - X_1)^2 \\ & + a_2 X_2^2 + b_2 (X_3 - X_2)^2 \\ & + \&c., \end{aligned}$$

as given in Arts. 9 and 10 of the first paper. If we express the X's in terms of the ξ's we shall see that the least value of p must be less than the least of the quantities

$$\frac{a_1+\lambda}{A_1^2},\ \frac{a_2+b_1+b_2}{A_2^2},\ \frac{a_3+b_2+b_3}{A_3^2},\ \&c.;$$

where the λ of Art. 10 has been written separately from the a_1. In the same way, the greatest value of p must be greater than the greatest of these. In the limit, for a differential equation of the second order, *the least and greatest values of the function* $\frac{a+2b}{A^2}$ *lie between the least and greatest values of* p.

If $a+2b$ *become negative between the limits, some of the values of* p *must be negative.*

55. Taking the same quadric, and supposing the a's and b's to be positive, we notice that, whatever the ratios of $X_1 X_2 \dots X_n$ may be, the sum of their squares will be increased if we remove the terms containing b from the equation. Hence the greatest diameter of the quadric $2U = 1$ is less than the greatest diameter of the quadric formed by omitting the terms containing the b's. *That is, the least value of* p *is greater than the least value of* $\frac{a}{A^2}$.*

* Dr. Salmon remarks, at the beginning of his sixth lesson on *Higher Algebra*, that, if in the symmetrical determinantal equation

$$\Delta_1 = \begin{vmatrix} a_{11}-p, & a_{12}, & \&c. \\ a_{12}, & a_{22}-p, & \&c. \\ \&c. & \&c. & \&c. \end{vmatrix} = 0,$$

we suppress the first row and the first column, we have a second symmetrical determinantal equation which we may write $\Delta_2 = 0$ whose roots are all real, and separate those of the first determinantal equation. We may extend this convenient theorem as follows.

If we border the original determinant with *any* given quantities $l_1, l_2 \dots$, and thus form a third equation, viz.,

$$\Delta_3 = \begin{vmatrix} a_{11}-p, & a_{12}, & \&c. & l_1 \\ a_{12}, & a_{22}-p, & \&c. & l_2 \\ \&c. & \&c. & & \&c. \\ l_1, & l_2, & \&c. & 0 \end{vmatrix} = 0,$$

56. *To show that, if the functions a, b (and of course A^2) be positive from one limit of integration to the other, then the function X, which corresponds to the least root of the equation to find p, keeps one sign from one limit to the other.*

As before, let $X_1 X_2 \dots X_n$ be the successive values of the function, and let the quadric (2) be

$$2U = a_1 X_1^2 + b_1 (X_2 - X_1)^2$$
$$+ \text{same with increased suffixes}$$
$$= 1.$$

Let us suppose that p has its least value, then, by equation (4), Art. 50, we have to make

$$R^2 = A_1^2 X_1^2 + A_2^2 X_2^2 + \dots + A_n^2 X_n^2$$

as great as possible.

If possible, let any of the X's, say $X_3 X_5 \dots$, have a different sign from the rest. Then, by changing the signs of these, leaving their numerical values unaltered, we clearly decrease the value of U. Since $2U$ is now less than unity, and $X_1 X_2 \dots X_n$ all have now the same sign, we can numerically increase all of them by the same quantity, so as to make $2U$ equal to unity. It follows that R has been increased. Hence R is not the greatest possible unless $X_1 X_2 \dots X_n$ all have the same sign.

57. This theorem is also true when the function X is obtained from the differential equation of an order higher than the second. The

---

then the roots of this will also be all real and will separate the roots of the first determinantal equation. If we make all the l's zero except one, we get the proposition as enumerated by Dr. Salmon.

To prove this we notice that, if $a_{11} a_{22} \dots$ be the minors of the constituents in the leading diagonal of Δ_1, then the third determinant (by Art. 37 of Dr. Salmon's *Algebra*) may be written in the form

$$\Delta_3 = \Delta'_3 = -\left\{l_1 \sqrt{a_{11}} + l_2 \sqrt{a_{22}} + \&c.\right\}^2$$

whenever p has such a value that $\Delta_1 = 0$. Suppose p to take in succession the values of the roots of $\Delta_1 = 0$, beginning at the least. Then, at each change in the value of p, the minors $a_{11} a_{22} \dots$ will change sign, because each is of the form Δ_2. Hence Δ'_3 will change sign also. Thus a root of the equation $\Delta_3 = 0$ lies between each adjacent two of the roots of $\Delta_1 = 0$.

This proposition may be stated in geometrical language. The roots of the determinantal equation $\Delta_1 = 0$ are the squares of the reciprocals of the principal semi-diameters of the quadric (2) in Art. 50, and what we have proved amounts to this,— the lengths of the principal diameters of any section, say,

$$l_1 X_1 + l_2 X_2 + \dots = 0,$$

separate the lengths of those of the original quadric. In this form the proposition admits of a very easy proof by referring tho quadric to its principal axes.

In the same way, if we border the determinant Δ_3 with a second set of arbitrary quantities, with zero in the corner, we get a new determinantal equation $\Delta_4 = 0$ whose roots are all real, and separate those of $\Delta_3 = 0$.

proof, however, is not of the same simple character as when we restrict the quadric to the form given in this article. I have, therefore, not thought it advantageous to give the proof at length.

58. When the function X is given by an equation of differences or a differential equation of the second order, the properties mentioned in Art. 49 are simple corollaries from the following lemma.

Consider the equation of differences

$$a_x X_x - \Delta (b_{x-1} \Delta X_{x-1}) = pA_x^2 X_x.$$

If q be another value of p, we have

$$a_x Y_x - \Delta (b_{x-1} \Delta Y_{x-1}) = qA_x^2 Y_x.$$

Eliminating the function a_x, we find

$$(q-p) A_x^2 X_x Y_x = b_x (X_{x+1} Y_x - X_x Y_{x+1}) - b_{x-1} (X_x Y_{x-1} - X_{x-1} Y_x).$$

This gives

$$(q-p) [A_m^2 X_m Y_m + \ldots + A_x^2 X_x Y_x] = b_x (X_{x+1} Y_x - X_x Y_{x+1}) \\ - b_{m-1} (X_m Y_{m-1} - X_{m-1} Y_m).$$

The right-hand side may also be written

$$b_x (Y_x \Delta X_x - X_x \Delta Y_x) - b_{m-1} (Y_{m-1} \Delta X_{m-1} - X_{m-1} \Delta Y_{m-1}).$$

59. Cor. 1.—Consider the full series of values $X_1 X_2 \ldots X_n$ arranged in order. We shall have ranges of positive and negative values succeeding each other. Let $X_m \ldots X_x$ be one of these ranges in which all the constituents have one sign, while those on each side, viz., X_{m-1} and X_{x+1}, have the opposite sign. *We shall prove that, if $q > p$, there is one change of sign at least in the corresponding range of Y's extending from Y_{m-1} to Y_{x+1} both inclusive.*

For, if possible, let all these Y's have one sign, then every one of the four terms on the right-hand side of the equality in the lemma has the sign opposite to that of the product $X_x Y_x$. Hence the lemma could not be true.

We have here made no assumption as to the function a_x, but b_x and A_x^2 have been supposed to have the same sign from one limit to the other.

60. Cor. 2.—Consider next a double range of values, say $X_l \ldots X_m \ldots X_x$, such that all the constituents from X_l to X_{m-1} have one sign, say, negative, and $X_m \ldots X_x$ have the other sign, while X_{l-1} and X_{x+1} have opposite signs to the adjacent constituents in order to make the double range complete. Then, by Cor. 1, if $q > p$, Y must change sign between Y_{l-1} and Y_l, and also between Y_{l-1} to Y_{x+1}. *We shall now prove that a single change of sign between Y_{l-1} and Y_l will not suffice for both these requirements.*

For, if it did, the products $X_l Y_l \dots X_x Y_x$ would all have the same sign. But, writing l for m in the equation of the lemma, every one of the four terms on the right-hand side has the sign opposite to that of the products $X_x Y_x$, and thus again the lemma could not be true.

In the same way, if we consider a complete triple range of values, say, $X_k \dots X_l \dots X_m \dots X_x$, so that X changes sign twice as x varies from one limit to the other; then, by Cor. (1), Y must change sign between X_{k-1} and X_{l+1}, X_{l-1} and X_{m+1}, X_{m-1} and X_{x+1}. But it follows exactly as before that two changes of sign will not suffice for all three requirements.

61. Cor. 3.—Consider the range of values $X_1 \dots X_x$ all of one sign, beginning at one extremity of the full series, and such that X_{x+1} has the opposite sign. *We shall prove that, if $q > p$, there is one change of sign at least in the corresponding range of Y's extending from Y_1 to Y_{x+1}.*

In this case the range begins at one extremity; we have therefore the conditions $b_0(X_1 - X_0) = \lambda X_1$, $b_0(Y_1 - Y_0) = \lambda Y_1$, which hold at that extremity. The equality in the lemma becomes therefore

$$(q-p)(A_1 X_1 Y_1 + \dots A_x X_x Y_x) = b_x(X_{x+1} Y_x - X_x Y_{x+1}).$$

If, then, all the Y's from Y_1 to Y_{x+1} had the same sign, every term on the left-hand side would have the same sign, and every term on the right-hand side would have the opposite sign, and thus the equality could not exist.

Similar remarks apply to a range terminating at the other extremity.

62. Cor. 4.—Lastly, consider all the n series

$$\begin{gathered} X_1 X_2 \dots X_n, \\ Y_1 Y_2 \dots Y_n, \\ \&c., \end{gathered}$$

corresponding to the n values of p, arranged in order of magnitude, beginning at the least.

By the preceding corollaries, each of these series must have at least one more change of sign than any series above it. As there are but n terms in each series, the lowest, or n^{th}, can have only $n-1$ changes of sign.

Hence the first series has no change of sign, the second has only one change, the third has only two, and so on. Also, the changes of sign in each series alternate, in the manner already explained, with the changes of sign in any series above it.

63. It should be noticed that, in Cor. 1 and 2, no use has been made

of the conditions at the limits. In these propositions, therefore, p and q are any arbitrary quantities, except that q must be greater than p.

In Cor. 3, the conditions at one limit are introduced, so that the results of these three corollaries are true, if only $\dfrac{X_1}{X_0} = \dfrac{Y_1}{Y_0}$ at one limit.

Finally, in Cor. 4, the conditions at both limits are assumed to be satisfied, and therefore p and q must now be different roots of the equation $$f(p) = 0.$$

64. If we take the second form of the lemma given in Art. 58, we may prove that similar theorems hold regarding the successive maxima and minima of the functions X and Y. But the mode of proof is very similar to that just used. There does not appear to be sufficient novelty to render it necessary to lengthen this paper by repeating the arguments with the necessary variations.

A Form of the Equations determining the Foci and Directrices of a Conic whose Equation in Cartesian Coordinates is given. By Prof. WOLSTENHOLME.

[*Read April 8th*, 1880.]

If the rational equation of a conic referred to coordinate axes inclined at an angle ω, be $u = 0$, let (x, y) be the coordinates of a focus, and (X, Y) current coordinates. Then, if we move the origin to the focus, the equation will become

$$u + X\frac{du}{dx} + Y\frac{du}{dy} + \tfrac{1}{2}\left(X^2\frac{d^2u}{dx^2} + Y^2\frac{d^2u}{dy^2} + 2XY\frac{d^2u}{dx\,dy}\right) = 0.$$

But, the origin being a focus, the equation must be of the form

$$X^2 + Y^2 + 2XY\cos\omega = (pX + qY + r)^2.$$

Hence $$\frac{p^2-1}{\dfrac{d^2u}{dx^2}} = \frac{q^2-1}{\dfrac{d^2u}{dy^2}} = \frac{r^2}{2u} = \frac{qr}{\dfrac{du}{dy}} = \frac{rp}{\dfrac{du}{dx}} = \frac{pq-\cos\omega}{\dfrac{d^2u}{dx\,dy}} \;(\equiv \lambda).$$

We can deduce the equations

$$\left(\lambda\frac{d^2u}{dx^2} + 1\right)2u = \lambda\left(\frac{du}{dx}\right)^2, \quad \left(\lambda\frac{d^2u}{dy^2} + 1\right)2u = \lambda\left(\frac{du}{dy}\right)^2,$$

$$\left(\lambda\frac{d^2u}{dx\,dy} + \cos\omega\right)2u = \lambda\frac{du}{dx}\frac{du}{dy};$$