22.

De transformatione integralium Abelianorum primi ordinis commentatio.

(Auctore *Fried. Jul. Richelot*, prof. in Academia Albertina Regiom.)

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Caput secundum.

De computatione integralium Abelianorum primi ordinis.

XI.

Quomodo integralia proposita indefinita et definita per transformationes repetitas determinentur.

Transformationes in prioribus capitibus ex vero ipsarum fonte derivatae, et in articulo praecedente directe demonstratae facilem algorithmum computationis integralium Abelianorum primi ordinis et definitorum et indefinitorum suppeditant. Id quod, quomodo fiat, hîc exponendum est.

Dum integralia definita, quippe quorum argumenta inter limites 0 et 1 continebantur, semper ad similia singula reduci vidimus, integralium indefinitorum singulum quodque ab 0 usque ad z integratum ad bina revenit, quorum argumentum alterum ab 0, alterum vero ab alio limite proficiscitur, et adeo in secunda transformatione non in primo, sed in quinto integralis novi intervallo continetur. Ut igitur integralia indefinita, aeque ac definita ad transformationem symmetricam reducamus, ante omnia secundum integrale per idoneam transformationem fundamentalem ita transformandum erit, ut ipsius argumentum in intervallo $0 \ldots . 1$, simul cum argumento dato z, ab 0 progrediatur.

Quem ad finem in prima transformatione ex art. IX. revocemus theorema prius, quod hac aequat. integrali (23.) exhibetur:

$$1. \quad \int_0^z \frac{(\alpha-\beta z)\,dz}{\sqrt{(\Delta z)}} = \int_0^{y_1} \frac{(\alpha'-\beta' y_1)\,dy_1}{\sqrt{(\Delta' y_1)}} - \int_1^{y_2} \frac{(\alpha'-\beta' y_2)\,dy_2}{\sqrt{(\Delta' y_2)}},$$

ubi moduli et formulae eodem loco leguntur. Argumentum y_2, argumento z ab 0 crescente, ab unitate in primo intervallo decrescit, quam ob causam integrale secundum per primam transformationem fundamentalem classis ($B.$) transformare velimus. Ponamus igitur:

$$y_2 = \frac{1-x}{1-k'^2 x},$$

qua formula introducta, erit, (vid. commentationis allatae art. III.)

$$2. \quad \int_1^{y_2} \frac{(\alpha'-\beta' y_2)\,dy_2}{\sqrt{[y_2(1-y_2)(1-k'^2 y_2)(1-\lambda'^2 y_2)(1-\mu'^2 y_2)]}}$$
$$= -\left(\frac{1}{\lambda'_1\mu'_1}\right)\int_0^x \frac{[(\alpha'-\beta')-(\alpha' k'^2-\beta')x]\,dx}{\sqrt{[x(1-x)(1-k'^2 x)(1-\lambda'^2 x)(1-\mu'^2 x)]}}.$$

Ut igitur argumentum x ex argumento z directe inveniamus, formulam antecedentem in formulam transformationis:

$$3. \quad \frac{1-(1-k'_1\lambda'_1)y_2}{1-(1+k'_1\lambda'_1)y_2} = -\sqrt{\left(\frac{(1-k^2 z)\left(1-\frac{\lambda^2\mu^2}{k^2}z\right)}{(1-z)(1-\lambda^2\mu^2 z)}\right)}$$

introducamus; unde fit:

$$4. \quad \frac{\lambda'_1-(\lambda'_1-k'_1)x}{\lambda'_1-(\lambda'_1+k'_1)x} = \sqrt{\left(\frac{(1-k^2 z)\left(1-\frac{\lambda^2\mu^2}{k^2}z\right)}{(1-z)(1-\lambda^2\mu^2 z)}\right)}.$$

Ut vero totam transformationem uno in conspectu videamus, his denotationibus uti placet. Sit:

$$z = \sin^2\varphi, \qquad y_1 = \sin^2\varphi'_1, \qquad x = \sin^2\varphi'_2,$$

$$5. \left\{\begin{array}{lllll} \mu = m, & \lambda = l, & k = c, & \sqrt{\left(\frac{k^2-\mu^2}{1-\mu^2}\right)} = L, & \sqrt{\left(\frac{k^2-\lambda^2}{1-\lambda^2}\right)} = M, \\ \mu' = m', & \lambda' = l', & k' = c', & \sqrt{\left(\frac{k'^2-\mu'^2}{1-\mu'^2}\right)} = L', & \sqrt{\left(\frac{k'^2-\lambda'^2}{1-\lambda'^2}\right)} = M', \\ \sqrt{(1-\mu^2)} = m_1, & \sqrt{(1-\lambda^2)} = l_1, & \sqrt{(1-k^2)} = c_1, & \frac{k_1}{\mu_1} = L_1, & \frac{k_1}{\lambda_1} = M_1, \\ \sqrt{(1-\mu'^2)} = m'_1, & \sqrt{(1-\lambda'^2)} = l'_1, & \sqrt{(1-k'^2)} = c'_1, & \sqrt{\left(\frac{1-k'^2}{1-\mu'^2}\right)} = L'_1, & \sqrt{\left(\frac{1-k'^2}{1-\lambda'^2}\right)} = M'_1. \end{array}\right.$$

Deinde ponamus:

$$\alpha = P_1, \quad \beta = Q_1, \quad \frac{\alpha-\beta}{\lambda_1\mu_1} = P_2, \quad \frac{\alpha k^2-\beta}{\lambda_1\mu_1} = Q_2;$$

unde sequuntur quatuor hae aequationes inter quantitates P et Q:

$$6. \quad \left\{\begin{array}{ll} P_1 - Q_1 = P_2 l_1 m_1, & P_2 - Q_2 = P_1 L_1 M_1, \\ P_1 c^2 - Q_1 = Q_2 l_1 m_1, & P_2 c^2 - Q_2 = Q_1 L_1 M_1, \end{array}\right.$$

quae docent, quantitates P_2 et Q_2 eodem modo compositas esse ex quantitatibus P_1, Q_1, c, l, m, ac quantitates P_1 et Q_1 ex quantitatibus P_2, Q_2, L, M.

Deinde ponamus:

$$\alpha' = P'_1, \quad \beta' = Q'_1, \quad \frac{\alpha'-\beta'}{\lambda'_1\mu'_1} = P'_2, \quad \frac{\alpha' k'^2-\beta'}{\lambda'_1\mu'_1} = Q'_2;$$

unde prorsus similes rationes inter novas quantitates P'_1, Q'_1, P'_2, Q'_2 oriuntur, ac antea, nimirum hae:

$$7. \quad \begin{cases} P'_1 - Q'_1 = P'_2\, l'_1\, m'_1, & P'_2 - Q'_2 = P'_1\, L'_1\, M'_1, \\ P'_1 c'^2 - Q'_1 = Q'_2\, l'_1\, m'_1, & P'_2 c'^2 - Q'_2 = Q'_1\, L'_1\, M'_1, \end{cases}$$

His denotationibus adhibitis, ex aequationibus (1.) et (2.) hanc nanciscimur:

$$8. \quad \int_0^{\varphi} \frac{(P_1 - Q_1 \sin^2\varphi)\, d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}}$$

$$= \int_0^{\varphi'_1} \frac{(P'_1 - Q'_1 \sin^2\varphi)\, d\varphi}{\sqrt{[(1-c'^2\sin^2\varphi)(1-l'^2\sin^2\varphi)(1-m'^2\sin^2\varphi)]}}$$

$$+ \int_0^{\varphi'_2} \frac{(P'_2 - Q'_2 \sin^2\varphi)\, d\varphi}{\sqrt{[1-c'^2\sin\varphi)(1-L'^2\sin^2\varphi)(1-M'^2\sin^2\varphi)]}}.$$

Moduli novi per has aequationes determinantur, quae ex formula (23.) mutatis mutandis sequuntur:

$$9. \quad \begin{cases} m'^2_1 = \left(\frac{c - lm}{c + lm}\right)^2, \\ l'^2_1 = \left(\frac{1-c}{1+c}\right)\left(\frac{c - lm}{c + lm}\right)\left(\frac{c + LM}{c - LM}\right), \\ c'^2_1 = \left(\frac{1-c}{1+c}\right)\left(\frac{c - lm}{c + lm}\right)\left(\frac{c - LM}{c + LM}\right), \\ L'^2_1 = \left(\frac{1-c}{1+c}\right)\left(\frac{c + lm}{c - lm}\right)\left(\frac{c - LM}{c + LM}\right), \\ M'^2_1 = \left(\frac{c - LM}{c + LM}\right)^2. \end{cases}$$

Coefficientes vero novi P', Q' ex coefficientibus P, Q, determinantur ope formularum:

$$10. \quad \begin{cases} P'_1 = \frac{1}{(c+lm)(1+c)} (P_1 c + Q_1), \\ Q'_1 = \frac{2c}{(c+lm)^2(1+c)} (P_1 lm + Q_1), \\ P'_2 = \frac{1}{(c+LM)(1+c)} (P_2 c + Q_2), \\ Q'_2 = \frac{2c}{(c+LM)^2(1+c)} (P_2 LM + Q_2). \end{cases}$$

Quarum priores ex formulis (24.) art. IX. mutatis mutandis sponte prodeunt, et posteriores ita eruuntur. Habemus ex aequationibus (7.)

$$P'_2 = \frac{P'_1 - Q'_1}{l'_1\, m'_1}, \qquad Q'_2 = \frac{P'_1 c'^2 - Q'_1}{l'_1\, m'_1},$$

unde nanciscimur, valoribus ipsorum P'_1 et Q'_1 (formul. 10.) substitutis:

$$11. \quad \begin{cases} P'_2 = \frac{1}{(c+lm)(1+c)} \cdot \frac{(P_1 c - Q_1)}{l'_1}, \\ Q'_2 = \frac{2c}{(c+lm)(1+c)^2(c+LM)} \left[\frac{P_1(c^2+LM) - Q_1(1+LM)}{l'_1}\right]. \end{cases}$$

Hîc valoribus ipsorum P_1, Q_1, ex formul. (6.) introductis, erit:

$$12.\quad \begin{cases} P'_2 = \dfrac{L'_1}{(c+lm)L_1 M_1}(P_2 c + Q_2), \\ Q'_2 = \dfrac{2L'_1 c}{(c+lm)(c+LM)L_1 M_1}(P_2 LM + Q_2). \end{cases}$$

Iam vero habemus ex aequat. (5.)

$$L_1 M_1 = \frac{c_1^2}{l_1 m_1}$$

atque:

$$l_1^2 m_1^2 (c^2 - L^2 M^2) = (1-c^2)(c^2 - l^2 m^2),$$

quibus aequationibus in valore ipsius L'_1 adhibitis, sequitur:

$$P'_2 = \frac{1}{(c+LM)(1+c)}(P_2 c + Q_2), \qquad Q'_2 = \frac{2c}{(c+LM)^2(1+c)}(P_2 LM + Q_2).$$

Iam vero magis arridet quantitates P', Q' per modulos novos exprimere. Quo facto habemus:

$$13.\quad \begin{cases} P'_1 = \left[\dfrac{1+m'_1}{4}\cdot\dfrac{m'_1+c'_1 l'_1}{m'_1-c'_1 l'_1}\right]\left[P_1\left(1-\dfrac{c'_1 l'_1}{m'_1}\right)+Q_1\left(1+\dfrac{c'_1 l'_1}{m'_1}\right)\right], \\ Q'_1 = \left[\dfrac{1+m'_1}{4}\cdot\dfrac{m'_1+c'_1 l'_1}{m'_1-c'_1 l'_1}\right]\left[P_1\left(1-\dfrac{c'_1 l'_1}{m'_1}\right)(1-m'_1)+Q_1\left(1+\dfrac{c'_1 l'_1}{m'_1}\right)(1+m'_1)\right], \\ P'_2 = \left[\dfrac{1+M'_1}{4}\cdot\dfrac{M'_1+c'_1 L'_1}{M'_1-c'_1 L'_1}\right]\left[P_2\left(1-\dfrac{c'_1 L'_1}{M'_1}\right)+Q_2\left(1+\dfrac{c'_1 L'_1}{M'_1}\right)\right], \\ Q'_2 = \left[\dfrac{1+M'_1}{4}\cdot\dfrac{M'_1+c'_1 L'_1}{M'_1-c'_1 L'_1}\right]\left[P_2\left(1-\dfrac{c'_1 L'_1}{M'_1}\right)(1-M'_1)+Q_2\left(1+\dfrac{c'_1 L'_1}{M'_1}\right)(1+M'_1)\right]. \end{cases}$$

Sive si P'_2 et Q'_2 per P_1, Q_1, c', l', m' exprimere malimus:

$$14.\quad \begin{cases} P'_2 = \left[\dfrac{1+m'_1}{4l'_1}\cdot\dfrac{m'_1+c'_1 l'_1}{m'_1-c'_1 l'_1}\right]\left[P_1\left(1-\dfrac{c'_1 l'_1}{m'_1}\right)-Q_1\left(1+\dfrac{c'_1 l'_1}{m'_1}\right)\right], \\ Q'_2 = \left[\dfrac{1+m'_1}{4l'_1}\cdot\dfrac{m'_1+c'_1 l'_1}{m'_1-c'_1 l'_1}\right]\left[P_1\left(1-\dfrac{c'_1 l'_1}{m'_1}\right)\left(1-\dfrac{c'^2_1}{m'_1}\right)-Q_1\left(1+\dfrac{c'_1 l'_1}{m'_1}\right)\left(1+\dfrac{c'^2_1}{m'_1}\right)\right]. \end{cases}$$

Anguli φ'_1, φ'_2, minimo positivo valore gaudentes e numero eorum erunt, quorum sinus dantur hac formula:

$$15.\quad \frac{1-(1-l'_1 c'_1)\sin^2\varphi'_1}{1-(1+l'_1 c'_1)\sin^2\varphi'_1} = \frac{\sqrt{\left[(1-c^2\sin^2\varphi)\left(1-\dfrac{l^2 m^2}{c^2}\sin^2\varphi\right)\right]}}{\cos\varphi\sqrt{(1-l^2 m^2\sin^2\varphi)}}$$
$$= \frac{1-(1-M'_1)\sin^2\varphi'_2}{1-(1+M'_1)\sin^2\varphi'_2},$$

atque inter argumenta φ'_1 et φ'_2, quae ambo per argumentum φ exprimuntur, haec aequatio datur:

$$16.\quad \operatorname{tang}\varphi'_2 = l'_1 \operatorname{tang}\varphi'_1.$$

Adnotemus adhuc, ex hac transformatione integralium indefinitorum duplicem integralis definiti determinationem oriri. Nimirum habemus, iisdem denotationibus adhibitis, has formulas:

$$17.\quad \begin{cases} \int_0^{\frac{\pi}{2}} \frac{(P_1 - Q_1 \sin^2\varphi)\, d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}} = \int_0^{\frac{\pi}{2}} \frac{(P_2 - Q_2 \sin^2\varphi)\, d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-L^2\sin^2\varphi)(1-M^2\sin^2\varphi)]}} \\ = \int_0^{\frac{\pi}{2}} \frac{(P'_1 - Q'_1 \sin^2\varphi)\, d\varphi}{\sqrt{[(1-c'^2\sin^2\varphi)(1-l'^2\sin^2\varphi)(1-m'^2\sin^2\varphi)]}} = \int_0^{\frac{\pi}{2}} \frac{(P'_2 - Q'_2 \sin^2\varphi)\, d\varphi}{\sqrt{[(1-c'^2\sin^2\varphi)(1-l'^2\sin^2\varphi)(1-m'^2\sin^2\varphi)]}}. \end{cases}$$

Sed etiam tertio modo integrale definitum exprimere licet. Nimirum in transformatione generali posito $\Phi = \frac{\pi}{2}$, nanciscimur hos valores argumentorum Φ'_1 et Φ'_2:

$$\sin\Phi'_1 = \frac{1}{\sqrt{(1+c'_1 l'_1)}}, \qquad \sin\Phi'_2 = \frac{1}{\sqrt{(1+M'_1)}},$$

sive:

$$\operatorname{tang}\Phi'_1 = \sqrt{\left(\frac{1}{c'_1 l'_1}\right)}, \qquad \operatorname{tang}\Phi'_2 = \sqrt{\frac{l'_1}{c'_1}} = \sqrt{\frac{1}{M'_1}}.$$

Aequatio vero integralis fit:

$$\int_0^{\frac{\pi}{2}} \frac{(P_1 - Q_1 \sin^2\varphi)\, d\varphi}{\sqrt{[1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}}$$

$$= \int_0^{\operatorname{arc\,tang} = \sqrt{\left(\frac{1}{c'_1 l'_1}\right)}} \frac{(P'_1 - Q'_1 \sin^2\varphi)\, d\varphi}{\sqrt{[(1-c'^2\sin^2\varphi)(1-l'^2\sin^2\varphi)(1-m'^2\sin^2\varphi)]}}$$

$$+ \int_0^{\operatorname{arc\,tang} = \sqrt{\frac{1}{M'_1}}} \frac{(P'_2 - Q'_2 \sin^2\varphi)\, d\varphi}{\sqrt{[(1-c'^2\sin^2\varphi)(1-L'^2\sin^2\varphi)(1-M'^2\sin^2\varphi)]}}.$$

Si vero numeratores ad hanc formam:

$$(R\cos^2\Phi + S\sin^2\Phi)$$

applicare velimus, ponamus necesse est:

$$18.\quad \begin{cases} P_1 = R_1, \quad Q_1 = R_1 - S_1, \quad P_2 = R_2, \quad Q_2 = R_2 - S_2, \\ P'_1 = R'_1, \quad Q'_1 = R'_1 - S'_1, \quad P'_2 = R'_2, \quad Q'_2 = R'_2 - S'_2, \end{cases}$$

quibus valoribus in formulis (6.), (7.), (10.), (11.), (13.) et (14.) substitutis, hae prodibunt aequationes:

$$19.\quad \begin{cases} S_1 = R_2 l_1 m_1, \qquad S'_1 = R'_2 l'_1 m'_1, \\ S_2 = R_1 L_1 M_1, \qquad S'_2 = R'_1 L'_1 M'_1, \end{cases}$$

$$20.\quad \begin{cases} R'_1 = \frac{(1+c)R_1 - S_1}{(1+c)(c+lm)} = \left(\frac{1+m'_1}{4m'_1}\right)\left(\frac{m'_1+c'_1 l'_1}{m'_1-c'_1 l'_1}\right)[2m'_1 R_1 - (m'_1 + c'_1 l'_1)S_1], \\ S'_1 = \left[\frac{S_1 - (1-c)R_1}{(1+c)(c+lm)}\right] m'_1 = \left(\frac{1+m'_1}{4}\right)\left(\frac{m'_1+c'_1 l'_1}{m'_1-c'_1 l'_1}\right)[(m'_1 + c'_1 l'_1)S_1 - 2c'_1 l'_1 R_1], \\ R'_2 = \frac{(1+c)R_2 - S_2}{(1+c)(c+LM)} = \frac{1+M'_1}{4M'_1}\cdot\frac{M'_1+c'_1 L'_1}{M'_1-c'_1 L'_1}[2M'_1 R_2 - (M'_1 + c'_1 L'_1)S_2], \\ S'_2 = \frac{S_2 - (1-c)R_2}{(1+c)(c+LM)} M'_1 = \frac{1+M'_1}{4}\cdot\frac{M'_1+c'_1 L'_1}{M'_1-c'_1 L'_1}[(M'_1 + c'_1 L'_1)S_2 - 2c'_1 L'_1 R_2], \end{cases}$$

vel si coefficientes R'_2 et S'_2 per $R_1 S_1$ exprimere malimus:

$$21. \begin{cases} R'_2 = \dfrac{S_1-(1-c)R_1}{(c+\mathrm{LM})\,l_1 m_1} = \dfrac{1+m'_1}{4l'_1 m'_1}\cdot\dfrac{m'_1+c'_1 l'_1}{m'_1-c'_1 l'_1}\,[m'_1+c'_1 l'_1)S_1 - 2c'_1 l'_1 R_1], \\ S'_2 = \dfrac{R_1(1+c)-S_1}{(c+\mathrm{LM})\,l_1 m_1}\cdot\dfrac{1-c}{1+c}\cdot\dfrac{c-\mathrm{LM}}{c+\mathrm{LM}} = \dfrac{1+m'_1}{4m'_1}\cdot\dfrac{m'_1+c'_1 l'_1}{m'_1-c'_1 l'_1}\,\mathrm{L}'_1\mathrm{M}'_1\,[2m'_1 R_1-(m'_1+c'_1 l'_1)\delta]. \end{cases}$$

Aequationes vero integrales (8.) et (17.), brevitatis gratia posito:

$$\Delta(c, l, m) = \sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]},$$

in has abeunt:

$$22. \quad \int_0^{\varphi} \frac{(R_1\cos^2\varphi + S_1\sin^2\varphi)}{\Delta(c, l, m)}$$

$$= \int_0^{\varphi'_1} \frac{(R'_1\cos^2\varphi + S'_1\sin^2\varphi)\,d\varphi}{\Delta(c', l', m')} + \int_0^{\varphi'_2} \frac{(R'_2\cos^2\varphi + S'_2\sin^2\varphi)\,d\varphi}{\Delta(c', \mathrm{L}', \mathrm{M}')},$$

$$23. \begin{cases} \displaystyle\int_0^{\frac{\pi}{2}} \frac{(R_1\cos^2\varphi + S_1\sin^2\varphi)\,d\varphi}{\Delta(c, l, m)} = \int_0^{\frac{\pi}{2}} \frac{(R_2\cos^2\varphi + S_2\sin^2\varphi)\,d\varphi}{\Delta(c, \mathrm{L}, \mathrm{M})} \\ \displaystyle = \int_0^{\frac{\pi}{2}} \frac{(R'_1\cos^2\varphi + S'_1\sin^2\varphi)\,d\varphi}{\Delta(c', l', m')} = \int_0^{\frac{\pi}{2}} \frac{(R'_2\cos^2\varphi + S'_2\sin^2\varphi)\,d\varphi}{\Delta(c', \mathrm{L}', \mathrm{M}')}. \end{cases}$$

Denotationes nostrae iam ita praeparatae sunt, ut transformatio per eundem algorithmum continuata facillime hîc proponi possit. Utrumque integrale indefinitum, ad quod in aequat. (8.) vel (22.) per primam transformationem devecti sumus, in duo nova integralia eadem ratione discerpitur, quorum argumenta coefficientes et moduli per easdem litteras, duplici virgula supra adiecta, designantur.

Primum igitur integrale termini secundi aequat. (8.)

$$\int_0^{\varphi'_1} \frac{(P'_1 - Q'_1\sin^2\varphi)\,d\varphi}{\sqrt{[(1-c'^2\sin^2\varphi)(1-l'^2\sin^2\varphi)(1-m'^2\sin^2\varphi)]}},$$

in haec duo nova transit:

$$24. \quad = \int_0^{\varphi''_{1,1}} \frac{(P''_1 - Q''_1\sin^2\varphi)\,d\varphi}{\Delta(c'', l'', m'')} + \int_0^{\varphi''_{1,2}} \frac{(P''_2 - Q''_2\sin^2\varphi)\,d\varphi}{\Delta(c'', \mathrm{L}'', \mathrm{M}'')},$$

ubi has aequationes ponimus:

$$25. \begin{cases} P''_1 = \dfrac{1}{(c'+l'm')(1+c')}(P'_1 c' + Q'_1), \\ Q''_1 = \dfrac{2c'}{(c'+l'm')^2(1+c')}(P'_1 l'm' + Q'_1), \\ P''_2 = \dfrac{1}{(c'+\mathrm{L}'\mathrm{M}')(1+c')}(P'_2 c' + Q'_2), \\ Q''_2 = \dfrac{2c'}{(c'+\mathrm{L}'\mathrm{M}')^2(1+c')}(P'_2 l'm' + Q'_2). \end{cases}$$

$$26. \quad \begin{cases} m_1''^2 = \left(\frac{c'-l'm'}{c'+l'm'}\right)^2, \\ l_1''^2 = \left(\frac{1-c'}{1+c'}\right)\left(\frac{c'-l'm'}{c'+l'm'}\right)\left(\frac{c'+L'M'}{c'-L'M'}\right), \\ c_1''^2 = \left(\frac{1-c'}{1+c'}\right)\left(\frac{c'-l'm'}{c'+l'm'}\right)\left(\frac{c'-L'M'}{c'+L'M'}\right), \\ L_1''^2 = \left(\frac{1-c'}{1+c'}\right)\left(\frac{c'+l'm'}{c'-l'm'}\right)\left(\frac{c'-L'M'}{c'+L'M'}\right), \\ M_1''^2 = \left(\frac{c'-L'M'}{c'+L'M'}\right)^2, \end{cases}$$

$$27. \quad \left(\frac{1-(1-l_1''c_1'')\sin^2\varphi_{1,1}''}{1-(1+l_1''c_1'')\sin^2\varphi_{1,1}''}\right)$$

$$= \frac{\sqrt{\left[(1-c'^2\sin^2\varphi_1')\left(1-\frac{l'^2m'^2}{c'^2}\sin^2\varphi_1'\right)\right]}}{\cos\varphi_1'\sqrt{(1-l'^2m'^2\sin^2\varphi_1')}} = \left(\frac{1-(1-M_1'')\sin^2\varphi_{1,2}''}{1-(1+M_1'')\sin^2\varphi_{1,2}''}\right),$$

unde fluit haec rursus aequatio:

$$28. \quad \operatorname{tang}\Phi_{1,2}'' = l_1''\operatorname{tang}\Phi_{1,1}''.$$

Secundum vero integrale termini secundi aequat. (8.):

$$\int_0^{\varphi_2} \frac{(P_2'-Q_2'\sin^2\varphi)\,d\varphi}{\sqrt{[1-c'^2\sin^2\varphi)(1-l'^2\sin^2\varphi)(1-m'^2\sin^2\varphi)]}}$$

in has transit:

$$29. \quad = \int_0^{\varphi_{2,1}''} \frac{(P_2''-Q_2''\sin^2\varphi)\,d\varphi}{\Delta(c'', L'', M'')} + \int_0^{\varphi_{2,2}''} \frac{(P_1''-Q_1''\sin^2\varphi)\,d\varphi}{\Delta(c'', l'', m'')}.$$

ita ut nec ad duos novos numeratores, nec ad alios perveniamus modulos, sed eadem duo integralia, nonnisi limite superiori mutato, adipiscamur. Quippe quae similitudo memorabiles eo demonstratur, ut formulae (25.) doceant, quantitates P_2'', Q_2'', m'', l'', c'', L'', M'' eodem modo ex quantitatibus, P_2', Q_2', m', l', c', $L'M'$ componi, ac P_1'', Q_1'', M'', L'', c'', l'', m'', ex P_1', Q_1', M', L', c', l', m' componuntur.

Argumenta vero $\Phi_{2,1}''$ et $\Phi_{2,2}''$ nova sunt, atque ex his aequat. emanant, aeque ac in formulis (15.):

$$30. \quad \left(\frac{1-(1-L_1''c_1'')\sin^2\varphi_{2,1}''}{1-(1+L_1''c_1'')\sin^2\varphi_{2,1}''}\right)$$

$$= \frac{\sqrt{\left[(1-c'^2\sin^2\varphi_2')\left(1-\frac{L'^2M'^2}{c'^2}\sin^2\varphi_2'\right)\right]}}{\cos\varphi_2'\sqrt{(1-L'^2M'^2\sin^2\varphi_2')}} = \left(\frac{1-(1-m_1'')\sin^2\varphi_{2,2}''}{1-(1+m_1'')\sin^2\varphi_{2,2}''}\right),$$

unde haec aequatio prodit:

$$31. \quad \operatorname{tang}\Phi_{2,2}'' = L_1''\operatorname{tang}\Phi_{2,1}''.$$

Omnibus collectis hanc habemus integralis indefiniti transformationem:

32. $$\begin{cases} \int_0^{\varphi} \frac{(P_1 - Q_1 \sin^2\varphi)\, d\varphi}{\Delta(c, l, m)} \\ = \int_0^{\varphi'_1} \frac{(P'_1 - Q'_1 \sin^2\varphi)\, d\varphi}{\Delta(c', l', m')} + \int_0^{\varphi'_2} \frac{(P'_2 - Q'_2 \sin^2\varphi)\, d\varphi}{\Delta(c', L', M')} \\ = \left\{ \begin{array}{l} \int_0^{\varphi''_{1,1}} \frac{(P''_1 - Q''_1 \sin^2\varphi)\, d\varphi}{\Delta(c'', l'', m'')} + \int_0^{\varphi''_{1,2}} \frac{(P''_2 - Q''_2 \sin^2\varphi)\, d\varphi}{\Delta(c'', L'', M'')} \\ + \int_0^{\varphi''_{2,2}} \frac{(P''_1 - Q''_1 \sin^2\varphi)\, d\varphi}{\Delta(c'', l'', m'')} + \int_0^{\varphi''_{2,1}} \frac{(P''_2 - Q''_2 \sin^2\varphi)\, d\varphi}{\Delta(c'', L'', M'')} \end{array} \right\}. \end{cases}$$

Neminem fugere potest in quaque sequente transformatione, nonnisi duo diversa integralia prodire, sed diversis argumentorum limitibus gaudentia. Adhibito vero theoremate fundamentali Abeliano, quippe per quod quatuor integralium, quae iisdem modulis atque numeratoribus gaudent, aggregatum, ad duorum talium aggregatum reducitur, evictum est memorabilissimum hoc theorema:

„Integrale Abelianum primi ordinis indefinitum continuo transfor„mari potest in aggregata quaternorum prorsus novorum integralium eius„dem generis, quorum bina iisdem modulis atque numeratore, sed diversis „limitibus gaudent." Id quod theorema iam ante quatuor annos a nobis inventum ante annum et quod excurrit geometris praedicavimus.

Prior methodus transformationis definiti integralis continuata haud ad quaterna, sed ad singulum integrale definitum perducit. Nimirum habebimus:

33. $$\begin{cases} \int_0^{\frac{\pi}{2}} \frac{(P_1 - Q_1 \sin^2\varphi)\, d\varphi}{\Delta(c, l, m)} = \int_0^{\frac{\pi}{2}} \frac{(P'_1 - Q'_1 \sin^2\varphi)\, d\varphi}{\Delta(c', l', m')} = \int_0^{\frac{\pi}{2}} \frac{(P''_1 - Q''_1 \sin^2\varphi)\, d\varphi}{\Delta(c'', l'', m'')} \text{ etc.} \\ \text{sive} \\ = \int_0^{\frac{\pi}{2}} \frac{(P_2 - Q_2 \sin^2\varphi)\, d\varphi}{\Delta(c, L, M)} = \int_0^{\frac{\pi}{2}} \frac{(P'_2 - Q'_2 \sin^2\varphi)\, d\varphi}{\Delta(c', L', M')} = \int_0^{\frac{\pi}{2}} \frac{(P''_2 - Q''_2 \sin^2\varphi)\, d\varphi}{\Delta(c'', L'', M'')} \text{ etc.} \end{cases}$$

Altera vero methodus, ubi primum integrale definitum in duo discerpitur integralia definita, ad quaterna ducit integralia definita.

Simili ratione alteram transformationem in capitibus antecedentibus inventam atque demonstratam tractare licet, quippe quae in aequatione integrali (30.) art. IX. continetur, nimirum fuit aequatio haec:

34. $$\int_0^z \frac{(\alpha(1-z) + \beta z)\, dz}{\sqrt{(\Delta z)}} = \pm \int_0^{y_1} \frac{\alpha^0(1-y) + \beta^0 y}{\sqrt{(\Delta^0 y)}} \pm \int_0^{y_2} \frac{(\alpha^0(1-y) + \beta^0 y)\, dy}{\sqrt{(\Delta^0 y)}} \Big],$$

cuius modulos, formulas limitesque argumentorum eodem loco invenis.

Duplex igitur hîc discrimen ab antecedente transformatione observatur; primum integrale datum, argumento z limitem $\frac{1}{1+k_1\lambda_1}$ transgresso, in duas partes dividendum erit, quarum prior in summam, posterior in differentiam duorum novorum integralium transit, deinde argumentum secundi novi integralis non in intervallo $0\ldots.1$, sed in intervallo $\infty\ldots.\frac{1}{\mu^{0 2}}$ versatur. Hanc ob causam, si integralis utriusque argumenta, simul cum argumento z, ab 0 proficisci velimus, secundum integrale per quintam transformationem fundamentalem classis (*B.*) mutandum est. Itaque ponamus:

$$y_2 = \frac{1}{\mu^{0 2}x}.$$

qua formula introducta, erit (vid. commentationis allatae art. III.)

$$35.\quad \int_\infty^{y_2} \frac{(\alpha^0(1-y)+\beta^0 y)\,dy}{\sqrt{[y(1-y)(1-k^{0 2}y)(1-\lambda^{0 2}y)(1-\mu^{0 2}y)]}}$$
$$= \frac{1}{k^0\lambda^0}\int_0^x \frac{((\alpha^0-\beta^0)(1-x)+(\alpha^0\mu_1^{0 2}-\beta^0)x)\,dx}{\sqrt{\left[x(1-x)\left(1-\left(\frac{\mu^{0 2}}{\lambda^{0 2}}\right)x\right)\left(1-\frac{\mu^{0 2}}{\lambda^{0 2}}x\right)(1-\mu^{0 2}x)\right]}}.$$

Ut argumentum x ex dato argumento z directe definiamus, formulam praecedentem in formulam transformationis argumenti y_2

$$\frac{1-\lambda^0\mu^0 y_2}{1+\lambda^0\mu^0 y_2} = -\sqrt{\left(\frac{(1-\mu^2 z)\left(1-\frac{\mu_1^2-k_1^2\lambda_1^2}{\mu_1^2}z\right)}{(1-(1-k_1^2\lambda_1^2)z)}\right)}$$

introducamus; quo facto habemus:

$$36.\quad \sqrt{\left(\frac{(1-\mu^2 z)\left(1-\frac{\mu_1^2-k_1^2\lambda_1^2}{\mu_1^2}z\right)}{(1-(1-k_1^2\lambda_1^2)z)}\right)} = \frac{1-\frac{\mu^0}{\lambda^0}x}{1+\frac{\mu^0}{\lambda^0}x}.$$

Etiam hîc denotationes novas, ad analogiam perspiciendam aptissimas introducere placet. Sit:

$z = \sin^2\varphi$, $y_1 = \sin^2\varphi_1^0$, $x = \sin^2\varphi_2^0$ (pro superioribus signis aequat. 34.),

$z = \sin^2\psi$, $y_1 = \sin^2\psi_1^0$, $x = \sin^2\psi_2^0$ (pro inferioribus signis aequat. 34.),

$$37.\left\{\begin{array}{lllll}
k = c, & \lambda = l, & \mu = m, & \frac{\mu}{k} = \mathfrak{l}, & \frac{\mu}{\lambda} = \mathfrak{k}, \\
k^0 = c^0, & \lambda^0 = l^0, & \mu^0 = m^0, & \frac{\mu^0}{k^0} = \mathfrak{l}^0, & \frac{\mu^0}{\lambda^0} = \mathfrak{k}^0, \\
\sqrt{(1-k^2)} = c_1, & \sqrt{(1-\lambda^2)} = l_1, & \sqrt{(1-\mu^2)} = m_1, & \sqrt{\left(\frac{k^2-\mu^2}{k^2}\right)} = \mathfrak{l}_1, & \sqrt{\left(\frac{\lambda^2-\mu^2}{\lambda^2}\right)} = \mathfrak{k}_1, \\
\sqrt{(1-k^{0 2})} = c_1^0, & \sqrt{(1-\lambda^{0 2})} = \lambda_1^0, & \sqrt{(1-\mu^{0 2})} = \mu_1^0, & \sqrt{\left(\frac{k^{0 2}-\mu^{0 2}}{\mu^{0 2}}\right)} = \mathfrak{l}_1^0, & \sqrt{\left(\frac{\lambda^{0 2}-\mu^{0 2}}{\lambda^{0 2}}\right)} = \mathfrak{k}_1^0.
\end{array}\right.$$

Deinde ponamus:

$$38.\quad \begin{cases} \alpha = \Pi_1, \quad \beta = K_1, \quad \dfrac{\alpha-\beta}{k\lambda} = \Pi_2, \quad \dfrac{\alpha\mu_1^2-\beta}{k\lambda} = K_2, \\ \alpha^0 = \Pi_1^0, \quad \beta^0 = K_1^0, \quad \dfrac{\alpha^0-\beta^0}{k^0\lambda^0} = \Pi_2^0, \quad \dfrac{\alpha^0\mu_1^{0\,2}-\beta^0}{k^0\lambda^0} = K_2^0. \end{cases}$$

ita ut habeamus aequationes symmetricas has:

$$39.\quad \begin{cases} \Pi_1 - K_1 = c\,l\,\Pi_2, & \Pi_2 - K_2 = \mathfrak{k}\,\mathfrak{l}\,\Pi_1, \\ \Pi_1 m_1^2 - K_1 = c\,l\,K_2, & \Pi_2 m_1^2 - K_2 = \mathfrak{k}\,\mathfrak{l}\,K_1, \\ \Pi_1^0 - K_1^0 = c^0 l^0 \Pi_2^0, & \Pi_2^0 - K_2^0 = \mathfrak{k}^0\mathfrak{l}^0\Pi_1^0, \\ \Pi_1^0 m_1^{0\,2} - K_1^0 = c^0 l^0 K_2^0, & \Pi_2^0 m_1^{0\,2} - K_2^0 = \mathfrak{k}^0\mathfrak{l}^0 K_1^0. \end{cases}$$

His denotationibus introductis ex aequationibus (34.) et (35.) adipiscimur has:

$$40.\quad \begin{cases} \displaystyle\int_0^{\varphi} \frac{(\Pi_1\cos^2\varphi + K_1\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)} \\ \displaystyle= \int_0^{\varphi_1^0} \frac{(\Pi_1^0\cos^2\varphi + K_1^0\sin^2\varphi)\,d\varphi}{\Delta(c^0,l^0,m^0)} + \int_0^{\varphi_2^0} \frac{(\Pi_2^0\cos^2\varphi + K_2^0\sin^2\varphi)\,d\varphi}{\Delta(\mathfrak{k}^0,\mathfrak{l}^0,m^0)}, \\ \displaystyle\int_{\arcsin = \frac{1}{\sqrt{(1+c_1 l_1)}}}^{\psi} \frac{(\Pi_1\cos^2\varphi + K_1\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)} \\ \displaystyle= -\int_{\frac{\pi}{2}}^{\psi_1^0} \frac{(\Pi_1^0\cos^2\psi + K_1^0\sin^2\psi)\,d\psi}{\Delta(c^0,l^0,m)} + \int_{\arcsin = \mathfrak{l}^0}^{\psi_2^0} \frac{(\Pi_2^0\cos^2\psi + K_2^0\sin^2\psi)\,d\psi}{\Delta(\mathfrak{k}^0,\mathfrak{l}^0,m^0)}. \end{cases}$$

Moduli novi his aequat. dantur:

$$41.\quad \begin{cases} c^{0\,2} = \left(\dfrac{m_1 - c_1 l_1}{m_1 + c_1 l_1}\right)^2, \\ l^{0\,2} = \left(\dfrac{1-m_1}{1+m_1}\right)\left(\dfrac{m_1 - c_1 l_1}{m_1 + c_1 l_1}\right)\left(\dfrac{m_1 + \mathfrak{k}_1\mathfrak{l}_1}{m_1 - \mathfrak{k}_1\mathfrak{l}_1}\right), \\ m^{0\,2} = \left(\dfrac{1-m_1}{1+m_1}\right)\left(\dfrac{m_1 - c_1 l_1}{m_1 + c_1 l_1}\right)\left(\dfrac{m_1 - \mathfrak{k}_1\mathfrak{l}_1}{m_1 + \mathfrak{k}_1\mathfrak{l}_1}\right), \\ \mathfrak{l}^{0\,2} = \left(\dfrac{1-m_1}{1+m_1}\right)\left(\dfrac{m_1 + c_1 l_1}{m_1 - c_1 l_1}\right)\left(\dfrac{m_1 - \mathfrak{k}_1\mathfrak{l}_1}{m_1 + \mathfrak{k}_1\mathfrak{l}_1}\right), \\ \mathfrak{k}^{0\,2} = \left(\dfrac{m_1 - \mathfrak{k}_1\mathfrak{l}_1}{m_1 + \mathfrak{k}_1\mathfrak{l}_1}\right)^2. \end{cases}$$

Numeratorum coefficientes ex his aequat. determinantur:

$$42.\quad \begin{cases} \Pi_1^0 = \dfrac{1}{(m_1 + c_1 l_1)(1+m_1)}(\Pi_1 m_1 + K_1), \\ K_1^0 = \dfrac{2m_1}{(m_1 + c_1 l_1)^2(1+m_1)}(\Pi_1 c_1 l_1 + K_1), \\ \Pi_2^0 = \dfrac{1}{(m_1 + \mathfrak{k}_1\mathfrak{l}_1)(1+m_1)}(\Pi_2 m_1 + K_2) \\ K_2^0 = \dfrac{2m_1}{(m_1 + \mathfrak{k}_1\mathfrak{l}_1)^2(1+m_1)}(\Pi_2 \mathfrak{k}_1\mathfrak{l}_1 + K_2). \end{cases}$$

Sive si hos coefficientes per novos modulos praeferamus exprimere:

$$43. \begin{cases} \Pi_1^0 = \left(\frac{1+c^0}{4}\cdot\frac{c^0+l^0m^0}{c^0-l^0m^0}\right)\left(\Pi_1\left(1-\frac{m^0l^0}{c^0}\right)+K_1\left(1+\frac{m^0l^0}{c^0}\right)\right), \\ K_1^0 = \left(\frac{1+c^0}{4}\cdot\frac{c^0+l^0m^0}{c^0-l^0m^0}\right)\left(\Pi_1\left(1-\frac{m^0l^0}{c^0}\right)(1-c^0)+K_1\left(1+\frac{m^0l^0}{c^0}\right)(1+c^0)\right), \\ \Pi_2^0 = \left(\frac{1+\mathfrak{k}^0}{4}\cdot\frac{\mathfrak{k}^0+\mathfrak{l}^0m^0}{\mathfrak{k}^0-\mathfrak{l}^0m^0}\right)\left(\Pi_2\left(1-\frac{m^0\mathfrak{l}^0}{\mathfrak{k}^0}\right)+K_2\left(1+\frac{m^0\mathfrak{l}^0}{\mathfrak{k}^0}\right)\right), \\ K_2^0 = \left(\frac{1+\mathfrak{k}^0}{4}\cdot\frac{\mathfrak{k}^0+\mathfrak{l}^0m^0}{\mathfrak{k}^0-\mathfrak{l}^0m^0}\right)\left(\Pi_2\left(1-\frac{m^0\mathfrak{l}^0}{\mathfrak{k}^0}\right)(1-\mathfrak{k}^0)+K_2\left(1+\frac{m^0\mathfrak{l}^0}{\mathfrak{k}^0}\right)(1+\mathfrak{k}^0)\right). \end{cases}$$

Anguli φ_1^0 et φ_2^0, ψ_1^0 et ψ_2^0, minimi positivi sunt, quorum sinus ex his formulis deducuntur:

$$44. \begin{cases} \left(\frac{1-l^0m^0\sin^2\varphi_1^0}{1+l^0m^0\sin^2\varphi_1^0}\right) = \sqrt{\left(\frac{(1-m^2\sin^2\varphi)\left(1-\frac{m_1^2-c_1^2l_1^2}{m_1^2}\sin^2\varphi\right)}{(1-(1-c_1^2l_1^2)\sin^2\varphi)}\right)} = \frac{1-\mathfrak{k}^0\sin^2\varphi_2^0}{1+\mathfrak{k}^0\sin^2\varphi_2^0}. \\ \left(\frac{1-l^0m^0\sin^2\psi_1^0}{1+l^0m^0\sin^2\psi_1^0}\right) = \sqrt{\left(\frac{(1-m^2\sin^2\psi)\left(1-\frac{m_1^2-c_1^2l_1^2}{m_1^2}\sin^2\psi\right)}{(1-(1-c_1^2l_1^2)\sin^2\varphi)}\right)} = \frac{1-\mathfrak{k}^0\sin^2\psi_2^0}{1+\mathfrak{k}^0\sin^2\psi_2^0}, \end{cases}$$

ubi argumentum φ ab 0 usque ad arc sin $=\frac{1}{\sqrt{(1+c_1l_1)}}$, argumentum vero ψ ab arc sin $\frac{1}{\sqrt{(1+c_1l_1)}}$ usque ad $\frac{\pi}{2}$ pergit. Inter argumenta denique nova regnat haec aequatio:

$$45. \quad \sin\varphi_2^0 = l^0\sin\varphi_1^0, \quad \sin\psi_2^0 = l^0\sin\psi_1^0.$$

Quod attinet ad integralium definitorum transformationem inde ex hac theoria sequentem, facile relationes deducimus has:

$$46. \begin{cases} \int_0^{\arcsin=\frac{1}{\sqrt{(1+c_1l_1)}}} \frac{(\Pi_1\cos^2\varphi+K_1\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)} \\ = \int_0^{\frac{\pi}{2}} \frac{(\Pi_1^0\cos^2\varphi+K_1^0\sin^2\varphi)\,d\varphi}{\Delta(c^0,l^0,m^0)} + \int_0^{\arcsin=\frac{m^0}{\mathfrak{k}^0}} \frac{(\Pi_2^0\cos^2\varphi+K_2\sin^2\varphi)\,d\varphi}{\Delta(\mathfrak{k}^0,\mathfrak{l}^0,m^0)}, \\ \int_{\arcsin=\frac{1}{\sqrt{(1+c_1l_1)}}}^{\frac{\pi}{2}} \frac{(\Pi_1\cos^2\varphi+K\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)} \\ = +\int_0^{\frac{\pi}{2}} \frac{(\Pi_1^0\cos^2\varphi+K_1^0\sin^2\varphi)\,d\varphi}{\Delta(c^0,l^0,m^0)} - \int_0^{\arcsin=\frac{m^0}{\mathfrak{k}^0}} \frac{(\Pi_2^0\cos^2\varphi+K_2^0\sin^2\varphi)\,d\varphi}{\Delta(\mathfrak{k}^0,\mathfrak{l}^0,m^0)}, \end{cases}$$

unde sequitur:

$$47. \quad \int_0^{\frac{\pi}{2}} \frac{(\Pi_1\cos^2\varphi+K_1\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)} = 2\int_0^{\frac{\pi}{2}} \frac{(\Pi_1^0\cos^2\varphi+K_1^0\sin^2\varphi)\,d\varphi}{\Delta(c^0,l^0,m^0)}.$$

Si numeratoribus formam:

$$(\mathrm{P} - \Sigma \sin^2 \varphi)$$

tribuere velimus, ponamus necesse est:

$$\Pi_1 = \mathrm{P}_1, \quad \mathrm{K}_1 = \mathrm{P}_1 - \Sigma_1, \quad \Pi_2 = \mathrm{P}_2, \quad \mathrm{K}_2 = \mathrm{P}_2 - \Sigma_2,$$
$$\Pi_1^0 = \mathrm{P}_1^0, \quad \mathrm{K}_1^0 = \mathrm{P}_1^0 - \Sigma_1^0, \quad \Pi_2^0 = \mathrm{P}_2^0, \quad \mathrm{K}_2^0 = \mathrm{P}_2^0 - \Sigma_2^0,$$

quibus valoribus in form. (39.), (40.), (42.), (43.), (47.) substitutis, habemus has aequationes et formulas:

$$48. \quad \begin{cases} \displaystyle\int_0^{\varphi} \frac{(\mathrm{P}_1 - \Sigma_1 \sin^2\varphi)\, d\varphi}{\Delta(c\, l\, m)} \\ \displaystyle = \int_0^{\varphi_1^0} \frac{(\mathrm{P}_1^0 - \Sigma_1^0 \sin^2\varphi)\, d\varphi}{\Delta(c^0, l^0, m^0)} + \int_0^{\varphi_2^0} \frac{(\mathrm{P}_2^0 - \Sigma_2^0 \sin^2\varphi)\, d\varphi}{\Delta(\mathfrak{k}^0, \mathfrak{l}^0, m^0)}, \\ \displaystyle\int_{\arcsin = \frac{1}{\sqrt{(1+c_1 l_1)}}}^{\psi} \frac{(\mathrm{P}_1 - \Sigma_1 \sin^2\varphi)\, d\varphi}{\Delta(c\, l\, m)} \\ \displaystyle = -\int_{\frac{\pi}{2}}^{\psi_1^0} \frac{(\mathrm{P}_1^0 - \Sigma_1^0 \sin^2\varphi)\, d\varphi}{\Delta(c^0, l^0, m^0)} + \int_{\arcsin = l^0}^{\psi_2^0} \frac{(\mathrm{P}_2^0 - \Sigma_2^0 \sin^2\varphi)\, d\varphi}{\Delta(\mathfrak{k}^0, \mathfrak{l}^0, m^0)}, \end{cases}$$

$$49. \quad \int_0^{\frac{\pi}{2}} \frac{(\mathrm{P}_1 - \Sigma_1 \sin^2\varphi)\, d\varphi}{\Delta(c, l, m)} = 2\int_0^{\frac{\pi}{2}} \frac{(\mathrm{P}_1^0 - \Sigma_1^0 \sin^2\varphi)\, d\varphi}{\Delta(c^0, l^0, m^0)},$$

$$50. \quad \begin{cases} \Sigma_1 = \mathrm{P}_2\, c\, l, & \Sigma_1^0 = \mathrm{P}_2^0\, c^0\, l^0, \\ \Sigma_2 = \mathrm{P}_1\, \mathfrak{k}\, \mathfrak{l}, & \Sigma_2^0 = \mathrm{P}_1\, \mathfrak{k}^0\, \mathfrak{l}^0, \end{cases}$$

$$51. \quad \begin{cases} \mathrm{P}_1^0 = \dfrac{(1+m_1)\mathrm{P}_1 - \Sigma_1}{(1+m_1)(m_1 + c_1 l_1)}, \\ \Sigma_1^0 = \left(\dfrac{\Sigma_1 - (1-m_1)\mathrm{P}_1}{(1+m_1)(m_1 + c_1 l_1)}\right)\left(\dfrac{m_1 - c_1 l_1}{m_1 + c_1 l_1}\right), \\ \mathrm{P}_2^0 = \dfrac{(1+m_1)\mathrm{P}_2 - \Sigma_2}{(1+m_1)(m_1 + \mathfrak{k}_1 \mathfrak{l}_1)}, \\ \Sigma_2^0 = \left(\dfrac{\Sigma_2 - (1-m_1)\mathrm{P}_2}{(1+m_1)(m_1 + \mathfrak{k}_1 \mathfrak{l}_1)}\right)\left(\dfrac{m_1 - \mathfrak{k}_1 \mathfrak{l}_1}{m_1 + \mathfrak{k}_1 \mathfrak{l}_1}\right). \end{cases}$$

Primo adspectu apparet, quantitates in hac transformatione adhibitas:

$$m^0, \quad l^0, \quad c^0, \quad \mathfrak{l}^0, \quad \mathfrak{k}^0, \quad \Pi_1^0, \quad \mathrm{K}_1^0, \quad \Pi_2^0, \quad \mathrm{K}_2^0,$$

ex similibus in ordinem prioris transformationis:

$$c_1', \quad l_1', \quad m_1', \quad L_1', \quad M_1', \quad P_1', \quad Q_1', \quad P_2', \quad Q_2',$$

protinus emergere, ubique loco quantitatum:

$$m, \quad l, \quad c, \quad \mathfrak{l}, \quad \mathfrak{k}, \quad \Pi_1, \quad \mathrm{K}_1, \quad \Pi_2, \quad \mathrm{K}_2 \quad \text{etc.}$$

positis in ordinem:

$$c_1, \quad l_1, \quad m_1, \quad L_1, \quad M_1, \quad P_1, \quad Q_1, \quad P_2, \quad Q_2, \quad \text{etc.}$$

Id quod ex natura ipsa utriusque transformationis sponte prodit. Formulae enim transformationis posterioris ex formulis prioris extemplo oriuntur, posito:

$$\sin\Phi = i \operatorname{tang}\Phi, \qquad \sin\Phi_1^0 = i\operatorname{tang}\Phi_1', \qquad \sin\Phi_2^0 = i\operatorname{tang}\Phi_2',$$

sive quod idem est, in utroque termino sexta classis *B.* transformatione fundamentali adhibita. Iam vero haec transformatio integrale

$$\int \frac{(P_1 - Q_1 \sin^2\varphi)\,d\varphi}{\Delta(c\,l\,m)}$$

commutat in integrale:

$$(\sqrt{-1})\int \frac{(P_1 \cos^{2\prime}\varphi + Q_1 \sin^{2\prime}\varphi)\,d'\varphi}{\Delta(m_1, l_1, c_1)}$$

unde illa quantitatum commutatio antea observata sponte oritur. Formulae transformationis, illa mutatione facta, una ex altera, prodeunt; nimirum sit:

$$\frac{1-(1-l_1' c_1')\sin^2\varphi_1'}{1-(1+l_1' c_1')\sin^2\varphi_1'} = \frac{1-l^0 m^0 \sin^2\varphi_1^0}{1+l^0 m^0 \sin^2\varphi_1^0},$$

$$\frac{1-(1-\mathfrak{M}_1')\sin^2\varphi_2'}{1-(1+\mathfrak{M}_1')\sin^2\varphi_2'} = \frac{1-\mathfrak{k}^0\sin^2\varphi_2^0}{1+\mathfrak{k}^0\sin^2\varphi_2^0},$$

$$\frac{(1-c^2\sin^2\varphi)\left(1-\frac{l^2 m^2}{c^2}\sin^2\varphi\right)}{\cos^2\varphi(1-m^2 l^2)\sin^2\varphi} = \frac{(1-m^2\sin^{2\prime}\varphi)\left(1-\frac{m_1^2-c_1^2 l_1^2}{m_1^2}\sin^{2\prime}\varphi\right)}{(1-(1-c_1^2 l_1^2)\sin^{2\prime}\varphi)}.$$

Etiam haec secunda transformatio continuata ad integralia definita integralia que indefinita adhibetur determinanda. Quaterni rursus nonnisi numeratoris coefficientes, quinternique moduli computandi sunt; nimirum in sequenti transformatione hae quantitates:

$$\Pi_1^{00},\quad K_1^{00},\quad \Pi_2^{00},\quad K_2^{00},\quad c^{00},\quad l^{00},\quad m^{00},\quad \mathfrak{l}^{00},\quad \mathfrak{k}^{00},$$

quae aeque ex quantitatibus:

$$\Pi_1^0,\quad K_1^0,\quad \Pi_2^0,\quad K_2^0,\quad c^0,\quad l^0,\quad m^0,\quad \mathfrak{l}^0,\quad \mathfrak{k}^0,$$

ac hae ex quantitatibus:

$$\Pi_1,\quad K_1,\quad \Pi_2,\quad K_2,\quad c,\quad l,\quad m,\quad \mathfrak{l},\quad \mathfrak{k}$$

computantur. Argumenta nova $\Phi_{1,1}^{00}$, $\Phi_{1,2}^{00}$, $\Phi_{2,1}^{00}$, $\Phi_{2,2}^{00}$ ex formulis

$$\frac{1-l^{00}m^{00}\sin^2\varphi_{1,1}^{00}}{1+l^{00}m^{00}\sin^2\varphi_{1,1}^{00}} = \sqrt{\left(\frac{(1-m_1^{02}\sin^2\varphi_1^0)\left(1-\frac{m_1^{02}-c_1^{02}l_1^{02}}{m_1^{02}}\sin^2\varphi_1^0\right)}{(1-(1-c_1^{02}l_1^{02})\sin^2\varphi_1^0)}\right)} = \frac{1-\mathfrak{k}^{00}\sin^2\varphi_{1,2}^{00}}{1+\mathfrak{k}^{00}\sin^2\varphi_{1,2}^{00}},$$

$$\frac{1-l^{00}m^{00}\sin^2\varphi_{2,1}^{00}}{1+l^{00}m^{00}\sin^2\varphi_{2,1}^{00}} = \sqrt{\left(\frac{(1-m_1^{02}\sin^2\varphi_2^0)\left(1-\frac{m_1^{02}-\mathfrak{k}_1^{02}\mathfrak{l}_1^{02}}{m_1^{02}}\sin^2\varphi_2^0\right)}{(1-(1-\mathfrak{k}_1^{02}\mathfrak{l}_1^{02})\sin^2\varphi_2^0)}\right)} = \frac{1-c^{00}\sin^2\varphi_{2,2}^{00}}{1+c^{00}\sin^2\varphi_{2,2}^{00}},$$

determinantur. Aequationes vero integrales, quia singulum quodque integrale, cuius argumentum limitem arc sin $= \frac{1}{\sqrt{(1+c_1 l_1)}}$ superat, non in summam sed in differentiam duorum novorum integralium commutatur, formam magis contortam assumere videntur, quam, quum primum de modulis disseruimus, simpliciorem reddemus, nec non alio loco rursus ope theorematis fundamentalis Abeliani semper ad quatuor summum integralium indefinitorum aggregatum reducemus. Contra rursus integrale definitum ita continuo transformari potest, ut habeamus:

$$\int_0^{\frac{\pi}{2}} \frac{(\Pi_1 \cos^2\varphi + \mathrm{K}_1 \sin^2\varphi)\, d\varphi}{\Delta(c, l, m)} = \int_0^{\frac{\pi}{2}} \frac{(\Pi_1^0 \cos^2\varphi + \mathrm{K}_1^0 \sin^2\varphi)\, d\varphi}{\Delta(c^0, l^0, m^0)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\Pi_1^{00} \cos^2\varphi + \mathrm{K}_1^{00} \sin^2\varphi)\, d\varphi}{\Delta(c^{00}, l^{00}, m^{00})} \text{ etc.}$$

XII.

De modulorum utriusque transformationis natura.

Iam saepius nobis animadversum est, transformationes nostras ad convergentem integralium Abelianorum computationis algorithmum viam sternere, quam rem in sequentibus adhuc exponere velimus. Hunc ad finem in naturam modulorum penetrare, qualemque sequantur legem moduli in ordinem alterius ex altera transformatione derivatae transformationis perscrutari debemus, adeo ut, quantopere, perpetuo transformationibus repetitis, crescant vel decrescant, strenue iudicari possit. Praemittamus vero plures utriusque transformationis modulorum relationes, quae facillime ex antecedentibus emergunt nec non ad finem propositum perducent. Series in laeva parte paginae, ab serie in dextera eo deducitur, ut ponamus loco ipsorum:

$$c,\ l,\ m,\ \mathfrak{l},\ \mathfrak{k} \text{ etc.}$$
$$m_1,\ l_1,\ c_1,\ L_1,\ M_1 \text{ etc.}$$

$$c^0 = \frac{m_1 - c_1 l_1}{m_1 + c_1 l_1}, \qquad m_1' = \frac{c - lm}{c + lm},$$

$$l^0 = \left(\frac{lc}{1+m}\right)\left(\frac{m_1 + \mathfrak{l}\mathfrak{k}}{m_1 + c_1 l_1}\right) = \left(\frac{1-m}{lc}\right)\left(\frac{m_1 - c_1 l_1}{m_1 - \mathfrak{k}_1 \mathfrak{l}_1}\right), \qquad l_1' = \left(\frac{l_1 m_1}{1+c}\right)\left(\frac{c + LM}{c + lm}\right) = \left(\frac{1-c}{l_1 m_1}\right)\left(\frac{c - lm}{c - LM}\right),$$

$$m^0 = \left(\frac{lc}{1+m}\right)\left(\frac{m_1 - \mathfrak{k}_1 \mathfrak{l}_1}{m_1 + c_1 l_1}\right) = \left(\frac{1-m_1}{lc}\right)\left(\frac{m_1 - c_1 l}{m_1 + \mathfrak{k}_1 \mathfrak{l}_1}\right), \qquad c_1' = \left(\frac{l_1 m_1}{1+c}\right)\left(\frac{c - LM}{c + lm}\right) = \left(\frac{1-c}{l_1 m_1}\right)\left(\frac{c - lm}{c + LM}\right),$$

$$\mathfrak{l}^0 = \left(\frac{lc}{1+m_1}\right)\left(\frac{m_1 - \mathfrak{l}_1 \mathfrak{k}_1}{m_1 - l_1 c_1}\right) = \left(\frac{1-m_1}{lc}\right)\left(\frac{m_1 + c_1 l_1}{m_1 + \mathfrak{k}_1 \mathfrak{l}_1}\right), \qquad L_1' = \left(\frac{l_1 m_1}{1+c}\right)\left(\frac{c - LM}{c - lm}\right) = \left(\frac{1-c}{l_1 m_1}\right)\left(\frac{c + lm}{c + LM}\right),$$

$$\mathfrak{k}^0 = \left(\frac{m_1 - \mathfrak{k}_1 \mathfrak{l}_1}{m_1 + \mathfrak{k}_1 \mathfrak{l}_1}\right), \qquad M_1' = \frac{c - LM}{c + LM},$$

6. $\frac{l^0 m^0}{c^0} = \frac{1-m_1}{1+m_1} = \frac{\mathfrak{l}^0 m^0}{\mathfrak{k}^0}$, $\frac{l'_1 c'_1}{m'_1} = \frac{1-c}{1+c} = \frac{L'_1 c'_1}{M'_1}$,

7. $l^0 m^0 = \left(\frac{1-m_1}{1+m_1}\right)\left(\frac{m_1-c_1 l_1}{m_1+c_1 l_1}\right)$, $l'_1 c'_1 = \left(\frac{1-c}{1+c}\right)\left(\frac{c-lm}{c+lm}\right)$,

8. $\mathfrak{l}^0 m^0 = \left(\frac{1-m_1}{1+m_1}\right)\left(\frac{m_1-\mathfrak{k}_1\mathfrak{l}_1}{m_1+\mathfrak{k}_1\mathfrak{l}_1}\right)$, $L'_1 c'_1 = \left(\frac{1-c}{1+c}\right)\left(\frac{c-LM}{c+LM}\right)$,

9. $\frac{1-l^0m^0}{1+l^0m^0} = \frac{m_1^2+c_1l_1}{m_1(1+c_1l_1)} = \frac{\mathfrak{k}^0-m^{02}}{\mathfrak{k}^0+m^{02}}$, $\frac{1-l'_1c'_1}{1+l'_1c'_1} = \frac{c^2+lm}{c(1+lm)} = \frac{M'_1-c'^2_1}{M'_1+c'^2_1}$,

10. $\frac{1-\mathfrak{l}^0 m^0}{1+\mathfrak{l}^0m^0} = \frac{m_1^2+\mathfrak{k}_1\mathfrak{l}_1}{m_1(1+\mathfrak{k}_1\mathfrak{l}_1)} = \frac{c^0-m^{02}}{c^0+m^{02}}$, $\frac{1-L'_1c'_1}{1+L'_1c'_1} = \frac{c^2+LM}{c(1+LM)} = \frac{m'_1-c'^2_1}{m'_1+c'^2_1}$,

11. $\frac{c^{02}-l^0m^0}{c^{02}+l^0m^0} = \frac{m_1^2-c_1l_1}{m_1(1-c_1l_1)} = \frac{\mathfrak{k}^0-\mathfrak{l}^{02}}{\mathfrak{k}^0+\mathfrak{l}^{02}}$, $\frac{m'^2_1-c'_1l'_1}{m'^2+c'_1l'_1} = \frac{c^2-lm}{c(1-lm)} = \frac{M'_1-L'^2_1}{M'_1+L'^2_1}$,

12. $\frac{\mathfrak{k}^{02}-\mathfrak{l}^0m^0}{\mathfrak{k}^{02}+\mathfrak{l}^0m^0} = \frac{m_1^2-\mathfrak{k}_1\mathfrak{l}_1}{m_1(1-\mathfrak{k}_1\mathfrak{l})} = \frac{c^0-l^{02}}{c^0+l^{02}}$, $\frac{M'^2_1-c'_1L'_1}{M'^2_1+c'_1L'_1} = \frac{c^2-LM}{c(1-LM)} = \frac{m'_1-l'^2_1}{m'_1+l'^2_1}$,

13. $c^0_1 = \frac{2\sqrt{(m,l,c)}}{m_1+c_1l_1}$, $m' = \frac{2\sqrt{(clm)}}{(c+lm)}$,

14. $\mathfrak{k}^0_1 = \frac{2\sqrt{(m,\mathfrak{l},\mathfrak{k})}}{(m_1+\mathfrak{l}_1\mathfrak{k})}$, $M' = \frac{2\sqrt{(cLM)}}{(c+LM)}$,

15. $l^{02}_1 = \frac{2m_1(c_1+l_1)}{(1+m_1)(m_1+c_1l_1)}\sqrt{\left[\left(\frac{m_1+\mathfrak{k}_1\mathfrak{l}_1}{m_1-\mathfrak{k}_1\mathfrak{l}_1}\right)\left(\frac{(m_1^2+c_1 l)-(1+c_1l_1)\mathfrak{k}_1\mathfrak{l}_1}{(m_1^2+c_1l_1)+(1+c_1l_1)\mathfrak{k}_1\mathfrak{l}_1}\right)\right]}$,

$$l'^2 = \frac{2c(m+l)}{(1+c)(c+lm)}\sqrt{\left[\left(\frac{c+LM}{c-LM}\right)\left(\frac{(c^2+lm)-(1+lm)LM}{(c^2+lm)+(1+lm)LM}\right)\right]},$$

16. $\mathfrak{l}^{02}_1 = \frac{2m_1(\mathfrak{k}_1+\mathfrak{l}_1)}{(1+m)(m_1+\mathfrak{k}_1\mathfrak{l}_1)}\sqrt{\left[\left(\frac{m_1+c_1l_1}{m_1-c_1l_1}\right)\left(\frac{(m_1^2+\mathfrak{k}_1\mathfrak{l})-(1+\mathfrak{k}_1\mathfrak{l}_1)c_1l_1}{(m_1^2+\mathfrak{k}_1\mathfrak{l}_1)+(1+\mathfrak{k}_1\mathfrak{l}_1)c_1l_1}\right)\right]}$,

$$L'^2 = \frac{2c(M+L)}{(1+c)(c+ML)}\sqrt{\left[\left(\frac{c+lm}{c-lm}\right)\left(\frac{(c^2+LM)-(1+LM)lm}{(c^2+LM)+(1+LM)lm}\right)\right.},$$

17. $m^{02}_1 = \frac{2m_1(c_1+l_1)}{(1+m_1)(m_1+c_1l_1)}\sqrt{\left[\left(\frac{m_1-\mathfrak{k}_1\mathfrak{l}_1}{m_1+\mathfrak{k}_1\mathfrak{l}}\right)\left(\frac{(m_1^2+c_1l_1)+(1+c_1l_1)\mathfrak{k}_1\mathfrak{l}_1}{(m_1^2+c_1l_1)-(1+c_1l_1)\mathfrak{k}_1\mathfrak{l}_1}\right)\right]}$,

$$c'^2 = \frac{2c(m+l)}{(1+c)(c+lm)}\sqrt{\left[\left(\frac{c-LM}{c+LM}\right)\left(\frac{(c^2+lm)+(1+lm)LM}{(c^2+lm)-(1+lm)LM}\right)\right.},$$

18. $m^0_1 l^0_1 = \frac{2m_1(c_1+l_1)}{(1+m_1)(m_1+c_1l_1)}$, $c'l' = \frac{2c(l+m)}{(1+c)(c+lm)}$,

19. $\frac{m^0_1}{\left(\frac{c^0_1}{l^0_1}\right)} = \frac{\left(\frac{2\sqrt{m_1}}{1+m_1}\right)}{\left(\frac{2\sqrt{\frac{c_1}{l_1}}}{1+\frac{c_1}{l_1}}\right)}$, $\frac{c'}{\left(\frac{m'}{l'}\right)} = \frac{\left(\frac{2\sqrt{c}}{1+c}\right)}{\left(\frac{2\sqrt{\frac{m}{l}}}{1+\frac{m}{l}}\right)}$,

20. $m^0_1\mathfrak{l}^0_1 = \frac{2m_1(\mathfrak{k}_1+\mathfrak{l}_1)}{(1+m_1)(m_1+\mathfrak{k}_1\mathfrak{l}_1)}$, $c'L' = \frac{2c(L+M)}{(1+c)(c+LM)}$,

$$21.\quad \frac{m_1^0}{\left(\frac{\mathfrak{k}_1^0}{\mathfrak{l}_1^0}\right)} = \frac{\left(\frac{2\sqrt{m_1}}{1+m_1}\right)}{\left(\frac{2\sqrt{\frac{\mathfrak{k}_1}{\mathfrak{l}_1}}}{1+\frac{\mathfrak{k}_1}{\mathfrak{l}_1}}\right)}, \qquad \frac{c'}{\left(\frac{M'}{L'}\right)} = \frac{\left(\frac{2\sqrt{c}}{1+c}\right)}{\left(\frac{2\sqrt{\frac{M}{L}}}{1+\frac{M}{L}}\right)},$$

$$22.\quad \frac{c_1^0}{l_1^0} : \frac{\mathfrak{k}_1^0}{\mathfrak{l}_1^0} = \frac{2\sqrt{\frac{c_1}{l_1}}}{\left(1+\frac{c_1}{l_1}\right)} : \frac{2\sqrt{\frac{\mathfrak{k}_1}{\mathfrak{l}_1}}}{\left(1+\frac{\mathfrak{k}_1}{\mathfrak{l}_1}\right)}, \qquad \frac{m'}{l'} : \frac{M'}{L'} = \frac{2\sqrt{\frac{m}{l}}}{\left(1+\frac{m}{l}\right)} : \frac{2\sqrt{\frac{M}{L}}}{\left(1+\frac{M}{L}\right)},$$

$$23.\quad \frac{l_1^0}{\mathfrak{l}_1^0} = \left(\frac{c_1+l_1}{\mathfrak{k}_1+\mathfrak{l}_1}\right)\left(\frac{m_1+\mathfrak{k}_1\mathfrak{l}_1}{m_1+c_1 l_1}\right), \qquad \frac{l'}{L'} = \left(\frac{l+m}{L+M}\right)\left(\frac{c+LM}{c+lm}\right),$$

$$24.\quad \sqrt{(c^{02}-l^{02})}\,\mathfrak{l}_1^0 = \frac{2m_1(l_1-c_1)}{(1+m_1)(m_1+c_1 l_1)}, \qquad \sqrt{(m_1'^2-l_1'^2)}\,L_1' = \frac{2c(l-m)}{(1+c)(c+lm)},$$

$$25.\quad \sqrt{(\mathfrak{k}^{02}-\mathfrak{l}^{02})}\,\mathfrak{l}_1^0 = \frac{2m_1(\mathfrak{l}_1-\mathfrak{k}_1)}{(1+m_1)(m_1+\mathfrak{k}_1\mathfrak{l}_1)}, \qquad \sqrt{(M_1'^2-L_1'^2)}\,l_1' = \frac{2c(L-M)}{(1+c)(c+LM)},$$

$$26.\quad \frac{\sqrt{(c^{02}-l^{02})}}{m_1^0}\left(\frac{\mathfrak{l}_1^0}{l_1^0}\right) = \left(\frac{l_1-c_1}{l_1+c_1}\right), \qquad \frac{\sqrt{(m_1'^2-l_1'^2)}}{c'}\cdot\frac{L'}{l'} = \left(\frac{l-m}{l+m}\right),$$

$$27.\quad \left(\frac{\sqrt{(\mathfrak{k}^{02}-\mathfrak{l}^{02})}}{m_1^0}\right)\left(\frac{l_1^0}{\mathfrak{l}_1^0}\right) = \left(\frac{\mathfrak{l}_1-\mathfrak{k}_1}{\mathfrak{l}_1+\mathfrak{k}_1}\right), \qquad \frac{\sqrt{(M_1'^2-L_1'^2)}}{c'}\cdot\frac{l'}{L'} = \left(\frac{L-M}{L+M}\right).$$

Quibus relationibus expositis, ad gradum approximationis modulorum ad nihil vel ad unitatem diiudicandum aggrediamur. Primum clarum est memorabile hoc

theorema:

„Moduli

$$„28.\quad \begin{cases} c, & c^0, & c^{00}, & c^{000} & \text{etc.} \\ \mathfrak{k}, & \mathfrak{k}^0, & \mathfrak{k}^{00}, & \mathfrak{k}^{000} & \text{etc.} \end{cases}$$

„tam rapide ad nihil convergunt, ut alius a quadrato alius antecedentis „semper superetur, sive semper erit:

$$„c^2 > c^0, \quad c^{02} > c^{00} \quad \text{etc.}$$
$$„\mathfrak{k}^2 > \mathfrak{k}^0, \quad \mathfrak{k}^{02} > \mathfrak{k}^{00} \quad \text{etc.“}$$

Demonstratio.

Habemus relationes per se claras:

$$l_1 > c_1, \quad 1+c^2 > m_1, \quad \mathfrak{l}_1 > \mathfrak{k}_1, \quad 1+\mathfrak{k}^2 > m_1,$$

unde sequuntur hae:

$$l_1(1+c^2) > c_1 m_1, \qquad \mathfrak{l}_1(1+\mathfrak{k}^2) > \mathfrak{k}_1 m_1,$$

sive:

$$\frac{l_1 c_1}{m_1} > \frac{1-c^2}{1+c^2}, \qquad \frac{\mathfrak{l}_1\mathfrak{k}_1}{m_1} > \frac{1-\mathfrak{k}^2}{1+\mathfrak{k}^2},$$

unde sequitur, fore:

$$\frac{m_1 - l_1 c_1}{m_1 + l_1 c_1} = c^0 < c^2, \qquad \frac{m_1 - \mathfrak{l}_1 \mathfrak{k}_1}{m_1 + \mathfrak{l}_1 \mathfrak{k}_1} = \mathfrak{k}^0 < \mathfrak{k}^2;$$

quae demonstratio ad omnes sequentes modulos extendi potest. Prorsus simili ratione demonstratur modulorum complementa in altera transformatione eandem legem sequi. Nimirum habemus

theorema.

„Modulorum complementa

$$29. \quad \begin{cases} m_1, & m'_1, & m''_1, & m'''_1 & \text{etc.} \\ M_1, & M'_1, & M''_1, & M'''_1 & \text{etc.} \end{cases}$$

„decrescentem seriem constituunt, cuius terminus quisque ab quadrato ter- „mini antecedentis superatur, sive semper erit:

$$„m_1^2 > m'_1, \qquad m'^2_1 > m''_1, \quad \text{etc.}$$
$$„M_1^2 > M'_1, \qquad M'^2_1 > M''_1, \quad \text{etc.''}$$

Iam vero adiiciamus alterum de modulis c et C

theorema.

„Semper erit:

$$„30. \quad \begin{cases} c^0 > \frac{1-c_1}{1+c_1}, & c^{00} > \frac{1-c^0_1}{1+c^0_1}, \quad \text{etc.} \\ \mathfrak{k}^0 > \frac{1-\mathfrak{k}_1}{1+\mathfrak{k}_1}, & \mathfrak{k}^{00} > \frac{1-\mathfrak{k}^0}{1+\mathfrak{k}^0}, \quad \text{etc.} \end{cases}$$

„id quod eo demonstratur, ut ex aequat. per se claris:

$$„\frac{m_1}{c_1 l_1} > \frac{1}{c_1}, \qquad \frac{m_1}{\mathfrak{k}_1 \mathfrak{l}_1} > \frac{1}{\mathfrak{k}},$$

„deducamus has:

$$„\frac{m_1 - c_1 l_1}{m_1 + c_1 l_1} > \frac{1-c_1}{1+c_1}, \qquad \frac{m_1 - \mathfrak{k}_1 \mathfrak{l}_1}{m_1 + \mathfrak{k}_1 \mathfrak{l}_1} > \frac{1-\mathfrak{k}_1}{1+\mathfrak{k}_1}.$$

Simile theorema in altera transformatione erit:

$$„31. \quad \begin{cases} m'_1 > \frac{1-m}{1+m}, & m''_1 > \frac{1-m'}{1+m'}, \quad \text{etc.} \\ M'_1 > \frac{1-M}{1+M}, & M''_1 > \frac{1-M'}{1+M'}, \quad \text{etc.''} \end{cases}$$

His quatuor theorematibus collatis, quatuor series modulorum complementorumque composuimus:

$$32.\quad \left\{\begin{array}{lllll}
1\ldots c'' < c'''^2, & c' < c''^2, & c < c'^2, & c^0 < c^2, & c^{00} < c^{02} \ldots\ldots 0,\\
1\ldots c'' > \frac{1-c_1'''}{1+c_1'''}, & c' > \frac{1-c_1''}{1+c_1''}, & c > \frac{1-c_1'}{1+c_1'}, & c^0 > \frac{1-c_1}{1+c_1}, & c^{00} > \frac{1-c_1^0}{1+c_1^0} \ldots 0,\\
1\ldots \mathfrak{k}'' < \mathfrak{k}'''^2, & \mathfrak{k}' < \mathfrak{k}''^2, & \mathfrak{k} < \mathfrak{k}'^2, & \mathfrak{k}^0 < \mathfrak{k}^2, & \mathfrak{k}^{00} < \mathfrak{k}^{02} \ldots\ldots 0,\\
1\ldots \mathfrak{k}'' > \frac{1-\mathfrak{k}_1'''}{1+\mathfrak{k}_1'''}, & \mathfrak{k}' > \frac{1-\mathfrak{k}_1''}{1+\mathfrak{k}_1''}, & \mathfrak{k} > \frac{1-\mathfrak{k}_1'}{1+\mathfrak{k}_1'}, & \mathfrak{k}^0 > \frac{1-\mathfrak{k}_1}{1+\mathfrak{k}_1}, & \mathfrak{k}^{00} > \frac{1-\mathfrak{k}_1^0}{1+\mathfrak{k}_1^0} \ldots 0,\\
1\ldots m_1^{00} < m_1^{0002}, & m_1^0 < m_1^{002}, & m_1 < m_1^{02}, & m_1' < m_1^2, & m_1'' < m_1'^2 \ldots\ldots 0,\\
1\ldots m_1^{00} > \frac{1-m^{000}}{1+m^{000}}, & m_1^0 > \frac{1-m^{00}}{1+m^{00}}, & m_1 > \frac{1-m^0}{1+m^0}, & m_1' > \frac{1-m}{1+m}, & m_1'' > \frac{1-m'}{1+m'} \ldots 0,\\
1\ldots M_1^{00} < M_1^{0002}, & M_1^0 < M_1^{002}, & M_1 < M_1^{02}, & M_1' < M_1^2, & M_1'' < M_1'^2 \ldots\ldots 0,\\
1\ldots M_1^{00} > \frac{1-M^{000}}{1+M^{000}}, & M_1^0 > \frac{1-M^{00}}{1+M^{00}}, & M_1 > \frac{1-M^0}{1+M^0}, & M_1' > \frac{1-M}{1+M}, & M_1'' > \frac{1-M'}{1+M'} \ldots 0.
\end{array}\right.$$

Haec tabula docet, quam rapide moduli c et $\mathfrak{k}$ in altera et complementa m_1, M_1 in altera transformatione ad nihil appropinquent. Exempli gratia posito: $c^2 = 0{,}5$, habebimus hanc seriem:

$c^2 = 0{,}5$, $c^{02} < 0{,}25$, $c^{002} < 0{,}1025$, $c^{0002} < 0{,}0105063$, $c^{00002} < 0{,}0001104$, ita ut quantitates c^{000002} in calculo usque ad sex nonnisi decimales appropinquato prorsus negligi possit.

Transeamus vero ad modulos l^0 et $\mathfrak{l}^0$ in altera cum modulis c et $\mathfrak{k}$ comparandos, dum in altera transformatione complementa l_1' et L_1' cum complementis m_1, M_1 comparare velimus. Primum formulae priores (24.) et (25.) docent, semper fore

$$c^0 > l^0 \quad \text{et} \quad \mathfrak{k}^0 > \mathfrak{l}^0.$$

Inde vero sequitur, quantitates l^0 et $\mathfrak{l}^0$ semper respective minores esse, quam quadrata primorum modulorum antecedentium c et $\mathfrak{k}$, ita ut his ad nihil convergentibus, ipsae ad nihil appropinquent. Habemus igitur has series modulorum:

$$33.\quad \left\{\begin{array}{l}
1\ldots l'' < c'''^2,\quad l' < c''^2,\quad l < c'^2,\quad l^0 < c^2,\quad l^{00} < c^{02},\ \ldots\ 0,\\
1\ldots \mathfrak{l}'' < \mathfrak{k}'''^2,\quad \mathfrak{l}' < \mathfrak{k}''^2,\quad \mathfrak{l} < \mathfrak{k}'^2,\quad \mathfrak{l}^0 < \mathfrak{k}^2,\quad \mathfrak{l}^{00} < \mathfrak{k}^{02},\ \ldots\ 0,
\end{array}\right.$$

In altera vero transformatione per formulas posteriores (24.) et (25.) demonstratur, fore:

$$l_1' < m_1',\qquad L_1' < M_1'$$

ergo

$$l_1' < m_1^2,\qquad L_1' < M_1^2,$$

ita ut complementa l_1 et L_1 simul respective cum complementis m_1 et M_1 ad nihil appropinquent. Habemus his has series:

$$34.\quad \left\{\begin{array}{l}
1\ldots l_1^{00} < m_1^{0002},\quad l_1^0 < m_1^{002},\quad l_1 < m_1^{02},\quad l_1' < m_1^2,\quad l_1'' < m_1'^2 \ldots 0,\\
1\ldots L_1^{00} < M_1^{0002},\quad L_1^0 < M_1^{002},\quad L_1 < M_1^{02},\quad L_1' < M_1^2,\quad L_1'' < M_1'^2 \ldots 0.
\end{array}\right.$$

Iam vero etiam placet, modulos l, l^0 etc., $\mathfrak{l}$, $\mathfrak{l}^0$ etc. atque complementa l, l'_1 etc., L, L'_1 etc. inter se ipsa comparare, quamvis lex hîc regnans non tam concinna, quam antea allatae, fiat. Habuimus formulam (2.):

$$l^0 = \left(\frac{lc}{1+m_1}\right)\left(\frac{m_1+\mathfrak{k}_1\mathfrak{l}_1}{m_1+c_1 l_1}\right),$$

unde sequitur:

$$\frac{l^0}{l} = \left(\frac{c}{1+m_1}\right)\left(\frac{m_1+\mathfrak{k}_1\mathfrak{l}_1}{m_1+c_1 l_1}\right).$$

Cum vero sit: $\mathfrak{k}_1 < \mathfrak{l}_1$, et $c_1 < l_1$, habemus:

$$\frac{c}{1+m_1}\cdot\frac{m_1+\mathfrak{k}_1\mathfrak{l}_1}{m_1+c_1 l_1} < \frac{c}{1+m_1}\cdot\frac{m_1+\mathfrak{l}_1^2}{m_1+c_1^2}.$$

Valore vero ipsius $\mathfrak{l}_1^2 = \frac{m_1^2-c_1^2}{c^2}$ substituto, erit:

$$\frac{l^0}{l} = \left(\frac{c}{1+m_1}\right)\left(\frac{m_1+\mathfrak{k}_1\mathfrak{l}_1}{m_1+c_1 l_1}\right) < \left(\frac{1}{c}\right)\left(\frac{m_1-c_1^2}{m_1+c_1^2}\right).$$

Cum denique sit:

$$(1+c)^2 > m_1,$$

sive

$$\frac{1-c}{1+c} < \frac{c_1^2}{m_1},$$

etiam erit:

$$c > \frac{m_1-c_1^2}{m_1+c_1^2},$$

unde sequitur fore:

$$\frac{1}{c}\cdot\frac{m_1-c_1^2}{m_1+c_1^2} < 1,$$

atque hanc ob rem:

$$l^0 < l.$$

Eodem modo formula (4.):

$$\mathfrak{l}^0 = \frac{1-m}{lc}\cdot\frac{m_1+c_1 l_1}{m_1+\mathfrak{k}_1\mathfrak{l}_1},$$

loco ipsius lc introducto valore aequivalenti $\frac{m^2}{\mathfrak{k}\mathfrak{l}}$, hanc suppeditat:

$$\frac{\mathfrak{l}^0}{\mathfrak{l}} = \left(\frac{\mathfrak{k}}{1+m_1}\right)\left(\frac{m_1+c_1 l_1}{m_1+\mathfrak{k}_1\mathfrak{l}_1}\right).$$

Iam vero erit:

$$\frac{\mathfrak{l}^0}{\mathfrak{l}} = \left(\frac{\mathfrak{k}}{1+m_1}\right)\left(\frac{m_1+c_1 l_1}{m_1+\mathfrak{k}_1\mathfrak{l}_1}\right) < \left(\frac{\mathfrak{k}}{1+m_1}\right)\left(\frac{m_1+l_1^2}{m_1+\mathfrak{k}_1^2}\right) = \frac{1}{\mathfrak{k}}\left(\frac{m_1-\mathfrak{k}_1^2}{m_1+\mathfrak{k}_1^2}\right).$$

Habemus rursus:

$$(1+\mathfrak{k})^2 > m_1,$$

sive:

$$\mathfrak{k} > \frac{m_1-\mathfrak{k}_1^2}{m_1+\mathfrak{k}_1^2},$$

unde sequitur fore:

$$\mathfrak{l}^0 < \mathfrak{l}.$$

Inde adhuc demonstratum est theorema hoc:

„Moduli

„35. $l,\ l^0,\ l^{00},\ l^{000}\ \dots\dots$ atque $\mathfrak{l},\ \mathfrak{l}^0,\ \mathfrak{l}^{00},\ \mathfrak{l}^{000}\ \dots\dots$

„iam ab initio seriem decrescentem constituunt, cuius terminus quisque a „quadrato moduli primi cohaerentis antecedentis superatur."

Simile theorema in altera transformatione:

„complementa modulorum secundorum:

$$„36.\quad \begin{cases} l_1, & l'_1, & l''_1, & l'''_1, & \text{etc.} \\ L, & L'_1, & L''_1, & L'''_1, & \text{etc.} \end{cases}$$

„ita ab initio iam decrescunt, ut quisque a quadrato complementi moduli „tertii cohaerentis superetur, eodem modo ex formulis posterioribus (2.) „et (4.) deducitur." Iam vero denique modulos minimos m, m^0 etc. in altera, et complementa minima c_1, c'_1 etc. in altera transformatione contemplemur. Habemus formulas ex form. (1.), (3.) et (5.) deductas:

$$m^{02} = \frac{1-m_1}{1+m_1}\cdot c^0\,\mathfrak{k}^0, \qquad c'^2_1 = \frac{1-c}{1+c}\cdot m'_1\,M'_1,$$

unde sequitur, cum sit:

$$\frac{1-m_1}{1+m_1} < m^2, \qquad c^0 < c^2, \qquad \mathfrak{k}^0 < \mathfrak{k}^2,$$

$$\frac{1-c}{1+c} < c_1^2, \qquad m'_1 < m_1^2, \qquad M'_1 < M_1^2,$$

fore:

$$m^0 < m.c.\mathfrak{k}, \qquad c'_1 < c_1.m_1 M_1$$

ita ut „moduli

$$„m,\quad m^0,\quad m^{00},\quad \dots\dots$$

„atque complementa

$$„37.\quad c_1,\quad c'_1,\quad c''_1,\quad \dots\dots$$

„seriem decrescentem constituant, eoque magis ad 0 convergentem, quo „minora producta:

$$„c\,\mathfrak{k},\qquad c^0\,\mathfrak{k}^0,\qquad c^{00}\,\mathfrak{k}^{00},\quad \text{etc.}$$

$$„m_1 M_1,\qquad m'_1 M'_1,\qquad m''_1 M''_1,\quad \text{etc.}$$

„fiant." Iam vero haec appropinquatio ad nihil multo arctius determinatur, sequenti consideratione adhibita. Habemus ex formulis (2.) et (3.):

$$m^0 = \left(\frac{l^0}{c^0}\right) c^0\,\mathfrak{k}^0, \qquad c'_1 = \left(\frac{l'_1}{m'_1}\right) m'_1\,M'_1,$$

unde, quia theoremata (28.) et (29.) ostendunt, fore:

$$c^0\,\mathfrak{k}^0 < c^2\,\mathfrak{k}^2, \qquad m'_1\,M'_1 < m_1^2\,M_1^2,$$

atque ex theorematibus (33.) et (34.) sequitur:

$$\frac{l^0}{c^0} < 1, \qquad \frac{l'_1}{m'_1} < 1,$$

prodeunt relationes:

$$38. \quad m^0 < c^2 \mathfrak{k}^2, \qquad c'_1 < m_1^2 M_1^2,$$

et haec theoremata:

„Moduli

$$„39. \quad m, \quad m^0, \quad m^{00},$$

„in altera transformatione repetita ita decrescunt, ut quisque ab quadrato „producti utriusque moduli c, $\mathfrak{k}$, antecedentis superetur, atque eodem modo „in altera transformatione

„Complementa

$$„40. \quad c_1, \quad c'_1, \quad c''_1,$$

„ita decrescunt, ut quisque a quadrato producti utriusque moduli tertii „m, M antecedentis superetur."

Iam vero generalem adhuc legem quinternorum modulorum ad nihil appropinquantium inde deducimus, ut animadvertamus, dum fractiones:

$$\frac{m}{c} = \mathfrak{l}, \quad \frac{m}{l} = \mathfrak{k}, \quad \frac{m}{\mathfrak{k}} = l, \quad \frac{m}{\mathfrak{l}} = c,$$

ad nihil convergant, fractionem:

$$\frac{l}{c} = \frac{\mathfrak{l}}{\mathfrak{k}}$$

mox ad unitatem appropinquaturam esse.

Similemque observationem de fractione $\frac{l_1}{m_1} = \frac{L_1}{M_1}$ faciamus. Habemus nimirum formulas sponte prodeuntes

$$\frac{l}{c} = \frac{\mathfrak{l}}{\mathfrak{k}}, \qquad \frac{l^0}{c^0} = \frac{\mathfrak{l}^0}{\mathfrak{k}^0}, \quad \text{etc.}$$

nec non ex formulis (26.) et (27.) sequuntur hae:

$$\sqrt{(c^{02} - l^{02})} = \frac{m_1^0 l_1^0}{\mathfrak{l}_1^0} \cdot \frac{c^2 - l^2}{(l_1 + c_1)^2}, \qquad \sqrt{(\mathfrak{k}^{02} - \mathfrak{l}^{02})} = \frac{m_1^0 \mathfrak{l}_1^0}{l_1^0} \cdot \frac{\mathfrak{k}^2 - \mathfrak{l}^2}{(\mathfrak{l}_1 + \mathfrak{k}_1)^2},$$

quibus inter se multiplicatis, post faciles reductiones nanciscimur hanc formulam:

$$\sqrt{\left(1 - \frac{l^{02}}{c^{02}}\right)} = \sqrt{\left(\frac{c^2 \mathfrak{k}^2}{c^0 \mathfrak{k}^0}\right)\left(\frac{m_1^0}{(l_1 + c_1)(\mathfrak{l}_1 + \mathfrak{k}_1)}\right)\left(1 - \frac{l^2}{c^2}\right)} = \sqrt{\left(1 - \frac{\mathfrak{l}^0}{\mathfrak{k}^0}\right)}.$$

Iam vero ex form. (32.) sequitur, semper fore:

$$c^0 > \frac{1 - c_1}{1 + c_1}, \qquad \mathfrak{k}^0 > \frac{1 - \mathfrak{k}_1}{1 + \mathfrak{k}_1},$$

quibus valoribus substitutis erit:

$$\sqrt{\left(1-\frac{l^{0\,2}}{c^{0\,2}}\right)} < \frac{(1+c_1)(1+\mathfrak{k}_1)}{(l_1+c_1)(\mathfrak{l}_1+\mathfrak{k}_1)}\cdot m_1^0\left(1-\frac{l^2}{c^2}\right).$$

Unde colligere possumus, quia, complementis c_1, $\mathfrak{k}_1$, l_1, $\mathfrak{l}_1$, m_1^0 ad unitatem appropinquantibus, factor $\frac{(1+c_1)(1+\mathfrak{k}_1)}{(l_1+c_1)(\mathfrak{l}_1+\mathfrak{k}_1)}m_1^0$ ipse ad unitatem accedit, quantitatem $\sqrt{\left(1-\frac{l^{0\,2}}{c^{0\,2}}\right)}$ per primam transformationem denique ordinis $\left(1-\frac{l^2}{c^2}\right)$ fieri, itaque postremo rapide usque ad nihil decrescere.

Eodem modo in altera transformatione complementa m_1' et l_1' similem legem sequi, ex formulis his demonstratur. Habemus:

$$\frac{l_1}{m_1} = \frac{L_1}{M_1}, \qquad \frac{l_1'}{m_1'} = \frac{L_1'}{M_1'}, \quad \text{etc.}$$

atque ex form. (26.) et (27.)

$$\sqrt{(m_1'^2-l_1'^2)} = \frac{c'l'}{L'}\cdot\frac{(m_1^2-l_1^2)}{(l+m)^2}, \qquad \sqrt{(M_1^2-L_1'^2)} = \frac{c'L'}{l'}\cdot\frac{(M_1^2-L_1^2)}{(L+M)^2},$$

quibus aequationibus coniunctis, erit:

$$\sqrt{\left(1-\frac{l_1'^2}{m_1'^2}\right)} = \sqrt{\left(\frac{m_1^2 M_1^2}{m_1' M_1'}\right)}\left(\frac{c'}{(l+m)(L+M)}\right)\left(1-\frac{l_1^2}{m_1^2}\right) = \sqrt{\left(1-\frac{L_1'^2}{M_1'^2}\right)}.$$

Iam vero antea habuimus relationes:

$$m_1' > \frac{1-m}{1+m}, \qquad M_1' > \frac{1-M}{1+M},$$

unde sequitur fore:

$$\sqrt{\left(1-\frac{l_1'^2}{m_1'^2}\right)} < \left(\frac{(1+m)(1+M)}{(l+m)(L+M)}\right)c'.\left(1-\frac{l_1^2}{m_1^2}\right);$$

quae relatio, cum moduli m, M, l, L, c in hac transformatione ad unitatem appropinquent, docet denique quantitatem:

$$\sqrt{\left(1-\frac{l_1'^2}{m_1'^2}\right)} = \sqrt{\left(1-\frac{L_1'^2}{M_1'^2}\right)},$$

fieri ordinis:

$$\left(1-\frac{l_1^2}{m_1^2}\right) = \left(1-\frac{L_1^2}{M_1^2}\right).$$

Itaque demonstravimus hoc **theorema**:

„fractiones

$$„41. \quad \frac{l}{c}, \quad \frac{l^0}{c^0}, \quad \frac{l^{00}}{c^{00}}$$

„et

$$„42. \quad \frac{l_1}{m_1}, \quad \frac{l_1'}{m_1'}, \quad \frac{l_1''}{m_1''}$$

„dum illic moduli:

$$„m, \quad m^0, \quad m^{00}, \quad \ldots.$$

„et hîc complementa:

$$„c_1, \quad c'_1, \quad c''_1, \quad \dots .$$

„ad nihil accedunt, mox ad unitatem appropinquant."

Inde ex hoc theoremate de modulis minimis adhuc, atque de complementis minimis in altera transformatione, memorabile deducere licet hoc

theorema:

„Moduli

$$„43. \quad m, \quad m^0, \quad m^{00}, \quad \dots .$$

„quos per primam transformationem repetitam adipiscimur, ita denique „semper decrescunt, ut quisque a quadrato antecedentis superetur, eam„que legem observant complementa:

$$„44. \quad c_1, \quad c'_1, \quad c''_1, \quad \text{etc.}$$

„in altera transformatione repetita."

Demonstratio.

Ex formulis (6.) hae sponte sequuntur:

$$m^0 = m^2 \frac{c^0}{l^0} \cdot \frac{1}{(1+m_1)^2}, \qquad c'_1 = c_1^2 \frac{m'_1}{l'_1} \cdot \frac{1}{(1+c)^2}.$$

Iam vero secundum theorema antecedens quantitates:

$$\frac{c^0}{l^0}, \quad \frac{c^{00}}{l^{00}}, \quad \dots . \qquad \frac{m'_1}{l'_1}, \quad \frac{m''_1}{m''_1}, \quad \dots .$$

ad unitatem appropinquant, dum igitur quantitatibus m_1 in altera, et c in altera ad 1 appropinquantibus fractiones

$$\frac{1}{(1+m_1)^2}, \quad \frac{1}{(1+m_1^0)^2}, \quad \text{etc.} \qquad \frac{1}{(1+c)^2}, \quad \frac{1}{(1+c')^2}, \quad \text{etc.}$$

ad valores $= \frac{1}{4}$ mox accedunt, sequitur, inde a termino quodam algorithmi, ubi primum $\frac{c^0}{l^0} \cdot \frac{1}{(1+m_1)^2}$ vel $\frac{m'_1}{l'_1} \cdot \frac{1}{(1+c)^2}$ unitate minor fiat, theorema quaesitum verum esse.

Adiiciamus adhuc duo theoremata de iisdem his modulis minimis m, m^0, ... atque de complementis c_1, c'_1, ... haud supervacanea, quae ex comparatione cum modulis, quos per transformationem secundi ordinis integralium ellipticorum adipiscimur, originem trahunt. Obtinemus enim, „fore in prima transformatione:

$$„45. \begin{cases} \frac{1-c_1}{1+c_1} > m^0, & \frac{1-c_1^0}{1+c_1^0} > m^{00}, & \frac{1-c^{00}}{1+c^{00}} > m^{000}, & \dots . \\ \frac{1-\mathfrak{k}_1}{1+\mathfrak{k}_1} > m^0, & \frac{1-\mathfrak{k}_1^0}{1+\mathfrak{k}_1^0} > m^{00}, & \frac{1-\mathfrak{k}^{00}}{1+\mathfrak{k}^{00}} > m^{000}, & \dots . \\ \frac{1-m_1}{1+m_1} < m^0, & \frac{1-m_1^0}{1+m_1^0} < m^{00}, & \frac{1-m_1^{00}}{1+m_1^{00}} < m^{000}, & \dots . \end{cases}$$

„sed fractiones:

$$„\frac{\frac{1-m_1}{1+m_1}}{m^0}, \qquad \frac{\frac{1-m_1^0}{1+m_1^0}}{m^{00}}, \qquad \text{etc.}$$

„denique ad unitatem quam proxime accedere, similiterque in altera trans-
„formatione fore:

$$„\frac{1-m}{1+m} > c_1', \quad \frac{1-m'}{1+m'} > c_1'', \quad \frac{1-m''}{1+m''} > c_1''', \quad \ldots\ldots$$

$$„\frac{1-M}{1+M} > c_1', \quad \frac{1-M'}{1+M'} > c_1'', \quad \frac{1-M_1''}{1+M_1''} > c_1''', \quad \ldots\ldots$$

$$„\frac{1-c}{1+c} < c_1', \quad \frac{1-c_1'}{1+c_1'} < c_1'', \quad \frac{1-c_1''}{1+c_1''} < c_1''', \quad \ldots\ldots$$

„atque fractiones:

$$„\frac{\frac{1-c}{1+c}}{c_1'}, \qquad \frac{\frac{1-c'}{1+c'}}{c_1''}, \qquad \text{etc.}$$

„rursus ad unitatem appropinquare."

Demonstratio.

Habemus relationes:

$$(m_1-c_1)(m_1-l_1) > 0, \qquad (m_1-\mathfrak{k}_1)(m_1-\mathfrak{l}_1) > 0$$

unde post faciles reductiones, sequitur:

$$\left(\frac{1-m_1}{1+m_1}\right)\left(\frac{m_1-c_1 l_1}{m_1+c_1 l_1}\right) < \left(\frac{1-c_1}{1+c_1}\right)\left(\frac{1-l_1}{1+l_1}\right), \quad \left(\frac{1-m_1}{1+m_1}\right)\left(\frac{m_1-\mathfrak{k}_1 \mathfrak{l}_1}{m_1+\mathfrak{k}_1 \mathfrak{l}_1}\right) < \left(\frac{1-\mathfrak{k}_1}{1+\mathfrak{k}_1}\right)\left(\frac{1-\mathfrak{l}_1}{1+\mathfrak{l}_1}\right).$$

Iam vero formulis (7.) et (8.):

$$m^0 l^0 = \left(\frac{1-m_1}{1+m_1}\right)\left(\frac{m_1-c_1 l_1}{m_1+c_1 l_1}\right), \qquad m^0 \mathfrak{l}^0 = \left(\frac{1-m_1}{1+m_1}\right)\left(\frac{m_1-\mathfrak{k}_1 \mathfrak{l}_1}{m_1+\mathfrak{k}_1 \mathfrak{l}_1}\right),$$

adhibitis erit:

$$m^0 l^0 < \left(\frac{1-c_1}{1+c_1}\right)\left(\frac{1-l_1}{1+l_1}\right), \qquad m^0 \mathfrak{l}^0 < \left(\frac{1-\mathfrak{k}_1}{1+\mathfrak{k}_1}\right)\left(\frac{1-\mathfrak{l}_1}{1+\mathfrak{l}_1}\right),$$

eoque fortius:

$$m^0 < \left(\frac{1-c_1}{1+c_1}\right), \qquad m^0 < \left(\frac{1-\mathfrak{k}_1}{1+\mathfrak{k}_1}\right).$$

Deinde habemus ex formulis (6.):

$$\frac{l^0}{c^0} = \frac{\frac{1-m_1}{1+m_1}}{m^0} = \frac{\mathfrak{l}^0}{\mathfrak{k}^0},$$

unde theoremate (41.) adiuti docemur, quantitates fractas:

$$\frac{\frac{1-m_1}{1+m_1}}{m^0}, \qquad \frac{\frac{1-m_1^0}{1+m_1^0}}{m^{00}}, \qquad \text{etc.}$$

simul cum $\frac{l^0}{c^0}$ ad unitatem accedere.

Altera pars theorematis eodem modo demonstratur.

Deinde hoc **theorema** proponimus:

„Series:

„47. $$\begin{cases} \frac{c}{\mathfrak{k}} = \frac{l}{\mathfrak{l}}, & \frac{c^0}{\mathfrak{k}^0} = \frac{l^0}{\mathfrak{l}^0}, \quad \frac{c^{00}}{\mathfrak{k}^{00}} = \frac{l^{00}}{\mathfrak{l}^{00}}, \quad \dots \\ \frac{m_1}{M_1} = \frac{l_1}{L_1}, & \frac{m_1'}{M_1'} = \frac{l_1'}{L_1'}, \quad \frac{m_1''}{M_1''} = \frac{l_1''}{L_1''}, \quad \dots \end{cases}$$

„si in illa $c < \mathfrak{k}$, in hac $m_1 < M_1$ fuerit, ita decrescunt, ut quisque termi-„nus a quadrato antecedentis superetur, dum contrario casu quantitates:

„48. $$\frac{\mathfrak{k}}{c}, \quad \frac{\mathfrak{k}^0}{c^0}, \quad \frac{\mathfrak{k}^{00}}{c^{00}}, \text{ etc.} \qquad \frac{M_1}{m_1}, \quad \frac{M_1'}{m_1'}, \quad \frac{M_1''}{m_1''}. \text{ etc.}$$

„eandem legem sequuntur."

Demonstratio.

Ex formula (21.) posteriori, sequitur haec:

$$\frac{c}{\mathfrak{k}} = \frac{\frac{2\sqrt{c^0}}{1+c^0}}{\frac{2\sqrt{\mathfrak{k}^0}}{1+\mathfrak{k}^0}} = \frac{l}{\mathfrak{l}},$$

unde deducimus hanc aequationem:

$$\frac{c^2}{\mathfrak{k}^2} : \frac{c^0}{\mathfrak{k}^0} = \left(\frac{1+\mathfrak{k}^0}{1+c^0}\right)^2,$$

ita ut posito $c^0 < \mathfrak{k}^0$ habeamus:

$$\frac{c^0}{\mathfrak{k}^0} < \frac{c^2}{\mathfrak{k}^2},$$

contra posito, $\mathfrak{k}^0 < c^0$, fiat:

$$\frac{\mathfrak{k}^0}{c^0} < \frac{\mathfrak{k}^2}{c^2}.$$

Iam vero fractio $\frac{2\sqrt{x}}{1+x}$ simul cum argumento x ab nihilo usque ad unitatem crescente, ipsa ab nihilo ad unitatem crescit, ita tamen, ut semper sit:

$$x > \frac{2\sqrt{x}}{1+x},$$

unde clarum fit, prout fuerit:

$$c < \mathfrak{k}, \qquad c = \mathfrak{k}, \qquad c > \mathfrak{k},$$

etiam fore:

$$c^0 < \mathfrak{k}^0, \quad c^0 = \mathfrak{k}^0, \quad c^0 > \mathfrak{k}^0.$$

quibus cum antecedentibus collatis, theorematis prior pars demonstrata est. Alteram partem per formulam hanc ex priori (21.) sequentem:

$$\frac{m_1'}{M_1'} = \frac{\left(\frac{2\sqrt{m_1}}{1+m_1}\right)}{\left(\frac{2\sqrt{M_1}}{1+M_1}\right)}$$

similiter ostendere possumus. Utramque vero etiam per has formulas, ex (1.), (2.), (4.), (5.) sequentes:

$$\frac{c^0}{\mathfrak{k}^0} = \left(\frac{m_1 - c_1 l_1}{m_1 + c_1 l_1}\right)\left(\frac{m_1 + \mathfrak{k}_1 \mathfrak{l}_1}{m_1 - \mathfrak{k}_1 \mathfrak{l}_1}\right) = \frac{m_1^2 - c_1^2 l_1^2}{m_1^2 - C_1^2 L_1^2} \cdot \frac{m + \mathfrak{k}_1 \mathfrak{l}_1}{m_1 + c\,\mathfrak{l}_1} = \frac{c^2}{C^2}\left(\frac{m_1 + \mathfrak{k}_1 \mathfrak{l}_1}{m_1 + c_1 l_1}\right)^2,$$

$$\frac{m_1'}{M_1'} = \left(\frac{c - lm}{c + lm}\right)\left(\frac{c + LM}{c - LM}\right) = \frac{c^2 - l^2 m^2}{c^2 - L^2 M^2}\left(\frac{c + LM}{c + lm}\right)^2 = \left(\frac{m_1}{M_1}\right)^2 \left(\frac{c + LM}{c + lm}\right)^2$$

demonstrare licet.

Adiicere placet adhuc, has aequationes ex (1.) et (5.) sequentes:

$$c_1^0 = \frac{\frac{2\sqrt{(c_1 l_1)}}{m_1}}{1 + \frac{c_1 l_1}{m_1}}, \qquad \mathfrak{k}_1^0 = \frac{\frac{2\sqrt{(\mathfrak{k}_1 \mathfrak{l}_1)}}{m}}{1 + \frac{\mathfrak{k}_1 \mathfrak{l}_1}{m}},$$

docere, algorithmum, quo moduli c et $\mathfrak{k}$ determinentur, denique ubi $c = l$, $\mathfrak{k} = \mathfrak{l}$, $m_1 = 1$ posuimus, his formulis concinnis exhiberi:

$$c_1^0 = \frac{2c_1}{1 + c_1^2}, \qquad \mathfrak{k}_1^0 = \frac{2\mathfrak{k}_1}{1 + \mathfrak{k}_1^2},$$

sive:

$$c^0 = \frac{1 - c_1^2}{1 + c_1^2}, \qquad \mathfrak{k}^0 = \frac{1 - \mathfrak{k}_1^2}{1 + \mathfrak{k}_1^2},$$

quae ad algorithmum notissimo medio arithmetico geometrico similem ducunt, dum secundum theorema (45.), sive secundum hanc formulam ex formula (6.) derivatam:

$$\sqrt{\left(1 - \frac{l^{02}}{c^{02}} m^{02}\right)} = \frac{2\sqrt{m_1}}{1 + m_1},$$

determinatio tertii moduli, dum $\frac{l^0}{c^0} = 1$ devenerit, ad illum algorithmum ipsum reducitur. Ad similes observationes formulae:

$$m' = \frac{2\sqrt{\left(\frac{ml}{c}\right)}}{1 + \frac{ml}{c}}, \quad M' = \frac{\frac{2\sqrt{(ML)}}{c}}{1 + \frac{ML}{c}}, \quad \sqrt{\left(1 - \frac{l_1'^2}{m_1'^2} c_1'^2\right)} = \frac{2\sqrt{c}}{1 + c}.$$

Priusquam disquisitiones de natura modulorum finiamus nonnullos casus speciales quinternorum modulorum ponere placet. Primum sit $\mathfrak{k} = 1$, sive $l = m$, et $M = L$.

Habebimus:

$$c^0 = \frac{1 - c_1}{1 + c_1}, \quad l^{02} = \left(\frac{1 - m_1}{1 + m_1}\right)\left(\frac{1 - c_1}{1 + c_1}\right), \quad m^0 = l^0, \quad \mathfrak{l}^{02} = \left(\frac{1 - m_1}{1 + m_1}\right)\left(\frac{1 + c_1}{1 - c_1}\right), \quad \mathfrak{k}^0 = 1,$$

$$m_1' = \frac{c - m^2}{c + m^2}, \quad l_1' = m_1', \qquad c_1' = \frac{1 - c}{1 + c}, \quad L_1' = \left(\frac{1 - c}{1 + c}\right)\left(\frac{c + m^2}{c - m^2}\right), \quad M_1' = L_1'.$$

Si $c = 1$, sive $m = \mathfrak{l}$ fuerit, habemus:

$$c^0 = 1, \quad l^{0\,2} = \left(\frac{1-m_1}{1+m_1}\right)\left(\frac{1+\mathfrak{k}_1}{1-\mathfrak{k}_1}\right), \quad m^0 = \mathfrak{l}^0, \quad \mathfrak{l}^{0\,2} = \left(\frac{1-m_1}{1+m_1}\right)\left(\frac{1-\mathfrak{k}_1}{1+\mathfrak{k}_1}\right), \quad \mathfrak{k}^0 = \frac{1-\mathfrak{k}_1}{1+\mathfrak{k}_1},$$

$$m_1' = \frac{1-lm}{1+lm}, \quad l_1' = \frac{l_1 m_1}{1+lm}, \quad c_1' = 0, \quad L_1' = 0, \quad M_1' = 0.$$

Si $M_1 = 1$ ponitur, sive $c_1 = l_1$, habemus:

$$c^0 = \frac{m_1 - l_1^2}{m_1 + l_1^2}, \quad l^0 = c^0, \quad m^0 = \frac{1-m_1}{1+m_1}, \quad l_0 = \frac{1-m_1}{1+m_1}\cdot\frac{m_1+c_1^2}{m_1-c_1^2}, \quad \mathfrak{k}^0 = \mathfrak{l}_0,$$

$$m_1' = \frac{1-m}{1+m}, \quad l_1'^2 = \left(\frac{1-c}{1+c}\right)\left(\frac{1-m}{1+m}\right), \quad c_1' = l_1', \quad L_1' = \left(\frac{1-c}{1+c}\right)\left(\frac{1+m}{1-m}\right), \quad M_1' = 1.$$

Si $m_1 = 1$ fuerit, habemus:

$$c^0 = \frac{1-l_1 c_1}{1+l_1 c_1}, \quad l^0 = \frac{lc}{1+l_1 c_1}, \quad m^0 = 0, \quad \mathfrak{l}^0 = 0, \quad \mathfrak{k}^0 = 0,$$

$$m_1' = 1, \quad l_1'^2 = \left(\frac{1-c}{1+c}\right)\left(\frac{1+M}{1-M}\right), \quad c_1' = L_1', \quad L_1'^2 = \left(\frac{1-c}{1+c}\right)\left(\frac{1-M}{1+M}\right), \quad M_1' = \frac{1-M}{1+M}.$$

XIII.

De computatione modulorum.

Moduli utriusque transformationis sequenti methodo facillime computantur. Altera in transformatione quantitatibus:

$$c^0 = \frac{m_1 - c_1 l_1}{m_1 + c_1 l_1}, \qquad \mathfrak{k}^0 = \frac{m_1 - \mathfrak{k}_1 \mathfrak{l}_1}{m_1 + \mathfrak{k}_1 \mathfrak{l}_1}$$

computatis, calculare licet modulos l^0, m^0, $\mathfrak{l}^0$, ... his formulis (2.), (3.), (4.) art. XII.:

$$l^0 = \sqrt{\left(\frac{1-m_1}{1+m_1}\cdot\frac{c^0}{\mathfrak{k}^0}\right)}, \quad \mathfrak{l}^0 = \sqrt{\left(\frac{1-m_1}{1+m_1}\cdot\frac{\mathfrak{k}^0}{c^0}\right)}, \quad m^0 = \left(\frac{1-m_1}{1+m_1}\cdot c^0\mathfrak{k}^0\right).$$

Quem calculum ita trigonometricum faciamus. Ponamus:

$$1. \begin{cases} c = \sin\alpha, \quad l = \sin\beta, \quad m = \sin\gamma, \quad \dfrac{m}{c} = \mathfrak{l} = \sin B, \quad \dfrac{m}{l} = \mathfrak{k} = \sin A, \\ c^0 = \sin\alpha^0, \quad l^0 = \sin\beta^0, \quad m^0 = \sin\gamma^0, \quad \mathfrak{l}^0 = \sin B^0, \quad \mathfrak{k}^0 = \sin A^0. \end{cases}$$

Determinentur aut anguli α^0 et A^0 aequationibus:

$$2. \quad \operatorname{tang}\left(45 - \frac{\alpha^0}{2}\right) = \sqrt{\left(\frac{\cos\alpha\cos\beta}{\cos\gamma}\right)}, \qquad \operatorname{tang}\left(45 - \frac{A^0}{2}\right) = \sqrt{\left(\frac{\cos A\cos B}{\cos\gamma}\right)},$$

aut moduli c^0 et $\mathfrak{k}^0$ ipsi angulis auxiliaribus ε et E introductis talibus, ut sit:

$$3. \quad \cos\varepsilon = \frac{\cos\alpha\cos\beta}{\cos\gamma}, \qquad \cos E = \frac{\cos A\cos B}{\cos\gamma},$$

per aequationes:

$$4. \quad c^0 = \operatorname{tang}^2\tfrac{1}{2}\varepsilon, \qquad \mathfrak{k}^0 = \operatorname{tang}^2\tfrac{1}{2}E.$$

Quo facto habebimus:

$$5.\quad \begin{cases} l^0 = \operatorname{tang}\tfrac{1}{2}\gamma\ \operatorname{tang}\tfrac{1}{2}\varepsilon \operatorname{cotang}\tfrac{1}{2}E = \operatorname{tang}\tfrac{1}{2}\gamma\sqrt{\left(\dfrac{\sin\alpha^0}{\sin A^0}\right)}, \\ \mathfrak{l}^0 = \operatorname{tang}\tfrac{1}{2}\gamma \operatorname{cotang}\tfrac{1}{2}\varepsilon\ \operatorname{tang}\tfrac{1}{2}E = \operatorname{tang}\tfrac{1}{2}\gamma\sqrt{\left(\dfrac{\sin A^0}{\sin\alpha^0}\right)}, \end{cases}$$

$$6.\quad m^0 = \operatorname{tang}\tfrac{1}{2}\gamma\ \operatorname{tang}\tfrac{1}{2}\varepsilon\ \operatorname{tang}\tfrac{1}{2}E = \operatorname{tang}\tfrac{1}{2}\gamma\sqrt{(\sin\alpha^0\sin A^0)}.$$

Si vero formulae (5.) et (6.) angulis α^0 vel A^0 ad nihil maxime appropinquantibus, haud magis exactos valores appropinquatos adipisci velis, hoc casu hanc aliam methodum trigonometricam adhibere licet. Habemus formulas (7.) et (18.) art. XII.:

$$7.\quad \begin{cases} m^0 l^0 = \dfrac{(1-m_1)(m_1-c_1 l_1)}{(1+m_1)(m_1+c_1 l_1)}, & m^0\mathfrak{l}^0 = \dfrac{(1-m_1)(m_1-\mathfrak{k}_1\mathfrak{l}_1)}{(1+m_1)(m+\mathfrak{k}_1\mathfrak{l}_1)}, \\ m_1^0 l_1^0 = \dfrac{2m_1(c_1+l_1)}{(1+m_1)(m_1+c_1 l_1)}, & m_1^0\mathfrak{l}_1^0 = \dfrac{2m_1(\mathfrak{k}_1+\mathfrak{l}_1)}{(1+m_1)(m_1+\mathfrak{k}_1\mathfrak{l}_1)}, \end{cases}$$

unde sequitur formulis (1.) substitutis:

$$\cos(\beta^0\pm\gamma^0) = \frac{2m_1(c_1+l_1)\mp(1-m_1)(m_1-c_1 l_1)}{(1+m_1)(m_1+c_1 l_1)},$$

$$\cos(B^0\pm\gamma^0) = \frac{2m_1(\mathfrak{k}_1+\mathfrak{l}_1)\mp(1-m_1)(m_1-\mathfrak{k}_1\mathfrak{l}_1)}{(1+m_1)(m_1+\mathfrak{k}_1 l_1)};$$

unde facili reductione facta hae formulae emanant:

$$8.\quad \begin{cases} \operatorname{tang}^2\tfrac{1}{2}(\beta^0+\gamma^0) = \dfrac{\sin^2\dfrac{\alpha}{2}.\sin^2\dfrac{\beta}{2}.\cos\gamma}{\left(\cos\dfrac{\alpha+\gamma}{2}\right)\left(\cos\dfrac{\alpha-\gamma}{2}\right)\left(\cos\dfrac{\beta+\gamma}{2}\right)\left(\cos\dfrac{\beta-\gamma}{2}\right)}, \\ \operatorname{tang}^2\tfrac{1}{2}(\beta^0-\gamma^0) = \dfrac{\left(\sin\dfrac{\alpha+\gamma}{2}\right)\left(\sin\dfrac{\alpha-\gamma}{2}\right)\left(\sin\dfrac{\beta+\gamma}{2}\right)\left(\sin\dfrac{\beta-\gamma}{2}\right)}{\cos^2\dfrac{\alpha}{2}\cos^2\dfrac{\beta}{2}\cos\gamma}, \\ \operatorname{tang}^2\tfrac{1}{2}(B^0+\gamma^0) = \dfrac{\sin^2\dfrac{A}{2}\sin^2\dfrac{B}{2}\cos\gamma}{\left(\cos\dfrac{A+\gamma}{2}\right)\left(\cos\dfrac{A-\gamma}{2}\right)\left(\cos\dfrac{B+\gamma}{2}\right)\left(\cos\dfrac{B-\gamma}{2}\right)}, \\ \operatorname{tang}^2\tfrac{1}{2}(B^0-\gamma^0) = \dfrac{\sin\dfrac{A+\gamma}{2}\ \sin\dfrac{A-\gamma}{2}\ \sin\dfrac{B+\gamma}{2}\ \sin\dfrac{B-\gamma}{2}}{\cos^2\dfrac{A}{2}\cos^2\dfrac{B}{2}\cos\gamma}, \end{cases}$$

quarum ope valores angulorum β^0, γ^0, vel B^0, γ^0 calculo logarithmico trigonometrico exacto determinantur.

In altera vero transformatione primum quantitatibus m_1' et M_1' ex formulis:

$$m'_1 = \frac{c-lm}{c+lm}, \qquad M'_1 = \frac{c-LM}{c+LM},$$

computatis, habemus:

$$l'_1 = \sqrt{\left(\frac{1-c}{1+c}\cdot\frac{m'_1}{M'_1}\right)}, \qquad L'_1 = \sqrt{\left(\frac{1-c}{1+c}\cdot\frac{M'_1}{m'_1}\right)}, \qquad c'_1 = \sqrt{\left(\frac{1-c}{1+c}\, m'_1 M'_1\right)}.$$

Ponamus rursus:

$$9. \quad \begin{cases} m_1 = \sin\alpha, \quad l_1 = \sin\beta, \quad c_1 = \sin\gamma, \quad \frac{c_1}{m_1} = L_1 = \sin B, \quad \frac{c_1}{l_1} = M_1 = \sin A, \\ m'_1 = \sin\alpha', \quad l'_1 = \sin\beta', \quad c'_1 = \sin\gamma', \quad \frac{c'_1}{m'_1} = L'_1 = \sin B', \quad \frac{c'_1}{l'_1} = M'_1 = \sin A', \end{cases}$$

atque nanciscimur valores angulorum α' et A' ex formulis:

$$10. \quad \tang\left(45 - \frac{\alpha'}{2}\right) = \sqrt{\left(\frac{\cos\alpha\cos\beta}{\cos\gamma}\right)}, \qquad \tang\left(45 - \frac{A'}{2}\right) = \sqrt{\left(\frac{\cos A\cos B}{\cos\gamma}\right)},$$

vel complementa m'_1 et M'_1 ipsa ex his:

$$11. \quad m'_1 = \tang^2 \tfrac{1}{2}\varepsilon, \qquad M'_1 = \tang^2 \tfrac{1}{2}E;$$

ubi anguli auxiliares dantur per formulas:

$$12. \quad \cos\varepsilon = \frac{\cos\alpha\cos\beta}{\cos\gamma}, \qquad \cos E = \frac{\cos A\cos B}{\cos\gamma}.$$

Tum cetera complementa erunt:

$$13. \quad \begin{cases} l'_1 = \tang\tfrac{1}{2}\gamma \;\; \tang\tfrac{1}{2}\varepsilon \,\text{cotang}\tfrac{1}{2}E = \tang\tfrac{1}{2}\gamma\sqrt{\left(\frac{\sin\alpha'}{\sin A'}\right)}, \\ L'_1 = \tang\tfrac{1}{2}\gamma \,\text{cotang}\tfrac{1}{2}\varepsilon \;\; \tang\tfrac{1}{2}E = \tang\tfrac{1}{2}\gamma\sqrt{\left(\frac{\sin A'}{\sin\alpha'}\right)}, \end{cases}$$

$$14. \quad c'_1 = \tang\tfrac{1}{2}\gamma \;\; \tang\tfrac{1}{2}\varepsilon \;\; \tang\tfrac{1}{2}E = \tang\tfrac{1}{2}\gamma\sqrt{(\sin\alpha'\sin A')}.$$

Tria ultima complementa, si angulus α vel A minimus devenerit, etiam his formulis definiuntur:

$$15. \quad \begin{cases} \tang^2\tfrac{1}{2}(\beta'+\gamma') = \dfrac{\sin^2\tfrac{1}{2}\alpha\,\sin^2\tfrac{1}{2}\beta\cos\gamma}{\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}}, \\ \tang^2\tfrac{1}{2}(\beta'-\gamma') = \dfrac{\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\beta-\gamma}{2}}{\cos^2\tfrac{1}{2}\alpha\,\cos^2\tfrac{1}{2}\beta\cos\gamma}, \\ \tang^2\tfrac{1}{2}(B'+\gamma') = \dfrac{\sin^2\tfrac{1}{2}A\,\sin^2\tfrac{1}{2}B\cos\gamma}{\cos\frac{A+\gamma}{2}\cos\frac{A-\gamma}{2}\cos\frac{B+\gamma}{2}\cos\frac{B-\gamma}{2}}, \\ \tang^2\tfrac{1}{2}(B'-\gamma') = \dfrac{\sin\frac{A+\gamma}{2}\sin\frac{A-\gamma}{2}\sin\frac{B+\gamma}{2}\sin\frac{B-\gamma}{2}}{\cos^2\tfrac{1}{2}A\,\cos^2\tfrac{1}{2}B\cos\gamma}, \end{cases}$$

unde anguli β' et γ' vel B' et γ' computari possunt, quibus adhibitis m'_1, l'_1, c'_1, L'_1, M'_1 determinantur.

Si anguli in antecedentibus adhibiti ad tam exiguos valores reducti sint, ut calculus logarithmico-trigonometricus haud satis sufficientem praebeat approximationem, alium algorithmum proponamus necesse est, quem in secunda transformatione praecipue adhibeamus necesse erit, quippe in qua exactissimi logarithmorum modulorum valores, ad valorem integralis ipsius computandum, desiderari postea elucebit. Ex theorematibus (47.) et (48.) art. XII. patet, binorum modulorum complementa vel m'_1 et l'_1 vel M'_1 et L'_1 denique multo minoribus altera quam altera gavisura esse valoribus, uno excepto casu, si fuerit $cl = m$. Itaque plurimis in casibus, complementis m'_1 et l'_1 ut ordinis δ^0_1 tertioque c'_1 ut ordinis δ positis, respective complementa M'_1 et L'_1 ordinis δ fiunt, et inverse. Quam ob rem ponamus illa maiora complementa fore m'_1 et l'_1, ipsaque ad aequalitatem appropinquare, atque quia anguli α, β, γ, B, A ultimi parvis valoribus gaudeant, nec per primam nec per secundam computationem logarithmico-trigonometricam logarithmos complementorum usque ad quaesitum decimalium numerum exacte computari posse. His statutis hac nova via ad computationem exactam modulorum progredi licet.

Primum complementum $m'_1 = \sin\alpha'$ per formulam veterem:

$$\tang\left(45 - \frac{\alpha'}{2}\right) = \sqrt{\left(\frac{ml}{c}\right)}$$

computetur, cetera vero per formulas:

$$16.\quad \begin{cases} l'_1 = \dfrac{l_1 m_1}{1+c}\cdot\dfrac{c+LM}{c+lm}, \\[2mm] c'_1 = \dfrac{c_1^2}{(1+c)^2}\cdot\dfrac{m'_1}{l'_1}, \\[2mm] L'_1 = \dfrac{L_1 M_1}{1+c}\cdot\dfrac{c+lm}{c+LM} = \dfrac{c'_1}{m'_1}, \\[2mm] M'_1 = \dfrac{c'_1}{l'_1}, \end{cases}$$

quae formulae ex form. (2.), (3.), (4.), (5.) art. XII. sponte emanant. Logarithmi enim horum modulorum facillime et quam exactissime hoc modo computantur. Inventis quantitatibus:

$$\log\frac{lm}{c},\quad \log\frac{LM}{c},\quad \log c,\quad \log l_1 m_1,\quad 2\log c_1$$

indeque computatis logarithmis:

$$\log\left(1+\frac{lm}{c}\right),\qquad \log\left(1+\frac{LM}{c}\right),\qquad \log(1+c)$$

habemus:

$$17.\quad\begin{cases}\log l'_1 = \log l_1 m_1 + \log\left(1+\frac{LM}{c}\right) + \log\text{com}(1+c) + \log\text{com}\left(1+\frac{lm}{c}\right),\\ \log c'_1 = 2\log c_1 + 2\log\text{com}(1+c) + \log\left(\frac{m'_1}{l'_1}\right),\\ \log L'_1 = \log L_1 M_1 + \log\left(1+\frac{lm}{c}\right) + \log\text{com}(1+c) + \log\text{com}\left(1+\frac{LM}{c}\right),\\ \log M'_1 = \log c'_1 + \log\text{com}\, l'_1.\end{cases}$$

Sed etiam logarithmi $\log m'_1$ et $\log M'_1$ ipsi, ad qualescunque valores decreverint, tamen haud minus exacte computari possunt per has formulas:

$$18.\quad\begin{cases} m'_1 = \frac{l_1^2}{c^2}\cdot\frac{\left(1+L^2\frac{m_1^2}{l_1^2}\right)}{\left(1+\frac{lm}{c}\right)^2}\left(1-\frac{m_1^2}{\left(1+L^2\frac{m_1^2}{l_1^2}\right)}\right),\\ M'_1 = \frac{L_1^2}{c^2}\cdot\frac{\left(1+l^2\frac{m_1^2}{l_1^2}\right)}{\left(1+\frac{lm}{c}\right)^2}\left(1-\frac{M_1^2}{\left(1+l^2\frac{m_1^2}{l_1^2}\right)^2}\right),\end{cases}$$

quia logarithmi singulorum factorum exactissime semper inveni possunt. Error ultimi decimalis, per additionem quinque logarithmorum ortus, ob exiguitatem complementorum nullius momenti fit; formulae vero illae ex formulis (1.) art. XII. facile deducuntur, atque, si m'_1 et M'_1 trigonometrice non amplius computari possunt ad computationem totius schematis adhibeantur necesse est. Transformatione eatenus continuata, ut quantitates M_1^2 et postea m_1^2 pro quaesito decimalium numero evanuerint, habemus has formulas:

$$19.\quad\begin{cases} m'_1 = \frac{l_1^2}{4}\left(1+\frac{m_1^2}{l_1^2}\right), & L'_1 = \frac{L_1 M_1}{2},\\ l'_1 = \frac{l_1 m_1}{2}, & M'_1 = \frac{L_1^2}{4}\left(1+\frac{m_1^2}{l_1^2}\right),\\ c'_1 = \frac{c_1^2}{(1+c)^2}\cdot\frac{m_1}{l_1}, & \end{cases}$$

quae, si adhuc $\log\frac{m_1^2}{l_1^2}$ ab 0 usque ad quaesitum decimalium numerum haud discrepat, in has abeunt:

$$20.\quad m'_1 = l'_1 = \frac{l_1^2}{2} = \frac{m_1^2}{2},\quad c'_1 = \frac{c_1^2}{4},\quad L'_1 = M'_1 = \frac{L_1^2}{2} = \frac{M_1^2}{2}.$$

Diversis his methodis apte adhibitis, per algorithmum continuatum complementa modulorum quantopere libet deminuere possumus. In altera transformatione, ubi, tantam certitudinem omnium modulorum logarithmorum haud desiderari, postea videbimus, tamen similes formulas proponamus. Nimirum habemus ex form. (1.), (2.), (3.), (4.), (5.) art. XII. superioribus has:

$$21.\quad \begin{cases} c^0 = \dfrac{l^2}{m_1^2}\cdot\dfrac{\left(1+\dfrac{\mathfrak{l}_1^2 c_1^2}{l^2}\right)}{\left(1+\dfrac{l_1 c_1}{m_1}\right)^2}\left(1-\dfrac{c^2}{1+\dfrac{\mathfrak{l}_1^2 c^2}{l^2}}\right) = \dfrac{m^0}{\mathfrak{l}^0}, \\[2ex] l^0 = \dfrac{c\,l}{(1+m_1)}\cdot\dfrac{\left(1+\dfrac{\mathfrak{k}\mathfrak{l}}{m_1}\right)}{\left(1+\dfrac{c_1 l_1}{m_1}\right)}, \\[2ex] m^0 = \dfrac{m^2}{(1+m_1)}\cdot\dfrac{c^0}{l^0}, \\[2ex] \mathfrak{l}^0 = \dfrac{\mathfrak{k}\mathfrak{l}}{(1+m_1)}\cdot\dfrac{\left(1+\dfrac{c_1 l_1}{m_1}\right)}{\left(1+\dfrac{\mathfrak{k}_1\mathfrak{l}_1}{m_1}\right)}, \\[2ex] \mathfrak{k}^0 = \dfrac{\mathfrak{l}^{02}}{m_1^2}\cdot\dfrac{\left(1+\dfrac{l_1^2 c^2}{l^2}\right)}{\left(1+\dfrac{\mathfrak{k}_1\mathfrak{l}_1}{m_1}\right)^2}\left(1-\dfrac{\mathfrak{k}^2}{\left(1+\dfrac{l_1^2 c^2}{l^2}\right)}\right), \end{cases}$$

quae, si quantitates c^2 et $\mathfrak{k}^2$ pro quaesito decimalium numero iam negligi possunt, in has abeunt:

$$c^0 = \frac{l^2}{4}\left(1+\frac{c^2}{l^2}\right), \qquad \mathfrak{l}^0 = \frac{\mathfrak{k}\mathfrak{l}}{2},$$

$$l^0 = \frac{c\,l}{2}, \qquad \mathfrak{k}^0 = \frac{l_1^2}{4}\left(1+\frac{c^2}{l^2}\right),$$

$$m^0 = \frac{m^2}{(1+m_1)}\cdot\frac{c}{l},$$

vel si adeo $\frac{c}{l} = 1$ poni possit, in has:

$$c^0 = l^0 = \frac{c^2}{2}, \quad m^0 = \frac{m^2}{4}, \quad \mathfrak{k}^0 = \mathfrak{l}^0 = \frac{\mathfrak{k}^2}{2}.$$

XIV.

De natura et computatione numeratorum.

Aggrediamur nunc ad numeratores integralium tractandos atque computandos, atque disquiramus, ad quos limites, transformationibus repetitis, illi appropinquent. Habuimus in priori transformatione quatuor formulas (10.) art. XI., quae facilibus reductionibus factis, in has abeunt:

$$1.\quad \begin{cases} P'_1 = \frac{2}{(1+c)(c+lm)}\left[\frac{c}{2}P_1+\frac{1}{2}Q_1\right], \\ Q'_1 = \frac{2}{(1+c)(c+lm)}\left[\frac{clm}{c+lm}P_1+\frac{c}{c+lm}Q_1\right], \\ P'_2 = \frac{2}{(1+c)(c+LM)}\left[\frac{c}{2}P_2+\frac{1}{2}Q_2\right], \\ Q'_2 = \frac{2}{(1+c)(c+LM)}\left[\frac{cLM}{c+LM}P_2+\frac{c}{c+LM}Q_2\right], \end{cases}$$

Adiiciamus adhuc has formulas concinnas:

$$2.\quad \begin{cases} P'_1-Q'_1 = \frac{2}{(1+c)(c+lm)}\left[\frac{c}{2}P_1-\frac{1}{2}Q_1\right]m'_1, \\ P'_2-Q'_2 = \frac{2}{(1+c)(c+LM)}\left[\frac{c}{2}P_2-\frac{1}{2}Q_2\right]M'_1. \end{cases}$$

Hîc quantitates P_1 et Q_1 coefficientes integralis dati erant, aeque ac P_2 et Q_2 coefficientes integralis complementarii ita determinantur:

$$3.\quad P_2 = \frac{P_1-Q_1}{l_1 m_1},\qquad Q_2 = \frac{P_1c^2-Q_1}{l_1 m_1}.$$

Inde ex formulis (1.) clarum fit, quia factores $\frac{2}{(1+c)(c+LM)}$ et $\frac{2}{(1+c)(c+lm)}$ transformationibus repetitis denique $=\frac{1}{2}$ fiunt, nec non alii factores $\left(\frac{P_1c}{2}-\frac{1}{2}Q_1\right)$ et $\left(\frac{P_2c}{2}-\frac{1}{2}Q_2\right)$ in $\frac{(P_1-Q_1)}{2}$ et $\frac{P_2-Q_2}{2}$ abeunt, tertii vero factores m'_1 et M'_1 denique evanescunt, etiam quantitates P_1 et Q_1 ad aequalitatem rapide denique convergere, eademque natura quantitates P_2 et Q_2 gaudere. Id quod hoc theoremate magis strenue pronunciatur.

„Quantitates:

$$„4.\quad 2^n(P_2^{(n)},\qquad 2^nQ_2^{(n)},\qquad 2^n\left(\frac{P_1^{(n)}-Q_1^{(n)}}{(m_1^{(n)})^2}\right),$$

„ad eundem finitum certum denique limitem appropinquant, similique ra-„tione quantitates:

$$„2^mP_1^{(m)},\qquad 2Q_1^{(m)},\qquad 2^m\left(\frac{P_2^{(m)}-Q_2^{(m)}}{(M_1^{(m)})^2}\right.$$

„ad alium limitem eundem accedunt."

Demonstratio.

Ponamus transformationibus toties repetitis, ut usque ad certum decimalium numerum habeamus:

$$m_1^{(n)} = l_1^{(n)} = \left(\frac{l_1^{(n-1)}}{2}\right)^2 = \left(\frac{m_1^{(n-1)}}{2}\right)^2,\qquad \frac{l^{(n)}m^{(n)}}{c^{(n)}} = 1,\qquad c^{(n)} = 1,$$

ubi n numerus transformationum desideratus fuit.

Itaque erit, id quod formulae (3.) docent:

$$P_2^{(n+1)} = Q_2^{(n+1)} = \frac{P_1^{(n+1)} - Q_1^{(n+1)}}{[m_1^{(n+1)}]^2},$$

atque

$$2^{(n+1)} \frac{P_1^{(n+1)} - Q_1^{(n+1)}}{(m_1^{(n+1)})^2} = 2^n \left(\frac{P_1^{(n)} - Q_1^{(n)}}{(m^{(n)})^2} \right),$$

id quod erat demonstrandum. Similiter altera pars theorematis demonstratur. Indeque clarum fit, denique fore:

$$P_1^{(n)} = Q_1^{(m)} = \frac{P_2^{(m)} - Q_2^{(m)}}{(M_1^{(m)})^2} = \tfrac{1}{2} P_1^{(n-1)} = \tfrac{1}{2} Q_1^{(m-1)} = \tfrac{1}{2} \frac{P_2^{(m-1)} - Q_2^{(m-1)}}{(M_1^{(m-1)})^2},$$

$$P_2^{(n)} = Q_2^{(n)} = \frac{P_1^{(n)} - Q_1^{(n)}}{(m_1^n)^2} = \tfrac{1}{2} P_2^{(n-1)} = \tfrac{1}{2} Q_2^{(n-1)} = \tfrac{1}{2} \frac{P_1^{(n-1)} - Q_1^{(n-1)}}{(m_1^{n-1})^2},$$

usque ad eundem certum decimalium numerum.

Si alteram formam numeratorum ($R_1 \cos \varphi + S \sin^2 \varphi$) praeferamus, habemus ex form. (20.) artic. XI.:

$$5. \quad \begin{cases} R'_1 = \dfrac{2}{(1+c)(c+lm)} \left[\dfrac{1+c}{2} R_1 - \tfrac{1}{2} S_1 \right], \\ S'_1 = \dfrac{2}{(1+c)(c+lm)} \left[\tfrac{1}{2} S_1 - \dfrac{1-c}{2} R_1 \right] m'_1. \end{cases}$$

Ex form. (19.) eiusdem art. sequitur adhuc fore:

$$6. \quad R'_2 = \frac{1}{l'_1 m'_1} S'_1, \qquad S'_2 = R'_1 L'_1 M'_1.$$

Hîc theorema antecedens in hoc abit:

„Quantitates

$$„7. \quad 2^n R_2^{(n)}, \qquad \frac{2^n}{[m_1^{(n)}]^2} S_1^{(n)},$$

„ad eundem limitem appropinquant finitum, similiterque quantitates

$$„2^n R_1^{(n)}, \qquad \frac{2^n}{[M_1^{(n)}]^2} S_2^{(n)},$$

„ad alium limitem finitum accedunt."

Iam vero nihil restat, nisi ut expeditam methodum adiiciamus, qua numeratores ad valorem integralis determinandum satis exacte computentur, quippe quam hic ita instituamus necesse est, ut in priori numeratoris forma logarithmi coefficientium ipsorum P et Q in quantitatibus P'_1, P'_2, $P'_1 - Q'_1$, $P'_2 - Q'_2$, in posteriori vero forma logarithmi coefficientium ipsorum R et S in quantitatibus:

$$R'_1, \quad S'_1, \quad R'_2, \quad S'_2$$

in quaque transformatione usque ad ultimum decimalem exacte determinentur.

Ponamus hunc ad finem illic:

$$8.\quad \begin{cases} n = 1, & P_1 = pP + qQ, \\ & P_1 - Q_1 = p_1P - q_1Q, \\ \dfrac{4}{(1+c)(c+lm)} = n', & 2P'_1 = n'[p'P + q'Q], \\ & 2(P'_1 - Q'_1) = n'[p'_1P - q'_1Q], \\ \dfrac{4}{(1+c')(c'+l'm')} = n'', & 2^2P''_1 = n'n''[p''P + q''Q], \\ & 2^2(P''_1 - Q''_1) = n'n''[p''_1P - q''_1Q], \\ \dfrac{4}{(1+c'')(c''+l''m'')} = n''', & 2^3P'''_1 = n'n''n'''[p'''P + q'''Q], \\ & 2^3(P'''_1 - Q'''_1) = n'n''n'''[p'''_1P - q'''_1Q], \\ & \text{etc.} \end{cases}$$

Ex his quantitatibus sine ullo negotio emanant quantitates P_2 et $P_2 - Q_2$ nimirum ex formulis (3.) adhibitis erit:

$$9.\quad \begin{cases} P_2 = \dfrac{1}{l_1 m_1}[p_1P - q_1Q], \\ (P_2 - Q_2) = L_1 M_1 [pP + qQ], \\ 2P'_2 = \dfrac{n'}{l'_1 m'}[p'_1P - q'_1Q], \\ 2(P'_2 - Q'_2) = n'L'_1 M'_1 [p'P + q'Q], \\ 2^2P''_2 = \dfrac{n'n''}{l''_1 m''_1}[p''_1P - q''_1Q], \\ 2^2(P''_2 - Q_2) = n'n''L''_1 M''_1 [p''P + q''Q], \\ \text{etc.} \end{cases}$$

Iam vero ad quantitates p, q computandas habemus has formulas ex (1.) et (2.) sequentes:

$$p = 1, \quad q = 0, \quad p_1 = 1, \quad q_1 = 1,$$

$$10.\quad \begin{cases} 2p' = p(1+c) - p_1 = c, \\ 2p'_1 = [p_1 - p(1-c)]m'_1 = cm'_1, \\ 2q' = q(1+c) + q_1 = 1, \\ 2q'_1 = [q_1 + q(1-c)]m'_1 = m'_1, \\ 2p'' = p'(1+c') - p'_1 = c\left[\dfrac{1+c'-m'_1}{2}\right], \\ 2p''_1 = [p'_1 - p'(1-c)]m''_1 = cm''_1\left[\dfrac{m'_1 - 1 + c'}{2}\right], \\ 2q'' = q'(1+c') + q'_1 = \dfrac{1+c'+m'_1}{2}, \\ 2q''_1 = [q'_1 + q'(1-c')]m''_1 = m''_1\left[\dfrac{m'_1 + 1 - c'}{2}\right], \\ \text{etc.} \end{cases}$$

ita ut habeamus post n transformationes:

$$11.\quad \begin{cases} 2p^{(n)} = p^{(n-1)}(1+c^{(n-1)}) - p_1^{(n-1)}, \\ 2p_1^{(n)} = [p_1^{(n-1)} - p^{(n-1)}(1-c^{(n-1)})]\, m_1^{(n)}, \\ 2q^{(n)} = q^{(n-1)}(1+c^{(n-1)}) + q_1^{(n-1)}, \\ 2q_1^{(n)} = [q_1^{(n-1)} + q^{(n-1)}(1-c^{(n-1)})]\, m_1^{(n)}, \end{cases}$$

quae formulae cum his ad computationem aptioribus commutari possunt:

$$12.\quad \begin{cases} 2p^{(n)} = 2p^{(n-1)} - p_1^{(n-1)}, \\ 2q^{(n)} = 2q^{(n-1)} - q_1^{(n-1)}, \\ 2p_1^{(n)} = p_1^{(n-1)} m_1^{(n)} \left[1 - \frac{p^{(n-1)}}{p_1^{(n-1)}}(1 - c_1^{(n-1)})\right], \\ 2q_1^{(n)} = q_1^{(n-1)} m_1^{(n)} \left[1 + \frac{q^{(n-1)}}{q_1^{(n-1)}}(1 - c_1^{(n-1)})\right], \end{cases}$$

quarum ultimae denique in has abeunt:

$$13.\quad 2p_1^{(n)} = p_1^{(n-1)} m_1^{(n)}, \qquad 2q_1^{(n)} = q_1^{(n-1)} m_1^{(n)}.$$

Si vero alteram numeratoris formam expeditissime computare adhibere velimus, ponamus:

$$14.\quad \begin{cases} R_1 = rR - sS, & 2R_1' = n'(r'R - s'S), & 2^2 R_1'' = n'n''(r''R - s''S), \\ S_1 = s_1 S - r_1 R, & 2S_1' = n'(s_1' S - r_1' R), & 2^2 S_1'' = n'n''(s_1'' S - r_1'' R), \end{cases}$$

unde sequuntur ex formulis (6.) hae:

$$15.\quad \begin{cases} R_2 = \frac{1}{l_1 m_1}(s_1 S - r_1 R), \\ S_2 = L_1 M_1 (rR - sS), \\ 2R_2' = \frac{n'}{l_1' m_1'}(s_1' S - r_1' R), \\ 2S_2' = n' L_1' M_1' (r'R - s'S), \\ 2^2 R_2'' = \frac{n_1' n_1''}{l_1'' m_1''}(s_1'' S - r_1'' R), \\ 2^2 S_2'' = n'n'' L_1'' M_1'' (r''R - s''S), \\ \text{etc.} \end{cases}$$

Coefficientes vero r, s, etc. ex his formulis, quae ex form. (5.) sequuntur, determinantur:

$$r = 1, \quad s = 0, \quad r_1 = 0, \quad s_1 = 1,$$

$$16.\quad \begin{cases} 2r' = r(1+c) + r_1 = 1+c, \\ 2r_1' = [r(1-c) + r_1]\, m_1' = (1-c)\, m_1', \\ 2s' = s(1+c) + s_1 = 1, \\ 2s_1' = [s(1-c) + s_1]\, m_1' = m_1', \end{cases}$$

$$16.\quad \begin{cases} 2r'' = r'(1+c')+r'_1 = \dfrac{(1+c)(1+c')+(1-c)m'_1}{2}, \\ 2r''_1 = [r'(1-c')+r'_1]\,m''_1 = \left(\dfrac{(1+c)(1-c')+(1-c)m'_1}{2}\right)m''_1, \\ 2s'' = s'(1+c')+s'_1 = \dfrac{(1+c')+m'_1}{2}, \\ 2s''_1 = [s'(1-c')+s'_1]\,m''_1 = \left(\dfrac{(1-c')+m'_1}{2}\right)m''_1, \\ \text{etc.} \end{cases}$$

ita ut post n transformationes habeamus:

$$17.\quad \begin{cases} 2r^{(n)} = r^{(n-1)}(1+c^{(n-1)})+r_1^{(n-1)}, \\ 2r_1^{(n)} = [r^{(n-1)}(1-c^{(n-1)})+r_1^{(n-1)}]\,m_1^{(n)}, \\ 2s^{(n)} = s^{(n-1)}(1+c^{(n-1)})+s_1^{(n-1)}, \\ 2s_1^{(n)} = [s^{(n-1)}(1-c^{(n-1)})+s_1^{(n-1)}]\,m_1^{(n)}; \end{cases}$$

quae formulae mox in has abeunt, computatu faciliores:

$$18.\quad \begin{cases} 2r^{(n)} = 2r^{(n-1)}+r_1^{(n-1)}, \\ 2s^{(n)} = 2s^{(n-1)}+s_1^{(n-1)}, \\ 2r_1^{(n)} = r_1^{(n-1)}m_1^{(n)}\left[1+\dfrac{r^{(n-1)}}{r_1^{(n-1)}}(1-c^{(n-1)})\right], \\ 2s_1^{(n)} = s_1^{(n-1)}m_1^{(n)}\left[1+\dfrac{s^{(n-1)}}{s_1^{(n-1)}}(1-c^{(n-1)})\right], \end{cases}$$

quarum ultimae denique fiunt:

$$19.\quad 2r_1^{(n)} = r_1^{(n-1)}m_1^{(n)}, \qquad 2s_1^{(n)} = s_1^{(n-1)}m_1^{(n)}.$$

Ex theorematibus (4.) et (7.) patet, terminos serierum

$$20.\quad \begin{cases} np, & nn'p', & nn'n''p'', & \text{etc.} \\ nq, & nn'q', & nn'n''q'', & \text{etc.} \\ \dfrac{np_1}{l_1m_1}, & \dfrac{nn'p'_1}{l'_1m'_1}, & \dfrac{nn'n''p''}{l''_1m''_1}, & \text{etc.} \\ \dfrac{nq_1}{l_1m_1}, & \dfrac{nn'q'_1}{l'_1m'_1}, & \dfrac{nn'n''q''}{l''_1m''_1}, & \text{etc.} \end{cases}$$

ut certos quosdam limites finitos appropinquare, quos respective per Π, K, Π_1, K_1 significabimus. Eodem modo termini serierum quatuor

$$21.\quad \begin{cases} nr, & nn'r', & nn'n''r'', & \text{etc.} \\ ns, & nn's', & nn'n''s'', & \text{etc.} \\ \dfrac{nr_1}{l_1m_1}, & \dfrac{nn'r'_1}{l'_1m'_1}, & \dfrac{nn'n''r''_1}{l''_1m''_1}, & \text{etc.} \\ \dfrac{ns_1}{l_1m_1}, & \dfrac{nn's'_1}{l'_1m'_1}, & \dfrac{nn'n''s''}{l''_1m''_1}, & \text{etc.} \end{cases}$$

ad certos quosdam limites accedunt, quos per P, Σ, P_1, Σ_1, significabimus.

Iam vero aggrediamur ad numeratores in altera transformatione aeque tractandos. Ex formulis (39.) et (42.) art. XI. hae formulae deducuntur, pro forma $\Pi\cos\Phi^2 + K\sin\Phi^2$ numeratoris valentes:

$$21. \quad \begin{cases} \Pi^0_1 = \dfrac{2}{(1+m_1)(m+c_1 l_1)}\left[\dfrac{m_1}{2}\Pi_1 + \tfrac{1}{2}K_1\right], \\ \Pi^0_1 - K^0_1 = \dfrac{2}{(1+m_1)(m_1+c_1 l_1)}\left[\dfrac{m_1}{2}\Pi_1 - \tfrac{1}{2}K_1\right] c^0, \end{cases}$$

$$22. \quad \Pi^0_2 = \frac{\Pi^0_1 - K^0_1}{l^0 c^0}, \qquad K^0_2 = \frac{\Pi^0_1 m^{0\,2}_1 - K^0_1}{l^0 c^0}.$$

Eodem modo pro altera forma numeratoris $(P - \Sigma\sin^2\Phi)$ ex form. (50.) et (51.) art. XI. hae sequuntur formulae:

$$23. \quad \begin{cases} P^0_1 = \dfrac{2}{(1+m_1)(m_1+c_1 l_1)}\left[\dfrac{1+m_1}{2}P_1 - \tfrac{1}{2}\Sigma_1\right], \\ \Sigma^0_1 = \dfrac{2}{(1+m_1)(m_1+c_1 l_1)}\left[\tfrac{1}{2}\Sigma - \dfrac{1-m_1}{2}P_1\right] c^0, \end{cases}$$

$$24. \quad P^0_2 = \frac{\Sigma^0_1}{c^0 l^0}, \qquad \Sigma^0_2 = P^0_1 \mathfrak{t}^0 \mathfrak{l}^0.$$

Rursus hic theoremata habemus haec:

„Quantitates:

$$„24. \quad \begin{cases} 2^{(n)}\Pi^{(n)}_2, & 2^{(n)}K^{(n)}_2, & 2^n \dfrac{\Pi^{(n)}_1 - K^{(n)}_1}{(c^{(n)})^2}, \\ 2^n\ \Pi^{(n)}_1, & 2^n\ K^{(n)}_1, & 2^n \dfrac{\Pi^{(n)}_2 - K^{(n)}_2}{(\mathfrak{t}^{(n)})^2}, \\ 2^n P^{(n)}_2, & \dfrac{2^n}{(c^{(n)})^2}\Sigma^{(n)}_1, & \\ 2^n P^{(n)}_1, & \dfrac{2^n}{(\mathfrak{t}^{(n)})^2}\Sigma^{(n)}_2, & \end{cases}$$

„ubi per indicem n adiectum numerus transformationum repetitarum de-„signatur, numero n crescente ad certos quosdam limites finitos accedunt."

Ad numeratores facile rursus comparandos ponamus rursus:

$$25. \quad \begin{cases} \nu = 1, & \Pi_1 = \pi\Pi + kK, \quad \text{vel:} & P_1 = \varrho P - \sigma\Sigma, \\ & \Pi_1 - K_1 = \pi_1\Pi - k_1 K, & \Sigma_1 = \sigma_1\Sigma - \varrho_1 P, \\ \nu^0 = \dfrac{4}{(1+m_1)(m_1+c_1 l_1)}, & 2\ \Pi^0_1 = \nu^0(\pi^0\Pi + k^0 K), & 2P^0_1 = \nu^0(\varrho^0 P - \sigma^0\Sigma), \\ & 2(\Pi^0_1 - K^0_1) = \nu^0(\pi^0_1\Pi - k^0_1 K), & 2\Sigma^0_1 = \nu^0(\sigma^0_1\Sigma - \varrho^0_1 P), \\ \nu^{00} = \dfrac{4}{(1+m^0_1)(m^0_1+c^0_1 l^0_1)}, & 2^2\ \Pi^{00}_1 = \nu^0\nu^{00}(\pi^{00}\Pi + k^{00}K), & 2P^{00}_1 = \nu^0\nu^{00}(\varrho^{00}P - \sigma^{00}\Sigma), \\ & 2^2(\Pi^{00}_1 - K^{00}_1) = \nu^0\nu^{00}(\pi^{00}_1\Pi - k^{00}_1 K), & 2\Sigma^{00}_1 = \nu^0\nu^{00}(\sigma^{00}_1\Sigma - \varrho^{00}_1 P), \\ & \text{etc.} & \text{etc.} \end{cases}$$

unde formulis adhibitis sponte hae prodeunt formulae:

$$26.\quad\begin{cases}
\Pi_2 = \frac{1}{lc}(\pi_1\Pi - k_1\mathrm{K}), & \mathrm{P}_2 = \frac{1}{lc}(\sigma_1\Sigma - \varrho_1\mathrm{P}),\\
\Pi_2 - \mathrm{K}_2 = \mathfrak{k}\mathfrak{l}(\pi\Pi + k\mathrm{K}), & \Sigma_2 = \mathfrak{k}\mathfrak{l}(\varrho\mathrm{P} - \sigma\Sigma),\\
2\,\Pi_2^0 = \frac{\nu^0}{l^0 c^0}(\pi_1^0\Pi + k_1^0\mathrm{K}), & \mathrm{P}_2^0 = \frac{\nu^0}{l^0 c^0}(\sigma_1^0\Sigma - \varrho_1^0\mathrm{P}),\\
2(\Pi_2^0 - \mathrm{K}_2^0) = \nu^0\mathfrak{k}^0\mathfrak{l}^0(\pi^0\Pi + k^0\mathrm{K}), & \Sigma_2^0 = \nu^0\mathfrak{k}^0\mathfrak{l}^0(\varrho^0\mathrm{P} - \sigma^0\Sigma),\\
2^2\,\Pi_2^{00} = \frac{\nu^0\nu^{00}}{l^{00}c^{00}}(\pi_1^{00}\Pi - k_1^{00}\mathrm{K}), & \mathrm{P}_2^{00} = \frac{\nu^0\nu^{00}}{l^{00}c^{00}}(\sigma_1^{00}\Sigma - \varrho_1^{00}\mathrm{P}),\\
2^2(\Pi_2^{00} - \mathrm{K}_2^{00}) = \nu^0\nu^{00}\mathfrak{k}^{00}\mathfrak{l}^{00}(\pi^{00}\Pi + k^{00}\mathrm{K}), & \Sigma_2^{00} = \nu^0\nu^{00}\mathfrak{k}^{00}\mathfrak{l}^{00}(\varrho^{00}\mathrm{P} - \sigma^{00}\Sigma),\\
\text{etc.} & \text{etc.}
\end{cases}$$

Coefficientes vero π, k, ex his formulis computantur:

$$27.\quad\begin{cases}
\pi = 1, \quad k = 0, \quad \pi_1 = 1, \quad k_1 = 1,\\
2\pi^0 = \pi(1+m_1) - \pi_1 = m_1,\\
2\pi_1^0 = [\pi_1 - \pi(1-m_1)]\,c^0 = m_1 c^0,\\
2k^0 = k(1+m_1) + k_1 = 1,\\
2k_1^0 = [k_1 + k(1-m_1)] - m_1,\\
\text{etc.}
\end{cases}$$

et generaliter:

$$2\pi^{(n)} = \pi^{(n-1)}(1+m_1^{(n-1)}) - \pi_1^{(n-1)},$$
$$2\pi_1^{(n)} = [\pi^{(n-1)} - \pi_1^{(n-1)}(1-m_1^{(n-1)})]\,c^{(n)},$$
$$2k^{(n)} = k^{(n-1)}(1+m_1^{(n-1)}) + k_1^{(n-1)},$$
$$2k_1^{(n)} = [k_1^{(n-1)} + k^{(n-1)}(1-m^{(n-1)})]\,c^{(n)},$$

quae formulae mox in has simplices abeunt:

$$28.\quad\begin{cases}
2\pi^{(n)} = 2\pi^{(n-1)} - \pi_1^{(n-1)},\\
2\,k^{(n)} = 2\,k^{(n-1)} - k_1^{(n-1)},\\
2\pi_1^{(n)} = \pi_1^{(n-1)}\,c^{(n)},\\
2\,k_1^{(n)} = k_1^{(n-1)}\,c^{(n)}.
\end{cases}$$

Contra quantitates ϱ, σ, in altera forma nominatorum computandorum adhibita, his formulis calculantur:

$$29.\quad\begin{cases}
\varrho = 1, \quad \sigma = 0, \quad \varrho_1 = 0, \quad \sigma_1 = 1,\\
2\varrho^0 = (1+m_1)\,\varrho + \varrho_1,\\
2\varrho_1^0 = ((1-m_1)\,\varrho + \varrho_1)\,c^0,\\
2\sigma^0 = (1+m_1)\sigma + \sigma_1,\\
2\sigma_1^0 = [(1-m_1)\sigma + \sigma_1]\,c^0,\\
\text{etc.}
\end{cases}$$

et generaliter:

$$2\varrho^{(n)} = \varrho^{(n-1)}(1+m_1^{(n-1)})+\varrho_1^{(n-1)},$$
$$2\varrho_1^{(n)} = [\varrho^{(n-1)}(1-m_1^{(n-1)})+\varrho_1^{(n-1)}]c^0,$$
$$2\sigma^{(n)} = \sigma^{(n-1)}(1+m_1^{(n-1)})+\sigma_1^{(n-1)},$$
$$2\sigma_1^{(n)} = \sigma^{(n-1)}(1-m_1^{(n-1)})+\sigma_1^{(n-1)},$$

quae formulae post nonnullas transformationes in has abeunt:

$$2\varrho^{(n)} = 2\varrho^{(n-1)}+\varrho_1^{(n-1)}, \qquad 2\varrho_1^{(n)} = \varrho_1^{(n-1)}m_1^{(n)},$$
$$2\sigma^{(n)} = 2\sigma^{(n-1)}+\sigma_1^{(n-1)}, \qquad 2\sigma_1^{(n)} = \sigma_1^{(n-1)}m_1^{(n)}.$$

Ex theoremate (24.) prodit hoc, „termini serierum:

$$„\nu\pi,\quad \nu\nu^0\pi^0,\quad \nu\nu^0\nu^{00}\pi^{00},\ \ldots \qquad „\nu\varrho,\quad \nu\nu^0\varrho,\quad \nu\nu^0\nu^{00}\varrho^{00},\ \ldots$$
$$„\nu k,\quad \nu\nu^0 k^0,\quad \nu\nu^0\nu^{00}k^{00},\ \ldots \qquad „\nu\sigma,\quad \nu\nu^0\sigma,\quad \nu\nu^0\nu^{00}\sigma^{00},\ \ldots$$

„ad certos finitos limites appropinquant, quos in sequentibus per

$$„P,\quad K,\quad R,\quad S,$$

„designabimus. Eodem modo termini serierum:

$$„\frac{\nu\pi}{lc},\quad \frac{\nu\nu^0\pi_1^0}{l^0c^0},\quad \frac{\nu\nu^0\nu^{00}\pi_1^{00}}{l^{00}c^{00}},\ \ldots \qquad „\frac{\nu\varrho_1}{lc},\quad \frac{\nu\nu^0\varrho_1^0}{l^0c^0},\quad \frac{\nu\nu^0\nu^{00}\varrho_1^{00}}{l^{00}c^{00}},\ \ldots$$
$$„\frac{\nu k_1}{lc},\quad \frac{\nu\nu^0 k_1^0}{l^0c^0},\quad \frac{\nu\nu^0\nu^{00}k_1^{00}}{l^{00}c^{00}},\ \ldots \qquad „\frac{\nu\sigma_1}{lc},\quad \frac{\nu\nu^0\sigma_1^0}{l^0c^0},\quad \frac{\nu\nu^0\nu^{00}\sigma_1^{00}}{l^{00}c^{00}},\ \ldots$$

„ad certos denique limites respective:

$$„P_1,\quad K_1,\quad R_1,\quad S_1,$$

„appropinquant." Quae theoremata etiam directe, mutatis mutandis, ex similibus in altera transformatione deducere licet.

XV.

Quomodo limites argumentorum, integraliaque indefinita et definita ipsa computentur.

Revocemus formulas (44.) et (45.) articuli XI., quibus limites Φ_1^0 et Φ_2^0, ψ_1^0 et ψ_2^0 determinandi erant, ut satisfieret aequationibus integralibus (40.) vel (48.) eiusdem articuli.

Ponamus brevitatis gratia hîc et in sequentibus:

$$1.\quad \sqrt{\left(\frac{(1-m^2\sin^2\varphi)\left(1-\frac{m_1^2-c_1^2l_1^2}{m_1^2}\sin^2\varphi\right)}{[1-(1-c_1^2l_1^2)\sin^2\varphi]}\right)} = D(\varphi, c, l, m),$$

quae quantitas simul cum m_1 ad unitatem appropinquat. Qua denotatione adhibita, ex formulis (44.) facilibus reductionibus factis prodeunt, hae formulae:

$$2.\quad \sin\Phi_1^0 = (1+m_1)\left(1+\frac{c_1l_1}{m_1}\right)\frac{\sin\varphi\cos\varphi}{\sqrt{(\cos^2\varphi+c_1^2l_1^2\sin^2\varphi)}}\cdot\frac{1}{1+D(\varphi,c,l,m)},$$

$$3.\quad \sin\Phi_2^0 = lc\quad\left(1+\frac{l_1f_1}{m_1}\right)\frac{\sin\varphi\cos\varphi}{\sqrt{(\cos^2\varphi+c_1^2l_1^2\sin^2\varphi)}}\cdot\frac{1}{1+D(\varphi,c,l,m)} = l^0\sin\Phi_1^0.$$

Iam vero duos casus, quibus utraque aequat. in (40.) vel in (48.) discernitur, accuratius disquiramus. Primum igitur posito:

$$4.\qquad \Phi < \arcsin \frac{1}{\sqrt{(1+c_1 l_1)}} > 0,$$

et minimis positivis valoribus ipsorum Φ_1^0 et Φ_2^0 e numero eorum, qui formulis (2.) et (3.) respondent, assumtis, habemus hanc aequat. integralem:

$$5.\qquad \int_0^{\varphi} \frac{(\Pi\cos^2\varphi + \mathrm{K}\sin^2\varphi)\,d\varphi}{\sqrt{(\Delta(c,l,m))}}$$
$$= \int_0^{\varphi_1^0} \frac{(\Pi_1^0\cos^2\varphi + \mathrm{K}_1^0\sin^2\varphi)\,d\varphi}{\sqrt{(\Delta(c^0,l^0,m^0))}} + \int_0^{\varphi_2^0} \frac{(\Pi_2^0\cos^2\varphi + \mathrm{K}_2^0\sin^2\varphi)\,d\varphi}{\sqrt{(\Delta(\mathfrak{k}^0,\mathfrak{l}^0,m^0))}}.$$

Tum vero posito:

$$6.\qquad \Phi > \arcsin \frac{1}{\sqrt{(1+c_1 l_1)}} < \frac{\pi}{2},$$

integrali dato in duas partes diviso, habemus:

$$7.\qquad \int_0^{\psi} \frac{(\Pi\cos^2\varphi + \mathrm{K}\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)}$$
$$= \int_0^{\arcsin\frac{1}{\sqrt{(1+c_1 l_1)}}} \frac{(\Pi\cos^2\varphi + \mathrm{K}\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)} + \int_{\arcsin\frac{1}{\sqrt{(1+c_1 l_1)}}}^{\psi} \frac{(\Pi\cos^2\varphi + \mathrm{K}\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)},$$

quarum priori per priorem, posteriori per posteriorem aequationem (40.) art. XI. transformata, habebimus:

$$8.\qquad \int_0^{\psi} \frac{(\Pi\cos^2\varphi + \mathrm{K}\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)}$$
$$= \int_0^{\frac{\pi}{2}} \frac{(\Pi_1^0\cos^2\varphi + \mathrm{K}_1^0\sin^2\varphi)\,d\varphi}{\Delta(c^0,l^0,m^0)} + \int_0^{\arcsin\frac{m^0}{\mathfrak{k}^0}} \frac{(\Pi_2^0\cos^2\varphi + \mathrm{K}_2^0\sin^2\varphi)\,d\varphi}{\Delta(\mathfrak{k}^0,\mathfrak{l}^0,m^0)}$$
$$- \int_{\frac{\pi}{2}}^{\psi_1^0} \frac{(\Pi_1^0\cos^2\varphi + \mathrm{K}_1^0\sin^2\varphi)\,d\varphi}{\Delta(c^0,l^0,m^0)} + \int_{\arcsin=\frac{m^0}{\mathfrak{k}^0}}^{\psi_2^0} \frac{(\Pi_2^0\cos^2\varphi + \mathrm{K}_2^0\sin^2\varphi)\,d\varphi}{\Delta(\mathfrak{k}^0,\mathfrak{l}^0,m^0)}.$$

Posteriora duo integralia termini secundi in unum hoc transeunt:

$$\int_0^{\psi_2^0} \frac{(\Pi_1^0\cos^2\varphi + \mathrm{K}_1^0\sin^2\varphi)\,d\varphi}{\Delta(\mathfrak{k}^0,\mathfrak{l}^0,m^0)}.$$

In prioribus vero integralibus novis loco argumenti Φ, posito $\pi-\Phi$, post faciles reductiones, si nunc per ψ_1^0, angulum $\pi-\psi_1^0$, qui igitur, maior proximus formulae (2.) respondentium, plerumque ad angulum 2Φ propius accedit, denotamus, habemus ex aequat. (8.) hanc aequationem eadem forma ac (5.) gaudentem:

$$9.\quad \int_0^{\psi}\frac{(\Pi\cos^2\varphi+K\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)}$$
$$=\int_0^{\psi_1^0}\frac{(\Pi_1^0\cos^2\varphi+K_1^0\sin^2\varphi)\,d\varphi}{\Delta(c^0,l^0,m^0)}+\int_0^{\psi_2^0}\frac{(\Pi_2^0\cos^2\varphi+K_2^0\sin^2\varphi)\,d\varphi}{\Delta(\mathfrak{k}^0,\mathfrak{l}^0,m^0)}.$$

Si porro $\varPhi>\frac{\pi}{2}$ et $<\pi$ fuerit, integrale datum ita representare licet, posito $\varPhi=\pi-{}'\varPhi$:

$$10.\quad \int_0^{\varphi}\frac{(\Pi\cos^2\varphi+K\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)}$$
$$=\int_0^{\pi}\frac{(\Pi\cos^2\varphi+K\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)}-\int_0^{'\varphi}\frac{(\Pi\cos^2\varphi+K\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)}.$$

Huius aequationis termino ultimo per formulam (9.), quia argumentum ${}'\varPhi$ minor quam $\frac{\pi}{2}$ est, transformato, atque hac aequatione, quae ex eadem aequatione (9.), $\psi=\frac{\pi}{2}$ posito, sequitur:

$$11.\quad \int_0^{\pi}\frac{(\Pi\cos^2\varphi+K\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)}=\int_0^{2\pi}\frac{(\Pi_1^0\cos^2\varphi+K_1^0\sin^2\varphi)\,d\varphi}{\Delta(c^0,l^0,m^0)}$$

adhibita, fit:

$$12.\quad \int_0^{\varphi}\frac{(\Pi\cos^2\varphi+K\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)}$$
$$=\int_0^{2\pi}\frac{(\Pi_1^0\cos^2\varphi+K_1^0\sin^2\varphi)\,d\varphi}{\Delta(c^0,l^0,m^0)}-\int_0^{\varphi_1^0}\frac{(\Pi_1^0\cos^2\varphi+K_1^0\sin^2\varphi)\,d\varphi}{\Delta(c^0,l^0,m^0)}$$
$$-\int_0^{\varphi_2^0}\frac{(\Pi_2^0\cos^2\varphi+K_2\sin^2\varphi)}{\Delta(\mathfrak{k}^0,\mathfrak{l}^0,m^0)},$$

ubi aut $\varPhi_1^0$ aut $(\pi-\varPhi_1^0)$ minimus positivus angulus formulae (2.) respondentium est, prout fuerit:

$$\text{aut,}\ \sin\varPhi<\frac{1}{\sqrt{(1+c_1 l_1)}}=\sin{}'\varPhi,\quad \text{aut,}\ \sin\varPhi>\frac{1}{\sqrt{(1+c_1 l_1)}}=\sin{}'\varPhi.$$

Ponamus vero, ut rursus hic ad formam (5.) et (9.) revertamus, loco ipsius:

$$13.\qquad \varPhi_1^0\ \ldots\ldots\ 2\pi-\varPhi_1^0,\qquad \varPhi_2^0\ \ldots\ldots\ -\varPhi_2^0,$$

quo facto ad aequationem rursus devehimur hanc:

$$14.\quad \int_0^{\varphi}\frac{(\Pi\cos^2\varphi+K\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)}$$
$$=\int^{\varphi_1^0}\frac{(\Pi_1^0\cos^2\varphi+K_1^0\sin^2\varphi)\,d\varphi}{\Delta\, c^0\, l^0\, m^0}+\int^{\varphi_2^0}\left(\frac{\Pi_2^0\cos^2\varphi}{\Delta\,\mathfrak{k}^0\,\mathfrak{l}^0\,m^0}\right).$$

Qua consideratione continuata sequitur aequationem integralem (5.) semper stare, si limites determinantur, quales in sequenti tabula expositos invenis.

Ponamus esse:

$$\delta,\quad \varphi_1^0,\quad \varphi_2^0,$$

minimos angulos positivos, quorum quadrata sinium et ex formulis (2.) et (3.) et ita determinentur:

$$\sin^2\delta = \frac{1}{(1+c_1 l_1)}.$$

$$15.\quad \begin{cases} \text{Posito: } \arg\varphi\,(>0<\delta), & (>\delta<\tfrac{\pi}{2}), & (>\tfrac{\pi}{2}<\pi-\delta), & (>\pi-\delta<\pi). \\ \text{Erit: } \begin{cases} \arg\varphi_1^0 = \varphi_1^0, \\ \arg\varphi_2^0 = \varphi_2^0, \end{cases} & \begin{matrix} =\pi-\varphi_1^0, \\ =\varphi_2^0, \end{matrix} & \begin{matrix} \pi+\varphi_1^0, \\ -\varphi_2^0, \end{matrix} & \begin{matrix} 2\pi-\varphi_1^0, \\ -\varphi_2^0, \end{matrix} \end{cases}$$

$$\begin{matrix} (>\pi<\pi+\delta), & (>\pi+\delta<\tfrac{3\pi}{2}), & (>3\tfrac{\pi}{2}<2\pi-\delta), & (>2\pi-\delta<2\pi), \\ 2\pi+\varphi_1^0, & 3\pi-\varphi_1^0, & 3\pi+\varphi_1^0, & 4\pi-\varphi_1^0, \\ \varphi_2^0, & \varphi_2^0, & -\varphi_2^0, & -\varphi_2^0, \end{matrix}$$

etc.

$$\begin{matrix} (>h\pi<h\pi+\delta), & (>h\pi+\delta<(h+\tfrac{1}{2})\pi), & (>h+\tfrac{1}{2}\pi<(h+1)\pi-\delta), & (>(h+1)\pi-\delta<(h+1)\pi), \\ 2h\pi+\varphi_1^0, & (2h+1)\pi-\varphi_1^0, & (2h+1)\pi+\varphi_1^0, & (2h+2)\pi-\varphi_1^0, \\ \varphi_2^0, & \varphi_2^0, & -\varphi_2^0, & -\varphi_2^0, \end{matrix}$$

ubi h quilibet numerus integer est. Prorsus similemque tabulam pro negativis valoribus anguli φ adipiscimur. Quia vero formula (2.) docet angulum φ_0' circiter duplicem esse anguli φ', inde patet, ut verum valorem argumentorum φ_1^0 et φ_2^0 adipiscamur, eum plerumque mox assumendum esse angulum φ_1^0, qui angulo 2φ proximus sit, angulumque φ_2^0 minimum esse ex formula:

$$\sin\varphi_2^0 = l^0 \sin\varphi_1^0$$

prodeuntem positivum vel negativum.

His positis, quomodo transformationem propositam continuemus, disquisitio sponte emanat. Quo facto, videmus primum, limitem δ, quo antea utraque aequatio integralis discernebatur,

$$16.\qquad \delta = \arc\sin = \frac{1}{\sqrt{(1+c_1 l_1)}},$$

appropinquare ad $\frac{\pi}{4}$. Tum videmus angulum φ_1^0, si $m=0$, et $l_1 = c_1$ devenerit ex formulis:

$$17.\qquad \sin\varphi_1^0 = 1 + c_1^2\frac{\sin\varphi\cos\varphi}{\sqrt{(1-c_1^4\sin^2\varphi)}},\qquad \varphi_2^0 = c^0\sin\varphi_1^0$$

determinari, atque integralia in hanc formam abire:

$$\int_0^{\varphi_1}\frac{(\Pi_1\cos^2\varphi+K_1\sin^2\varphi)\,d\varphi}{(1-c^2\sin^2\varphi)},\qquad \int_0^{\varphi_2}\frac{(\Pi_2\cos^2\varphi+K_2\sin^2\varphi)\,d\varphi}{(1-\mathfrak{k}^2\sin^2\varphi)}.$$

Postremo vero si adhuc c evanuerit, erit

$$18. \qquad \Phi_1^0 = 2\Phi, \qquad \Phi_2^0 = 0.$$

Inde generalis argumentorum lex memorabilis prodit, quam ut explicemus, totamque transformationem brevi in conspectu ponamus, hanc denotationem in art. XI. propositam atque confirmatam adhibere velimus.

Ponamus datum angulum numeratoris coefficientes et modulos respective esse:

$$19. \left\{\begin{array}{llll}
 & \Phi, \ldots\ldots\ldots\ldots & \Pi,\ K, & c,\ l,\ m; \\
 & \text{post primam transformationem habebimus:} & & \\
\text{arg.} & & & \\
 & \Phi_1^0, \ldots\ldots\ldots\ldots & \Pi_1^0 + K_1^0, & c^0,\ l^0,\ m^0; \\
 & \Phi_2^0, \ldots\ldots\ldots\ldots & \Pi_2^0 + K_2^0, & \mathfrak{k}^0,\ \mathfrak{l}^0,\ m^0; \\
 & \text{post secundam transformationem habebimus:} & & \\
\text{arg.} & & & \\
 & \Phi_{1,1}^{00},\ \Phi_{2,2}^{00}, \ldots\ldots\ldots & \Pi_1^{00}\ K_1^{00}, & c^{00},\ l^{00},\ m^{00}; \\
 & \Phi_{1,2}^{00},\ \Phi_{2,1}^{00}, \ldots\ldots\ldots & \Pi_2^{00}\ K_2^{00}, & \mathfrak{k}^{00},\ \mathfrak{l}^{00},\ m^{00}; \\
 & \text{post tertiam transformationem habebimus:} & & \\
\text{arg.} & & & \\
 & \Phi_{1,1,1}^{000},\ \Phi_{1,2,2}^{000},\ \Phi_{2,2,1}^{000},\ \Phi_{2,1,2}^{000}, \ldots & \Pi_1^{000}\ K_1^{000}, & c^{000},\ l^{000},\ m^{000}; \\
 & \Phi_{1,1,2}^{000},\ \Phi_{1,2,1}^{000},\ \Phi_{2,2,2}^{000},\ \Phi_{2,1,1}^{000}, \ldots & \Pi_2^{000}\ K_2^{000}, & \mathfrak{k}^{000},\ \mathfrak{l}^{000},\ m^{000}; \\
 & & \text{etc.} &
\end{array}\right.$$

ubi indices apud argumenta Φ infra adiecti originem eorum ex antecedentibus argumentis accuratissime indicat. Iam vero sequitur hoc theorema: „Termini serierum *horizontalium* angulorum

$$\begin{array}{lllllllllll}
\Phi, & \frac{\varphi_1^0}{2}, & \frac{\varphi_{1,1}^{00}}{4}, & \frac{\varphi_{1,1,1}^{000}}{8}, & \frac{\varphi_{1,1,1,1}^{0000}}{16} & \text{etc.} & \frac{\varphi_{1,2,2}^{000}}{8}, & \frac{\varphi_{1,2,2,1}^{0000}}{16} & \text{etc.} & \frac{\varphi_{1,1,2,2}^{0000}}{16} & \text{etc.} \\
 & \frac{\varphi_2^0}{2}, & \frac{\varphi_{2,1}^{00}}{4}, & \frac{\varphi_{2,1,1}^{000}}{8}, & \frac{\varphi_{2,1,1,1}^{0000}}{16} & \text{etc.} & \frac{\varphi_{2,2,2}^{000}}{8}, & \frac{\varphi_{2,2,2,1}^{0000}}{16} & \text{etc.} & \frac{\varphi_{2,1,2,2}^{0000}}{16} & \text{etc.} \\
 & & \frac{\varphi_{1,2}^{00}}{4}, & \frac{\varphi_{1,2,1}^{000}}{8}, & \frac{\varphi_{1,2,1,1}^{0000}}{16} & \text{etc.} & & \frac{\varphi_{1,1,1,2}^{0000}}{16} & \text{etc.} & \frac{\varphi_{1,2,2,2}^{0000}}{16} & \text{etc.} \\
 & & \frac{\varphi_{2,2}^{00}}{4}, & \frac{\varphi_{2,2,1}^{000}}{8}, & \frac{\varphi_{2,2,1,1}^{0000}}{16} & \text{etc.} & & \frac{\varphi_{2,1,1,2}^{0000}}{16} & \text{etc.} & \frac{\varphi_{2,2,2,2}^{0000}}{16} & \text{etc.} \\
 & & & \frac{\varphi_{1,1,2}^{000}}{8}, & \frac{\varphi_{1,1,2,1}^{0000}}{16} & \text{etc.} & & \frac{\varphi_{1,2,1,2}^{0000}}{16} & \text{etc.} & & \\
 & & & \frac{\varphi_{2,1,2}^{000}}{8}, & \frac{\varphi_{2,1,2,1}^{0000}}{16} & \text{etc.} & & \frac{\varphi_{2,2,1,2}^{0000}}{16} & \text{etc.} & &
\end{array}$$

ad certos denique limites appropinquant, et adeo, si post n transformationes $l^{(n)}$ evanuerit, ii anguli, quorum $n$tus index 2 est, ipsi evanescant.

Ponamus illos limites respective:

19. $$= \Phi,\quad \Phi_2,\quad \Phi_{1,2},\quad \Phi_{2,2},\quad \Phi_{1,1,2},\quad \Phi_{2,1,2},\quad \Phi_{1,2,2},\quad \Phi_{2,2,2}.$$

ita ut brevitatis gratia omnes indices (1.) sequentes omittantur. Quo theoremate adiuti, cum integrale quodque, quantitate $(\Pi - \mathrm{K})$ ex theor. (24.) art. XIV. simul cum c ad nihil appropinquante, ad $\Pi.\varphi$ denique revenire videamus, sequitur hoc

theorema.

Valor appropinquatus integralis indefiniti Abeliani primi ordinis:

$$\int_0^{\varphi} \frac{(\Pi\cos\varphi + \mathrm{K}\sin\varphi)\,d\varphi}{\sqrt{[1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}} =$$

erit:

20. $$\begin{cases} [\Phi + \Phi_{2,2} + \Phi_{2,1,2} + \Phi_{1,2,2} \text{ etc.}]\,(P\Pi + K\mathrm{K}) \\ + [\Phi_2 + \Phi_{1,2} + \Phi_{1,1,2} + \Phi_{2,2,2} \text{ etc.}]\,(P_1\Pi - K_1\mathrm{K}), \end{cases}$$

ubi P, K, $P_1 K_1$ limites in artic. XIV. determinati sunt. Eodem modo pro altera numeratoris forma hanc adipiscimur aequationem:

21. $$\begin{cases} \int_0^{\varphi} \frac{(\mathrm{P} - \Sigma\sin^2\varphi)\,d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}} = \\ [\Phi + \Phi_{2,2} + \Phi_{2,1,2} + \Phi_{1,2,2} \text{ etc.}]\,(R\mathrm{P} - S\Sigma) \\ + [\Phi + \Phi_{1,2} + \Phi_{1,1,2} + \Phi_{2,2,2} \text{ etc.}]\,(S_1\Sigma - R_1\mathrm{P}). \end{cases}$$

Animadvertatur in prima serie eos limites Φ stare, quorum indices parem numerum signi (2.), in secunda vero, quorum indices imparem numerum signi (2.) continent. In utraque serie hactenus progrediendum est, ut limites Φ decrescentes usque ad quaesitum decimalium numerum evanuerint. In his generalibus valoribus integralis indefiniti posito $\varphi = \frac{\pi}{2}$, omnes limites evanescunt, primo Φ excepto, qui in $\frac{\pi}{2}$ transit, ita ut habeamus:

22. $$\begin{cases} \int_0^{\frac{\pi}{2}} \frac{(\Pi\cos\varphi^2 + \mathrm{K}\sin^2\varphi)\,d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}} = (P\Pi + K\mathrm{K})\frac{\pi}{2}, \\ \int_0^{\frac{\pi}{2}} \frac{(\mathrm{P} - \Sigma\sin^2\varphi)\,d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}} = (R\mathrm{P} - S\Sigma)\frac{\pi}{2}. \end{cases}$$

Adnotemus adhuc has aequationes ex formulis (25.) art. XIV. prodeuntes:

23. $$F'(\pi k c l m) = \nu^0 F(\pi^0 k^0 c^0 l^0 m^0) = \nu^0 \nu^{00} F(\pi^{00} k^{00} c^{00} l^{00} m^{00}) \text{ etc.}$$

ubi brevitatis gratia posuimus.

$$F'(\pi k c l m) = \int_0^{\frac{\pi}{2}} \frac{(\pi\Pi + k\mathrm{K}) - (\pi_1\Pi + k_1\mathrm{K})\sin^2\varphi)\,d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}},$$

atque eodem modo:

$$24. \quad f'(\varrho\sigma c l m) = \nu^0 f'(\varrho^0 \sigma^0 c^0 l^0 m^0) = \nu^0 \nu^{00} f'(\varrho^{00} \sigma^{00} c^{00} l^{00} m^{00})$$

etc.

ubi positum est:

$$f'(\varrho\sigma c l m) = \int_0^{\frac{\pi}{2}} \frac{[(\varrho P - \sigma\Sigma) - (\sigma_1 \Sigma - \varrho_1 P)\sin^2\varphi)\,d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}}.$$

Quantitates π, k, ϱ, σ, in form. (27.) et (29.) explicantur.

Iam vero ad alterius aggrediamur transformationes argumenta accuratissime computanda. Quem ad finem revocemus formulas (15.) art. XI., quarum ope argumenta Φ'_1 et φ'_2 determinabantur, ita ut satisfacerent aequationibus integralibus (8.) vel (22.), quae nonnisi forma numeratoris inter se differunt. Ponamus hic brevitatis gratia:

$$25. \quad \frac{\sqrt{\left[(1-c^2\sin^2\varphi)\left(1-\frac{l^2 m^2}{c^2}\sin^2\varphi\right)\right]}}{\cos\varphi\sqrt{(1-l^2 m^2 \sin^2\varphi^2)}} = E(\Phi, c, l, m),$$

quam quantitatem, simul cum modulo c ad unitatem appropinquare patet. Facilibus reductionibus factis, ex his formulis (15.) art. XI.

$$26. \quad \frac{1-(1-l'_1 c'_1)\sin^2\varphi'_1}{1-(1+l'_1 c'_1)\sin^2\varphi'_1} = E(\Phi\, c, l, m) = \frac{1-(1-M'_1\sin^2\varphi'_2}{1-(1+M'_1\sin^2\varphi'_2},$$

hae prodeunt:

$$27. \quad \begin{cases} \operatorname{tang}\Phi'_1 = (1+c)\left(1+\frac{lm}{c}\right)\frac{\operatorname{tang}\varphi}{\sqrt{(1-l^2 m^2\sin^2\varphi)}}\cdot\frac{1}{(1+E\varphi)}, \\ \operatorname{tang}\varphi'_2 = l_1\, m_1\left(1+\frac{LM}{c}\right)\frac{\operatorname{tang}\varphi}{\sqrt{(1-l^2 m^2\sin^2\varphi)}}\cdot\frac{1}{(1+E\varphi)} = l'_1 \operatorname{tang}\Phi'_1, \end{cases}$$

ubi φ'_1 et Φ'_2 semper ut anguli minimi positivi his formulis respondentes assumendi sunt. Hinc methodus minus elegans atque minus expedita integralia nostra indefinita computandi derivatur, quam ita instituendam exponimus. Ponamus rursus fore:

datum angulum,	numeratoris coefficientes		et modulos respective:
$=\Phi$,	$=P$,	$=Q$,	$=c, l, m$,

28. post primam transformationem adipiscamur:

hos angulos:	numeratoris coefficientes		et modulos:
Φ'_1,	P'_1,	Q'_1,	c', l', m',
Φ'_2,	P'_2,	Q'_2,	c', L', M',

post secundam transformationem hos:

$\Phi''_{1,1}$, $\Phi''_{2,2}$,	P''_1,	Q''_1,	c'', l'', m'',
$\Phi''_{1,2}$, $\Phi''_{2,1}$,	P''_2,	Q''_2,	c'', L'', M'',

28. $\left\{\begin{array}{llll} \text{post tertiam transformationem:} & & & \\ & \text{hos angulos:} & \text{numeratoris coefficientes:} & \text{modulos:} \\ & \varphi'''_{1,1,1},\ \varphi'''_{1,2,2},\ \varphi'''_{2,1,2},\ \varphi'''_{2,2,1}, & P'''_1,\ Q'''_1, & c''',\ l''',\ m''', \\ & \varphi'''_{1,1,2},\ \varphi'''_{1,2,1},\ \varphi'''_{2,1,1},\ \varphi'''_{2,2,2}, & P'''_2,\ Q'''_2, & c''',\ L''',\ M''', \\ \text{etc.} & & & \end{array}\right.$

ubi indices apud angulos rursus originem ex antecedentibus ordinemque transformationis indicant.

Transformationibus repetitis denique ponere licet:

$$c_1 = 0 \qquad \text{et:} \qquad m_1 = l_1$$

integraliaque in integralia elliptica tertiae speciei quorum modulus proxime ad unitatem accedit, parameterque forma $-c^2\sin^2\theta$ gaudet, reducuntur, quae secundum articulum (98.) primi tomi libri illustrissimi *„Traité des fonctions elliptiques par Legendre"* computantur. Quem ad finem, n esse numerum transformationum desideratarum, ponamus, ita ut sit:

$$c_1^{(n)} = 0, \qquad l_1^{(n)} = m_1^{(n)}.$$

Denotemus porro per $\Phi_{\text{I}}^{(n)}$ quemlibet eorum angulorum $n$tae transformationis, quorum indices inferiores parem numerum signi: 2 continent, et qui igitur ad numeratorem

$$P_1^{(n)} - Q_1^{(n)}\sin^2\varphi_{\text{I}}^{n}$$

pertinent, eodemque modo per $\Phi_{\text{II}}^{(n)}$ quemlibet eorum angulorum $n$tae transformationis, quorum indices numerum 2, aut semel aut ter etc. continent, et qui ad numeratorem alterum

$$P_2^{(n)} - Q_2^{(n)}\sin^2\varphi_{\text{II}}^{(n)}$$

pertinent. His positis dico, valorem integralis indefiniti exprimi per aggregatum expressionum huius formae:

$$29.\quad \left\{\begin{array}{l} \dfrac{P_1^{(n)}-Q_1^{(n)}}{m_1^2}\log\text{nat tang}\left(45°+\dfrac{\varphi_1^{(n)}}{2}\right) - \dfrac{P_1^{(n)}m^{(n)2}-Q_1^{(n)}}{2mm_1^2}\log\text{nat}\dfrac{1+m^{(n)}\sin\varphi_{\text{I}}^{(n)}}{1-m^{(n)}\sin\varphi_{\text{I}}^{(n)}} \\ + \dfrac{P_2^{(n)}-Q_2^{(n)}}{M_1^2}\log\text{nat tang}\left(45°+\dfrac{\varphi_2^{(n)}}{2}\right) - \dfrac{P_2^{(n)}M^{(n)2}-Q_2^{(n)}}{2MM_1^2}\log\text{nat}\dfrac{1+M^{(n)}\sin\varphi_{\text{II}}^{(n)}}{1-M^{(n)}\sin\varphi_{\text{II}}^{(n)}}, \end{array}\right.$$

quae formulae, si angulus $\varphi_1^{(n)}$ vel $\varphi_2^{(n)}$ proxime ad unitatem accedit, in has abeunt:

$$30.\quad \left\{\begin{array}{l} \dfrac{P_1^{(n)}c^{(n)}-Q_1^{(n)}}{c^{(n)}-m^{(n)2}}\displaystyle\int_0^{\varphi_1^n}\dfrac{d\varphi}{\sqrt{(1-c^{(n)2}\sin^2\varphi)}} - \dfrac{P_1^{(n)}m^{(n)2}-Q_1^{(n)}}{2m^{(n)}m_1^{(n)2}}\log\text{nat}\dfrac{1+m^{(n)}\sin\varphi_{\text{I}}^{(n)}}{1-m^{(n)}\sin\varphi_{\text{I}}^{(n)}} \\ + \dfrac{P_2^{(n)}c^{(n)}-Q_2^{(n)}}{c^{(n)}-M^{(n)2}}\displaystyle\int_0^{\varphi_2^n}\dfrac{d\varphi}{\sqrt{(1-c^{(n)2}\sin^2\varphi)}} - \dfrac{P_2^{(n)}M^{(n)2}-Q_2^{(n)}}{2M^{(n)}M_1^{(n)2}}\log\text{nat}\dfrac{1+M^{(n)}\sin\varphi_{\text{II}}^{(n)}}{1-M^{(n)}\sin\varphi_{\text{II}}^{(n)}}, \end{array}\right.$$

Iam vero, si praeferre placet, transformationes toties adhuc per formulas:

31. $\operatorname{tang}\Phi'_1 = (1+m^2)\frac{\operatorname{tang}\varphi}{\sqrt{(1-m^4\sin^2\varphi)}}$, $\quad\operatorname{tang}\Phi'_2 = m'_1\operatorname{tang}\Phi'_1$

repetere possumus, ut usque ad limites $\Pi\,\Pi_1\,\mathrm{K}\,\mathrm{K}_1$ in capite XIV. descriptos perveniamus, quantitatesque ordinis M'^2_1 et m'^2_1 negligere possimus; quarum transformationum numero per n rursus denotato, valor integralis indefiniti ex terminis huius formae componitur:

$$32.\quad\left\{\begin{aligned}&\frac{\Pi_1 P-\mathrm{K}_1 Q}{2^{n+1}}\log\operatorname{nat}\left(\frac{1+\sin\varphi_1^{(n)}}{1-\sin\varphi_1^{(n)}}\right)\left(\frac{1+m^{(n)}\sin\varphi_1^{(n)}}{1-m^{(n)}\sin\varphi_1^{(n)}}\right)+\frac{\Pi P-\mathrm{K}Q}{2^{n+1}}\log\operatorname{nat}\left(\frac{1+m^{(n)}\sin\varphi_1^{(n)}}{1-m^{(n)}\sin\varphi_1^{(n)}}\right)\\&+\frac{\Pi P+\mathrm{K}Q}{2^{n+1}}\log\operatorname{nat}\left(\frac{1+\sin\varphi_{11}^{(n)}}{1-\sin\varphi_{11}^{(n)}}\right)\left(\frac{1+M^{(n)}\sin\varphi_{11}^{(n)}}{1-M^{(n)}\sin\varphi_{11}^{(n)}}\right)+\frac{\Pi_1 P-\mathrm{K}_1 Q}{2^{n+1}}\log\operatorname{nat}\left(\frac{1+M^{(n)}\sin\varphi_{11}^{(n)}}{1-M^{(n)}\sin\varphi_{11}^{(n)}}\right),\end{aligned}\right.$$

quae expressiones, si anguli Φ^n proxime ad unitatem accedunt, cum his commutentur necesse est

$$33.\quad\left\{\begin{aligned}&\left(\frac{\Pi_1 P-\mathrm{K}Q}{2^n}\right)\left(\int_0^{\varphi_1^n}\frac{d\varphi}{\sqrt{(1-c^{(n)2}\sin^2\varphi)}}-\tfrac{1}{2}\log\operatorname{nat}\frac{1+m^{(n)}\sin\varphi_1^{(n)}}{1-m^{(n)}\sin\varphi_1^{(n)}}\right)\\&\qquad\qquad+\frac{\Pi P+\mathrm{K}Q}{2^{n+1}}\log\operatorname{nat}\frac{1+m^{(n)}\sin\varphi_1^{(n)}}{1-m^{(n)}\sin\varphi_1^{(n)}}\\&+\left(\frac{\Pi P+\mathrm{K}Q}{2^n}\right)\left(\int_0^{\varphi_{11}^n}\frac{d\varphi}{\sqrt{(1-c^{(n)2}\sin^2\varphi)}}-\tfrac{1}{2}\log\operatorname{nat}\frac{1+M^{(n)}\sin\varphi_{11}^{(n)}}{1-M^{(n)}\sin\varphi_{11}^{(n)}}\right)\\&\qquad\qquad+\frac{\Pi_1 P-\mathrm{K}_1 Q}{2^{n+1}}\log\operatorname{nat}\frac{1+M^{(n)}\sin\varphi_{11}^{(n)}}{1-M^{(n)}\sin\varphi_{11}^{(n)}}.\end{aligned}\right.$$

Prorsus similes expressiones pro altera numeratoris forma adipiscimur.

Integralia vero definita, quippe in quorum transformatione bina semper integralia nova in unum coniungi possunt, multo expeditiorem praebent determinationem. Habemus enim posito $\Phi = \frac{\pi}{2}$, ex formulis (30.) has:

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\frac{(P-Q\sin^2\varphi)\,d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}}\\&=\frac{P_1^{(n)}c^{(n)}-Q_1^{(n)}}{c^{(n)}-m^{(n)2}}\log\operatorname{nat}\frac{4}{c_1^{(n)}}-\frac{P_1^{(n)}m^{(n)2}-Q_1^{(n)}}{m^{(n)}m^{(n)2}}\log\operatorname{nat}\frac{2}{m_1^{(n)}}\\&=\frac{P_2^{(n)}c^{(n)}-Q_2^{(n)}}{c^{(n)}-M^{(n)2}}\log\operatorname{nat}\frac{4}{c_1^{(n)}}-\frac{P_2^{(n)}M^{(n)2}-Q_2^{(n)}}{M_1^{(n)}.M_1^{(n)2}}\log\operatorname{nat}\frac{2}{M_1^{(n)}},\end{aligned}$$

atque transformationibus satis continuatis ex formulis (33.) hanc:

$$\int_0^{\frac{\pi}{2}}\frac{(P_1-Q_1\sin^2\varphi)\,d\varphi}{\Delta(c,l,m)}=\frac{\Pi_1 P-\mathrm{K}_1 Q}{2^n}\log\operatorname{nat}\frac{2}{M_1^{(n)}}+\frac{\Pi P+\mathrm{K}Q}{2^n}\log\operatorname{nat}\frac{2}{m_1^{(n)}}.$$

Facile demonstratur transformatione sequenti peracta eundem valorem integralis definiti prodire. Novus enim valor:

$$\frac{\Pi_1 P - K_1 Q}{2^{n+1}} \log \text{nat} \frac{2}{M_1^{(n+1)}} + \frac{\Pi P + K Q}{2^{(n+1)}} \log \text{nat} \frac{2}{m_1^{(n+1)}},$$

formulis (20.) artic. XIII. substitutis his:

$$M_1^{(n+1)} = \frac{M_1^{(n)2}}{2}, \qquad m_1^{(n+1)} = \frac{m_1^{(n)2}}{2}$$

in valorem antecedentem transit.

Expressiones integralis propositi antecedentes, ex loco citato facillime derivantur, tamen ad computationem integralis indefiniti multo minus quam illae aptae sunt, quas ex altera nostra transformatione prodire, vidimus. Repetamus vero computationem integralium indefinitorum propositorum, aliam multo faciliorem atque elegantiorem ope theorematis Abeliani, ex his nostris disquisitionibus deductam, alio loco geometris nos communicaturos esse.

XVI.

E x e m p l a.

Quaecunque de computatione modulorum, numeratorum integraliumque ipsorum diximus, his duobus exemplis adhuc magis explanare placet, ubi duo integralia definita, quorum moduli datis valoribus gaudent, secundum utramque methodum in articulis praecedentibus expositam computata invenis.

Exemplum primum.

Datum sit hoc schema modulorum per algorithmum repetitum diminuendorum.

Valores dati modulorum:

$\alpha = 11^0$	$0'\ 0''$	$\log c = 9{,}2805988$	$\log c_1 = 9{,}9919466$
$\beta = 4^0$	$0'\ 0''$	$\log l = 8{,}8435845$	$\log l_1 = 9{,}9989408$
$\gamma = 3^0$	$0'\ 0''$	$\log m = 8{,}7188002$	$\log m_1 = 9{,}9994044$

$$\tfrac{1}{2} \log \frac{c_1 l_1}{m_1} = 9{,}9957415, \qquad \frac{\alpha^0}{2} = 0^0\, 16'\, 51'',26,$$

$$\log\left(1 + \frac{c_1 l_1}{m_1}\right) = 0{,}2967923, \qquad \log(1 + m_1) = 0{,}3007323.$$

Valores modulorum primae transformationis:

$\alpha = 0^0$	$33'\ 42'',52$	$\log c^0 = 7{,}991460$	$\log c_1^0 = 9{,}9999791$
$\beta = 0^0$	$18'\ 55'',12$	$\log l^0 = 7{,}740615$	$\log l_1^0 = 9{,}9999934$
$\gamma = 0^0$	$4'\ 12'',003$	$\log m^0 = 7{,}086981$	$\log m_1^0 = 9{,}9999997$

$$\log \nu^0 = 0{,}0051310,$$

$$\log \pi^0 = 9{,}6983744, \qquad \log k^0 = 9{,}6989700,$$

$$\log \nu^0 \pi^0 = 9{,}7035054, \qquad \log \nu^0 k^0 = 9{,}7041010,$$

$$\log \frac{\mathfrak{l}^{02}}{4} = 4{,}8792, \qquad \log\left(1 + \frac{c_1^0 l_1^0}{m_1^0}\right) = 0{,}3010164,$$

$$\log\left(1 + \frac{\mathfrak{l}^{02} c^{02}}{l^{02}}\right) = 0{,}6155, \qquad \log(1 + m_1^0) = 0{,}3010299,$$

Valores secundae transformationis:

$$\alpha^{00} = 0^0\,0'\,6'',4427, \qquad \log c^{00} = 5{,}4946,$$

$$\beta^{00} = 0^0\,0'\,5'',4746, \qquad \log l^{00} = 5{,}4239,$$

$$\gamma^{00} = 0^0\,0'\,0'',09058, \qquad \log m^{00} = 3{,}643,$$

$$\log \nu_1^{00} = 0{,}0000140,$$

$$\log \pi^{00} = 9{,}6962398, \qquad \log k^{00} = 9{,}7010940,$$

$$\log \nu_1^0 \nu_1^{00} \pi^{00} = 9{,}7013848 = \log P, \qquad \log \nu_1^0 \nu_1^{00} k^{00} = 9{,}7062390 = \log K.$$

Valor integralis:

$$\int_0^{\frac{\pi}{2}} \frac{(\Pi \cos^2\varphi - \mathrm{K} \sin^2\varphi)\, d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}} = 0{,}7897772\,\Pi + 0{,}7986541\,\mathrm{K}.$$

Ut legem, quam moduli sequuntur, exemplo hoc comprobemus, adiicere placet adhuc calculum exactum priorum et sequentium transformationum, quae ad computationem integralis propositi haud amplius adhibentur.

$\log c = 9{,}2805988,$	$\log c^0 = 7{,}9914604,$	$\log c^{00} = 5{,}4946461,$
$\log l = 8{,}8435845,$	$\log l^0 = 7{,}7406152,$	$\log l^{00} = 5{,}4239323,$
$\log m = 8{,}7188002,$	$\log m^0 = 7{,}0869810,$	$\log m^{00} = 3{,}6426158,$
$\log \mathfrak{l} = 9{,}4382014,$	$\log \mathfrak{l}^0 = 9{,}0955206,$	$\log \mathfrak{l}^{00} = 8{,}1479697,$
$\log \mathfrak{k} = 9{,}8752157,$	$\log \mathfrak{k}^0 = 9{,}3463658,$	$\log \mathfrak{k}^{00} = 8{,}2186835,$
	$\log \nu^0 = 0{,}0051310,$	$\log \nu^{00} = 0{,}0000140,$
	$\log \pi^0 = 9{,}6983744,$	$\log \pi^{00} = 9{,}6962398,$
	$\log k^0 = 9{,}6989700,$	$\log k^{00} = 9{,}7010940,$
	$\log \pi_1^0 = 7{,}6898348,$	$\log \pi_1^{00} = 2{,}8834179,$
	$\log k_1^0 = 7{,}6904304,$	$\log k_1^{00} = 2{,}8840795,$
$\log c^{000} = 0{,}6232302-10,$	$\log c^{0000} = 0{,}939786-20,$	$\log c^{00000} = 0{,}578504-39,$
$\log l^{000} = 0{,}6175482-10,$	$\log l^{0000} = 0{,}939749-20,$	$\log l^{00000} = 0{,}578504-39,$
$\log m^{000} = 0{,}6888536-14,$	$\log m^{0000} = 0{,}775685-28,$	$\log m^{00000} = 0{,}949309-56,$
$\log \mathfrak{l}^{000} = 6{,}0656234-10,$	$\log \mathfrak{l}^{0000} = 0{,}835899-9,$	$\log \mathfrak{l}^{00000} = 0{,}370805-17,$
$\log \mathfrak{k}^{000} = 6{,}0713054-10,$	$\log \mathfrak{k}^{0000} = 0{,}835936-9,$	$\log \mathfrak{k}^{00000} = 0{,}370805-17,$

$\log \nu^{000} = 0,$ $\log \nu^{0000} = 0,$ $\log \nu^{00000} = 0,$

$\log \pi^{000} = \log \pi^{00},$ $\log \pi^{0000} = \log \pi^{000},$ $\log \pi^{00000} = \log \pi^{0000},$

$\log k^{000} = \log k^{00},$ $\log k^{0000} = \log k^{000},$ $\log k^{00000} = \log k^{0000},$

$\log \pi_1^{000} = 0{,}2056179-17,$ $\log \pi_1^{0000} = 0{,}8443735-37,$ $\log \pi_1^{00000} = 0{,}1218475-75.$

$\log k_1^{000} = 0{,}2062800-17,$ $\log k_1^{0000} = 0{,}8450356-37,$ $\log k_1^{00000} = 0{,}1225096-75,$

Exemplum secundum.

Datum sit hoc schema modulorum per alterum algorithmum repetitum augendorum.

Valores dati:

$\alpha = 87^0\ 0'\ 0'',00,$ $\log m_1 = 9{,}9994044 = \log\sin\alpha,$ $\log m = 8{,}7188002 = \log\cos\alpha,$

$\beta = 86^0\ 0'\ 0'',00,$ $\log l_1 = 9{,}9989408 = \log\sin\beta,$ $\log l = 8{,}8435845 = \log\cos\beta,$

$\gamma = 79^0\ 0'\ 0'',00,$ $\log c_1 = 9{,}9919466 = \log\sin\gamma,$ $\log c = 9{,}2805988 = \log\cos\gamma,$

$B = 79^0 24' 43'',85,$ $\log L_1 = 9{,}9925422 = \log\sin B,$ $\log B = 9{,}2642094 = \log\cos B,$

$A = 79^0 44' 41'',03,$ $\log M_1 = 9{,}9930058 = \log\sin A,$ $\log A = 9{,}2505030 = \log\cos A,$

$\log np = 0,$ $\log nq = -\infty,$

$\log \frac{np_1}{l_1 m_1} = 0{,}0016548,$ $\log \frac{nq_1}{l_1 m_1} = 0{,}0016548,$

$\frac{1}{2}\log\frac{ml}{c} = 9{,}1408930,$ $\varepsilon = 88^0 54' 13'',27,$ $\frac{1}{2}\log\frac{ML}{c} = 9{,}6170568,$ $E' = 80^0 7' 42'',29,$

$\frac{\alpha'}{2} = 37^0 7' 28'',84,$ $\log\operatorname{tang}\frac{1}{2}\varepsilon' = 9{,}9916896,$ $\frac{A'}{2} = 22^0 30' 28'',12,$ $\operatorname{tang}\frac{1}{2}E' = 9{,}9248017,$

$\log\left(1+\frac{ml}{c}\right) = 0{,}0082309,$ $\log\left(1+\frac{ML}{c}\right) = 0{,}0687202,$ $\log(1+c) = 0{,}0758421.$

Valores primae transformationis:

$\alpha' = 74^0 14' 57'',68,$ $\log m_1' = 9{,}9833791,$ $\log m' = 9{,}4336919,$

$\beta' = 74^0\ 4' 10'',33,$ $\log l_1' = 9{,}9829924,$ $\log l' = 9{,}4384959,$

$\gamma' = 42^0 51' 15'',40,$ $\log c_1' = 9{,}8325958,$ $\log c' = 9{,}8651549,$

$B' = 44^0 57' 52'',66,$ $\log L_1' = 9{,}8492167,$ $\log L' = 9{,}8497530,$

$A' = 45^0\ 0' 50'',24,$ $\log M_1' = 9{,}8496034,$ $\log M' = 9{,}8493666,$

$\log n' = 1{,}2373882,$

$\log p' = 8{,}9795688,$ $\log q' = 9{,}6989700,$

$\log p_1' = 8{,}9629479,$ $\log q_1' = 9{,}6823491,$

$\log nn'p' = 0{,}2169570,$ $\log nn'q' = 0{,}9363582,$

$\log\frac{nn'p_1'}{l_1' m_1'} = 0{,}2339646,$ $\log\frac{nn'q_1'}{l_1' m_1'} = 0{,}9533658,$

$\frac{1}{2}\log\frac{m'l'}{c'} = 9{,}5035165,$ $\varepsilon'' = 84^0 10' 0'',52,$ $\frac{1}{2}\log\frac{M'L'}{c'} = 9{,}9169824,$ $E'' = 46^0 58' 39'',64,$

$\frac{\alpha''}{2} = 27^0 19' 4'',04,$ $\operatorname{tang}\frac{1}{2}\varepsilon'' = 9{,}9557086,$ $\frac{A''}{2} = 5^0 26' 35'',32,$ $\operatorname{tang}\frac{1}{2}E'' = 9{,}6380705,$

$\log\left(1+\frac{m'l'}{c'}\right) = 0{,}0420368,$ $\log\left(1+\frac{M'L'}{c'}\right) = 0{,}2258992,$ $\log(1+c') = 0{,}2388202.$

Valores secundae transformationis:

$\alpha'' = 54^0 38' \; 8'',06,$ $\log m''_1 = 9,9114172,$ $\log m'' = 9,7625098,$

$\beta'' = 54^0 38' \; 5'',87,$ $\log l''_1 = 9,9114139,$ $\log l'' = 9,7625163,$

$\gamma'' = 8^0 51' 34'',23,$ $\log c''_1 = 9,1875549,$ $\log c'' = 9,9947873,$

$B'' = 10^0 53' 10'',34,$ $\log L''_1 = 9,2761377,$ $\log L'' = 9,9921134,$

$A'' = 10^0 53' 10'',64,$ $\log M''_1 = 9,2761410,$ $\log M'' = 9,9921132,$

$\log n'' = 0,4560481,$

$\log p'' = 8,5653869,$ $\log q'' = 9,8285855,$

$\log p''_1 = 8,4322769,$ $\log q''_1 = 9,3990384,$

$\log n n' n'' p'' = 0,2588232,$ $\log n n' n'' q'' = 1,5220218,$

$\log \frac{n n' n'' p''_1}{l''_1 m''_1} = 0,3028821,$ $\log \frac{n n' n'' q''_1}{l''_1 m''_1} = 1,2696436,$

$\frac{1}{2}\log \frac{m'' l''}{c''} = 9,7651194,$ $\varepsilon''' = 70^0 10' 55'',85,$ $\frac{1}{2}\log \frac{M'' L''}{c''} = 9,9947197,$ $E''' = 12^0 35' 31'',61,$

$\frac{\alpha'''}{2} = 14^0 47' 21'',59,$ $\log \operatorname{tang} \frac{1}{2}\varepsilon''' = 9,8466955,$ $\frac{A'''}{2} = 0^0 20' 53'',895,$ $\log \operatorname{tang} \frac{E'''}{2} = 9,0424286,$

$\log\left(1 + \frac{m'' l''}{c''}\right) = 0,1267906,$ $\log\left(1 + \frac{M'' L''}{c}\right) = 0,2957818,$ $\log(1 + c'') = 0,2984315,$

$\frac{1}{2}\log\left[\cos\gamma'' \sin\frac{\alpha''}{2} \sin\frac{\beta''}{2}\right] = 9,3208734,$ $\frac{1}{2}\log\left[\cos\gamma'' \cos\frac{\alpha''}{2} \cos\frac{\beta''}{2}\right] = 9,8946851,$

$$\log\left[\sin\frac{\alpha''+\gamma''}{2} \sin\frac{\alpha''-\gamma''}{2} \sin\frac{\beta''+\gamma''}{2} \sin\frac{\beta''-\gamma''}{2}\right] = 8,6220015,$$

$$\log\left[\cos\frac{\alpha''+\gamma''}{2} \cos\frac{\alpha''-\gamma''}{2} \cos\frac{\beta''+\gamma''}{2} \cos\frac{\beta''-\gamma''}{2}\right] = 9,7879939,$$

$\log \operatorname{tang} \frac{1}{2}(\beta''' + \gamma''') = 9,4268764,$ $\log \operatorname{tang} \frac{1}{2}(\beta''' - \gamma''') = 9,4163157,$

$\beta''' = 29^0 34' 43'',178,$ $\gamma''' = 0^0 20' 37'',8640.$

Valores tertiae transformationis:

$\alpha''' = \beta''' = 29^0 34' 43'',18,$ $\log m'''_1 = 9,6933910 = \log l'''_1,$ $\log m''' = \log l''' = 9,9393589,$

$\gamma''' = 0^0 20' 37'',868,$ $\log c'''_1 = 7,7782468,$ $\log c''' = 9,9999922,$

$A''' = B''' = 0^0 41' 47'',793,$ $\log M'''_1 = 8,0848558 = \log L'''_1,$ $\log M''' = \log L''' = 9,9999679,$

$\log n''' = 0,1820506,$

$\log p''' = 8,3619780,$ $\log q''' = 9,9004659,$

$\log p'''_1 = 7,8175403,$ $\log q'''_1 = 8,8051125,$

$\log n n' n'' n''' p''' = 0,2374649,$ $\log n n' n'' n''' q''' = 1,7759528,$

$\log n n' n'' n''' p'''_1 = 0,3062452,$ $\log n n' n'' n''' q'''_1 = 1,2938174,$

$\frac{1}{2}\log \frac{m''' l'''}{c'''} = 9,9393628,$ $\varepsilon^{IV} = 40^0 51' 21'',98,$ $\frac{1}{2}\log \frac{M''' L'''}{c} = 9,9999718,$ $E^{IV} = 0^0 55' 23'',3,$

$\frac{\alpha^{IV}}{2} = 3^0 59' 13'',03,$ $\log \operatorname{tang} \frac{1}{2}\varepsilon^{IV} = 9,5710728,$ $\frac{A^{IV}}{2} = 0^0 0' 6'',70,$ $\log \operatorname{tang} \frac{1}{2}E^{IV} = 7,9059\ldots.,$

$\log\left(1 + \frac{m''' l'''}{c}\right) = 0,2446123,$ $\log\left(1 + \frac{M''' L'''}{c'''}\right) = 0,3010019,$ $\log(1 + c''') = 0,3010262.$

Valores quartae transformationis:

$\alpha^{\text{IV}} = \beta^{\text{IV}} = 7^0\,58'\,26'',06,$ $\log m_1^{\text{IV}} = \log l_1^{\text{IV}} = 9{,}1421456,$ $\log m^{\text{IV}} = \log l^{\text{IV}} = 9{,}9957805,$

$\gamma^{\text{IV}} = 0^0\;\,0'\;\,1'',857233,$ $\log c_1^{\text{IV}} = 4{,}9544413,$ $\log c^{\text{IV}} = 0{,}0\ldots,$

$A^{\text{IV}} = B^{\text{IV}} = 0^0\;\,0'\,13'',38816,$ $\log M_1^{\text{IV}} = \log L_1^{\text{IV}} = 5{,}8122957,$ $\log M^{\text{IV}} = \log L^{\text{IV}} = 0{,}0,$

$$\log n^{\text{IV}} = 0{,}0564293,$$

$\log p^{\text{IV}} = 8{,}2950882,$ $\log q^{\text{IV}} = 9{,}9175556,$

$\log p_1^{\text{IV}} = 6{,}6586285,$ $\log q_1^{\text{IV}} = 7{,}6463255,$

$\log n\,n'\,n''\,n'''\,n^{\text{IV}}\,p^{\text{IV}} = 0{,}2270044,$ $\log n\,n'\,n''\,n'''\,n^{\text{IV}}\,q^{\text{IV}} = 1{,}8494718,$

$\log n\,n'\,n''\,n'''\,n^{\text{IV}}\,p_1^{\text{IV}} = 0{,}3062535 = \log \Pi_1,$ $\log n\,n'\,n''\,n'''\,n^{\text{IV}}\,q_1^{\text{IV}} = 1{,}2939505 = \log \mathrm{K}_1,$

Valores quintae transformationis:

$\log m_1^{\text{V}} = \log l_1^{\text{V}} = 7{,}9874602,$ $\log m^{\text{V}} = \log l^{\text{V}} = 9{,}9999795,$

$\log c_1^{\text{V}} = 0{,}3068226-11,$ $\log c^{\text{V}} = 0,$

$\log M_1^{\text{V}} = \log L_1^{\text{V}} = 1{,}3193624-10,$ $\log M^{\text{V}} = \log L^{\text{V}} = 0,$

$$\log n^{\text{V}} = 0{,}0041990,$$

$\log p^{\text{V}} = 8{,}2900459,$ $\log q^{\text{V}} = 9{,}9187169,$

$\log p_1^{\text{V}} = 4{,}3450587,$ $\log q_1^{\text{V}} = 5{,}3327557,$

$\log n\,n'\,n''\,n'''\,n^{\text{IV}}\,n^{\text{V}}\,p^{\text{V}} = 0{,}2261611,$ $\log n\,n'\,n''\,n'''\,n^{\text{IV}}\,n^{\text{V}}\,q^{\text{V}} = 1{,}8548321,$

$\log n\,n'\,n''\,n'''\,n^{\text{IV}}\,n^{\text{V}}\,p_1^{\text{V}} = \log \Pi_1,$ $\log n\,n'\,n''\,n'''\,n^{\text{IV}}\,n^{\text{V}}\,q_1^{\text{V}} = \log \mathrm{K}_1,$

Valores sextae transformationis:

$$\log m_1^{\text{VI}} = \log l_1^{\text{VI}} = 5{,}6739109-10,$$

$$\log c_1^{\text{VI}} = 0{,}0115852-22,$$

$$\log n^{\text{VI}} = 0{,}0000205,$$

$\log p^{\text{VI}} = 8{,}2900190,$ $\log q^{\text{VI}} = 9{,}9187223,$

$\log p_1^{\text{VI}} = 0{,}7179396-11,$ $\log q_1^{\text{VI}} = 0{,}7056365-10,$

$\log n\,n'n''n'''n^{\text{IV}}n^{\text{V}}n^{\text{VI}}p^{\text{VI}} = 0{,}2261547 = \log \Pi,$ $\log n\,n'n''n'''n^{\text{IV}}n^{\text{V}}n^{\text{VI}}q^{\text{VI}} = 1{,}8548580 = \log \mathrm{K}.$

Valor integralis:

$$\int_0^{\frac{\pi}{2}} \frac{(P-Q\sin^2\varphi)\,d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}} - (1{,}588429)\,P - 0{,}798650\,Q.$$

Exemplum tertium.

Datum sit hoc schema modulorum per transformationem alteram diminuendorum:

Valores dati:

$\alpha = 75^0\,0'\,0'',00,$ $\log c = 9{,}9849438,$ $\log c_1 = 9{,}4129962,$

$\beta = 54^0\,0'\,0'',00,$ $\log l = 9{,}9079576,$ $\log l_1 = 9{,}7692187,$

$\gamma = 25^0\,0'\,0'',00,$ $\log m = 9{,}6259483,$ $\log m_1 = 9{,}9572757,$

$$\tfrac{1}{2}\log\frac{c_1 l_1}{m_1} = 9{,}6124696, \qquad \varepsilon^0 = 80^0 20' 12'',34,$$

$$\frac{\alpha^0}{2} = 22^0 43' 15'',31, \qquad \log\operatorname{tang}\tfrac{1}{2}\varepsilon^0 = 9{,}9264042,$$

$$\log\left(1+\frac{c_1 l_1}{m_1}\right) = 0{,}0673897, \qquad \log(1+m_1) = 0{,}2801930.$$

Valores primae transformationis:

$$\alpha^0 = 45^0 26' 30'',62, \qquad \log c^0 = 9{,}8528084, \qquad \log c_1^0 = 9{,}8461100,$$

$$\beta^0 = 40^0 23' 20'',22, \qquad \log l^0 = 9{,}8115569, \qquad \log l_1^0 = 9{,}8817630,$$

$$\gamma^0 = 3^0\ 5' 53'',185, \qquad \log m^0 = 8{,}7327621, \qquad \log m_1^0 = 9{,}9993648,$$

$$\log \nu^0 = 0{,}2972016,$$

$$\log \pi^0 = 9{,}6562457, \qquad \log k^0 = 9{,}6989700,$$

$$\log \pi_1^0 = 9{,}5090541, \qquad \log k_1^0 = 9{,}5517784,$$

$$\log \nu\nu^0\pi^0 = 9{,}9534473, \qquad \log \nu\nu^0 k^0 = 9{,}9961716,$$

$$\tfrac{1}{2}\log\frac{c_1^0 l_1^0}{m_1^0} = 9{,}8642541, \qquad \varepsilon^{00} = 57^0 38' 35'',47,$$

$$\frac{\alpha^{00}}{2} = 8^0 48' 43'',30, \qquad \log\operatorname{tang}\tfrac{1}{2}\epsilon = 9{,}7405565,$$

$$\log\left(1+\frac{c_1^0 l_1^0}{m_1^0}\right) = 0{,}1861623, \qquad \log(1+m_1^0) = 0{,}3007125.$$

Valores secundae transformationis:

$$\alpha^{00} = 17^0 37' 26'',61, \qquad \log c^{00} = 9{,}4811130, \qquad \log c_1^{00} = 9{,}9791219,$$

$$\beta^{00} = 17^0 28' 17'',23, \qquad \log l^{00} = 9{,}4774550, \qquad \log l_1^{00} = 9{,}9794876,$$

$$\gamma^{00} = 0^0\ 2' 32'',118, \qquad \log m^{00} = 6{,}8677572, \qquad \log m_1^{00} = 9{,}9999999,$$

$$\log \nu^{00} = 0{,}1158204,$$

$$\log \pi^{00} = 9{,}4644565, \qquad \log k^{00} = 9{,}8310818,$$

$$\log \pi_1^{00} = 8{,}688245, \qquad \log k_1^{00} = 8{,}732751,$$

$$\log \nu\nu^0\nu^{00}\pi^{00} = 9{,}8774785, \qquad \log \nu\nu^0\nu^{00} k^{00} = 0{,}2441038,$$

$$\left(1+\frac{c_1^{00} l_1^{00}}{m_1^{00}}\right) = 0{,}2808277, \quad \log(1+m_1^{00}) = 0{,}3010300, \quad \log\left(1+\frac{l_1^{00\,2} c^{00\,2}}{l^{00\,2}}\right) = 0{,}2845025.$$

Valores tertiae transformationis:

$$\alpha^{000} = 2^0 43' 45'',29, \qquad \log c^{000} = 8{,}677756, \qquad \log c_1^{000} = 9{,}9995071,$$

$$\beta^{000} = 2^0 43' 44'',77, \qquad \log l^{000} = 8{,}677739, \qquad \log l_1^{000} = 9{,}9995071,$$

$$\log m^{000} = 3{,}133, \qquad \log m_1^{000} = 0{,}0,$$

$$\log \nu^{000} = 0{,}0202024,$$

$$\log \pi^{000} = 9{,}4264912, \qquad \log k^{000} = 9{,}8480606,$$

$$\log \pi_1^{000} = 7{,}06497, \qquad \log k_1^{000} = 7{,}10948,$$

$$\log \nu\nu^0\nu^{00}\nu^{000}\pi^{000} = 9{,}8597156, \qquad \log \nu\nu^0\nu^{00}\nu^{000} k^{000} = 0{,}2812850,$$

$$\log\left(1+\frac{c_1^{000} l_1^{000}}{m_1^{000}}\right) = 0{,}3005374, \qquad \log\left(1+\frac{f_1^{000} l_1^{000}}{m_1^{000}}\right) = 0{,}3010300 = \log(1+m_1^{000})$$

Valores quartae transformationis:

$$\log c^{0000} = \log l^{0000} = 7{,}05496, \qquad \log c_1^{0000} = \log l_1^{0000} = 9{,}9999997,$$
$$\log \nu^{0000} = 0{,}0004926,$$
$$\log \pi^{0000} = 9{,}4255456, \qquad \log k^{0000} = 9{,}8484569,$$
$$\log \pi_1^{0000} = 3{,}819 - 10, \qquad \log k_1^{0000} = 3{,}863 - 10,$$
$$\log \nu\nu^0\nu^{00}\nu^{000}\nu^{0000}\pi^{0000} = 9{,}8592626, \qquad \log \nu\nu^0\nu^{00}\nu^{000}\nu^{0000}k^{0000} = 0{,}2821739,$$
$$\log\left(1 + \frac{c_1^{0000}\, l_1^{0000}}{m_1^{0000}}\right) = 0{,}3010297.$$

Valores quintae transformationis:

$$\alpha^{00000} = \beta^{00000} = 0^0\,0'\,0'',13\ldots, \qquad \log l^{00000} = \log c^{00000} = 3{,}809,$$
$$\log \nu^{00000} = 0{,}0000003,$$
$$\log \pi^{00000} = 9{,}4255451, \qquad \log k^{00000} = 9{,}8484571,$$
$$\log \nu\nu^0\nu^{00}\nu^{000}\nu^{0000}\nu^{00000}\pi^{00000} = 9{,}8592624 = \log P, \quad \log \nu\nu^0\nu^{00}\nu^{000}\nu^{0000}\nu^{00000}k^{00000} = 0{,}2821744 = \log K.$$

Valor integralis:

$$\int^{\frac{\pi}{2}} \frac{(\Pi\cos^2\varphi + \mathrm{K}\sin^2\varphi)\,d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}} = (1{,}136011)\Pi + (3{,}008114)\mathrm{K}.$$

Exemplum quartum.

Datum sit hoc schema modulorum per secundam transformationem augendorum:

Valores dati:

$$\alpha = 65^0\ 0'\ 0'',00, \qquad \log m_1 = 9{,}9572757, \qquad \log m = 9{,}6259483,$$
$$\beta = 36^0\ 0'\ 0'',00, \qquad \log l_1 = 9{,}7692187, \qquad \log l = 9{,}9079576,$$
$$\gamma = 15^0\ 0'\ 0'',00, \qquad \log c_1 = 9{,}4129962, \qquad \log c = 9{,}9849438,$$
$$B = 10^0\,35'\,35'',64, \qquad \log L_1 = 9{,}4557205, \qquad \log L = 9{,}9815270,$$
$$A = 26^0\ 7'\,29'',61, \qquad \log M_1 = 9{,}6437775, \qquad \log M = 9{,}9531972,$$
$$\log np = 0, \qquad \log nq = -\infty,$$
$$\log\frac{np_1}{l_1 m_1} = 0{,}2735056, \qquad \log\frac{nq_1}{l_1 m_1} = 0{,}2735056,$$
$$\tfrac{1}{2}\log\frac{lm}{c} = 9{,}7744811, \qquad \varepsilon = 69^0\,16'\,11'',58, \qquad \tfrac{1}{2}\log\frac{LM}{c} = 9{,}9748902,$$
$$\frac{\alpha'}{2} = 14^0\,14'\,57'',95, \qquad \log\operatorname{tang}\tfrac{1}{2}\varepsilon = 9{,}8393236, \qquad \frac{A'}{2} = 1^0\,39'\,19'',52,$$
$$\log\left(1 + \frac{ml}{c}\right) = 0{,}1316079, \qquad \log\left(1 + \frac{ML}{c}\right) = 0{,}2766457, \qquad \log(1+c) = 0{,}2935671.$$

Valores primae transformationis:

$\alpha' = 28^0 29' 55'',91$, $\log m'_1 = 9,6786471$, $\log m' = 9,9439032$,
$\beta' = 22^0 14' \; 7'',30$, $\log l'_1 = 9,5779651$, $\log l' = 9,9664408$,
$\gamma' = 1^0 15' \; 8'',16\ldots$, $\log c'_1 = 8,3395401$, $\log c' = 9,9998963$,
$B' = 2^0 37' 30'',83\ldots$, $\log L'_1 = 8,6608930$, $\log L' = 9,9995439$,
$A' = 3^0 18' 39'',05\ldots$, $\log M'_1 = 8,7615750$, $\log M' = 9,9992745$,

$\log n' = 0,1919412$,

$\log p' = 9,6839138$, $\log q' = 9,6989700$,
$\log p'_1 = 9,3625609$, $\log q'_1 = 9,3776171$,
$\log n n' p' = 9,8758550$, $\log n n' q' = 9,8909112$,
$\log \frac{n n' p'_1}{l'_1 m'_1} = 0,2978899$, $\log \frac{n n' q'_1}{l'_1 m'_1} = 0,3129461$,
$\frac{1}{2} \log \frac{l' m'}{c'} = 9,99552239$, $\varepsilon'' = 35^0 32' 38'',54$,
$\frac{\alpha''}{2} = 2^0 56' 54'',25$, $\text{tang}\, \frac{1}{2}\varepsilon = 9,5058636$,

$\log\left(1 + \frac{l' m'}{c'}\right) = 0,2585580$, $\log\left(1 + \frac{L' M'}{c'}\right) = 0,3004915$, $\log(1 + c') = 0,3009782$.

Valores secundae transformationis:

$\alpha'' = 5^0 53' 48'',50$, $\log m''_1 = 9,0117272$, $\log m'' = 9,9976958$,
$\beta'' = 5^0 42' 25'',22\ldots$, $\log l''_1 = 8,9975674$, $\log l'' = 9,9978420$,
$\gamma'' = 0^0 \; 0' 25'',45\ldots$, $\log c''_1 = 6,0912836$, $\log c'' = 0,0$,
$B'' = 0^0 \; 4' \; 7'',73\ldots$, $\log L''_1 = 7,0795564$, $\log L'' = 9,9999997$,
$A'' = 0^0 \; 4' 15'',94\ldots$, $\log M''_1 = 7,0937162$, $\log M'' = 9,9999997$,

$\log n'' = 0,0426275$,

$\log p'' = 9,5654752$, $\log q'' = 9,7918491$,
$\log p''_1 = 8,0730456$, $\log q''_1 = 8,0885315$,
$\log n n' n'' p'' = 9,8000439$, $\log n n' n'' q'' = 0,0264178$,
$\log \frac{n n' n'' p''_1}{l''_1 m''_1} = 0,2983197$, $\log \frac{n n' n'' q''_1}{l''_1 m''_1} = 0,3138056$,
$\log(1 + l'' m'') = 0,2988047$, $\log(1 + L'' M'') = 0,3010297$,

Valores tertiae transformationis:

$\alpha''' = 0^0 17' 39'',61\ldots$, $\log m'''_1 = 7,7107228$, $\log m''' = 9,9999943$,
$\beta''' = 0^0 17' 39'',05\ldots$, $\log l'''_1 = 7,7104896$, $\log l''' = 9,9999943$,
$\gamma''' = 0^0 \; 0' \; 0'',00\ldots$, $\log c'''_1 = 0,5807404 - 9$, $\log c''' = 0,0$,
$B''' = 0^0 \; 0' \; 0'',15\ldots$, $\log L'''_1 = 3,8700176$, $\log L''' = 0,0$,
$A''' = 0^0 \; 0' \; 0'',15\ldots$, $\log M'''_1 = 3,8702508$, $\log M''' = 0,0$,

$$\log n''' = 0{,}0022253,$$

$$\log p''' = 9{,}5584309, \qquad \log q''' = 9{,}7961276,$$

$$\log p_1''' = 5{,}4827333, \qquad \log q_1''' = 5{,}4982245,$$

$$\log n\,n'\,n''\,n'''\,p''' = 9{,}7952249, \qquad \log n\,n'\,n''\,n'''\,q''' = 0{,}0329216,$$

$$\log \frac{n\,n'\,n''\,n'''\,p_1'''}{l_1'''\,m_1'''} = 0{,}2983149 = \log \Pi_1, \quad \log \frac{n\,n'\,n''\,n'''\,q_1'''}{l_1'''\,m_1'''} = 0{,}3138061 = \log K_1,$$

$$\log(1 + l'''\,m''') = 0{,}3010244$$

Valores quartae transformationis:

$$\alpha^{\text{IV}} = \beta^{\text{IV}} = 0^0\,0'\,0'',72\ldots, \qquad \log m_1^{\text{IV}} = \log l_1^{\text{IV}} = 5{,}1201880,$$

$$\gamma^{\text{IV}} = 0^0\,0'\,0'',00, \qquad \log c_1^{\text{IV}} = 0{,}5594208 - 18,$$

$$A^{\text{IV}} = B^{\text{IV}} = 0^0\,0'\,0'',00, \qquad \log M^{\text{IV}} = \log L^{\text{IV}} = 0{,}4392328 - 13,$$

$$\log n^{\text{IV}} = 0{,}0000056,$$

$$\log p^{\text{IV}} = 9{,}5584127, \qquad \log q^{\text{IV}} = 9{,}7961384,$$

$$\log p_1^{\text{IV}} = 0{,}3018913 - 10, \qquad \log q_1^{\text{IV}} = 0{,}3173825 - 10,$$

$$\log n\,n'\,n''\,n'''\,n^{\text{IV}}\,p^{\text{IV}} = 9{,}7952123 = \log \Pi, \quad \log n\,n'\,n''\,n'''\,n^{\text{IV}}\,q^{\text{IV}} = 0{,}0329380 = \log K,$$

$$\log \frac{n\,n'\,n''\,n'''\,n^{\text{IV}}\,p_1^{\text{IV}}}{l_1^{\text{IV}}\,m_1^{\text{IV}}} = \log \Pi_1, \qquad \log \frac{n\,n'\,n''\,n'''\,n^{\text{IV}}\,q_1^{\text{IV}}}{l_1^{\text{IV}}\,m_1^{\text{IV}}} = \log K_1,$$

Valor integralis:

$$\int_0^{\frac{\pi}{2}} \frac{(P - Q\sin^2\varphi)\,d\varphi}{\sqrt{[(1-c^2\sin^2\varphi)(1-l^2\sin^2\varphi)(1-m^2\sin^2\varphi)]}} = (4{,}144126)\,P - (3{,}008114)\,Q.$$

Quia moduli primi et secundi exempli iidem sunt, nec non moduli in tertio et quarto exemplo, si in secundi et quarti exempli integralibus ponamus:

$$\Pi = P, \qquad K = P - Q,$$

ad eosdem valores perveniamus necesse est, quos in primo et tertio exemplo invenimus. Id quod calculo usque ad ultimum decimalem comprobatur. Habemus enim inde in secundo exemplo:

$$(0{,}789779)\,\Pi_1 + (0{,}798650)\,K_1,$$

et in quarto exemplo:

$$(1{,}136012)\,\Pi_1 + (3{,}008114)\,K_1$$

pro appropinquatis integralium propositorum valoribus, quippe qui valores usque ad ultimum decimalem cum valoribus eorumdem integralium in primo et tertio exemplo congruunt. Quae cum ita sint, in utroque exemplo egregiam theoriae nostrae confirmationem per se ipsam adepti sumus.
