On Quaternion Number-Systems.

By

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Introduction.

This paper continues and concludes the solution of the general enumeration problem of hypercomplex number-systems which are associative and have a modulus. The enumeration of non-quaternion systems is given in Mathematische Annalen vol. 58. The problem consists in finding all non-equivalent, non-reciprocal, irreducible number-systems with moduli, where the terms used are defined as follows.

Def. 1. Two systems having the units

$$e_1, e_2, \cdots, e_n \quad \text{and} \quad e_1', e_2', \cdots, e_n'$$

respectively are *equivalent* if linear relations exist of the type

$$e_k' = \sum_{i=1}^{n} a_{ki} e_i \qquad (k = 1, 2, \cdots, n)$$

where the determinant

$$|a_{ki}| \neq 0 \qquad (k, i = 1, 2, \cdots, n).$$

The a's are assumed to be ordinary complex numbers.

Def. 2. A system is *reducible* if its units may be divided into two or more subsystems such that the product of two units in the same subsystem is in that subsystem, while the product of units in different subsystems vanishes.

Def. 3. Two systems are *reciprocal* to each other when the multiplication table of one can be obtained from that of a system which is equivalent to the other by an interchange of rows and columns.

Def. 4. The *modulus* of a system is a number μ such that for an arbitrary number x of the system,

$$\mu x = x \mu = x.$$

The division of number-systems into quaternion and non-quaternion classes is due to Scheffers*) who defines a quaternion system as one in which three numbers independent of the modulus and of each other exist which satisfy the following equations:

$$(1) \qquad \begin{aligned} e_1 e_2 - e_2 e_1 &= 2e_3, \\ e_2 e_3 - e_3 e_2 &= 2e_1, \\ e_3 e_1 - e_1 e_3 &= 2e_2. \end{aligned}$$

The simplest quaternion system is Hamilton's quaternions which is symbolized by (H). The necessary and sufficient condition that a given system is quaternion, is that (H) occurs in it as a subsystem, that is, that four independent numbers of the system may be so chosen that their multiplication table is identical with that of (H). That this condition is sufficient is evident, since the three units of (H) that are distinct from the modulus fulfil equations (1) when the multiplication table for (H) is taken in the form

	1	2	3	4
1	− 4	− 3	2	1
2	3	− 4	− 1	2
3	− 2	1	− 4	3
4	1	2	3	4

where k is written for e_k. That this condition is necessary was proved by Scheffers for $n \leqq 8$, and appears for general n from a memoir by Molien.**)

§ 1.

Normal Forms.

In order to enumerate the various systems of distinct types it is necessary to find a normal form into which any quaternion system may be thrown and from an inspection of which its characteristic properties appear. Normal forms for any system, whether quaternion or not, have been given by Molien***) and the writer†) and will be symbolized by (M) and (P) respectively. A normal form for the multiplication table of non-quaternion

*) Mathematische Annalen, Vol. 39.

**) Mathematische Annalen, Vol. 41.

***) loc. cit.

†) Transactions of the American Mathematical Society, Vol. 3.

systems has been given by Scheffers*), which will be symbolized by (S). The features of these normal forms will now be given, followed by proofs that these three forms are compatible, that is, the multiplication table of a given system may be thrown into all three forms simultaneously, thus affording a form which comprises the advantages of all. This generalized normal form is called (N).

1. Normal Form (M).

Def. 5. A *primitive* system is one in m^2 units,

$$e_{11}, e_{12}, \cdots, e_{1m}, \quad e_{21}, e_{22}, \cdots, e_{2m}, \cdots, e_{mm}$$

such that the units obey the multiplicative law

$$e_{jh} e_{lk} = 0 \qquad \text{when} \qquad h \neq l,$$
$$e_{jh} e_{lk} = e_{jk} \qquad \text{when} \qquad h = l.$$

For $m = 1$ we have the system

$$e_1^2 = e_1.$$

For $m = 2$ we have the system which is equivalent to (H),

1	2	0	0
0	0	1	2
3	4	0	0
0	0	3	4

where

$$e_{11} = e_1, \; e_{12} = e_2, \; e_{21} = e_3, \; e_{22} = e_4.$$

When $m = 3$ we have the multiplication tables of nonians. Molien**) shows that if a system S contains several primitive subsystems

$$P_1, P_2, \cdots, P_\lambda,$$

the units of S may be so chosen that all these subsystems appear in the multiplication table, showing that the units which comprise

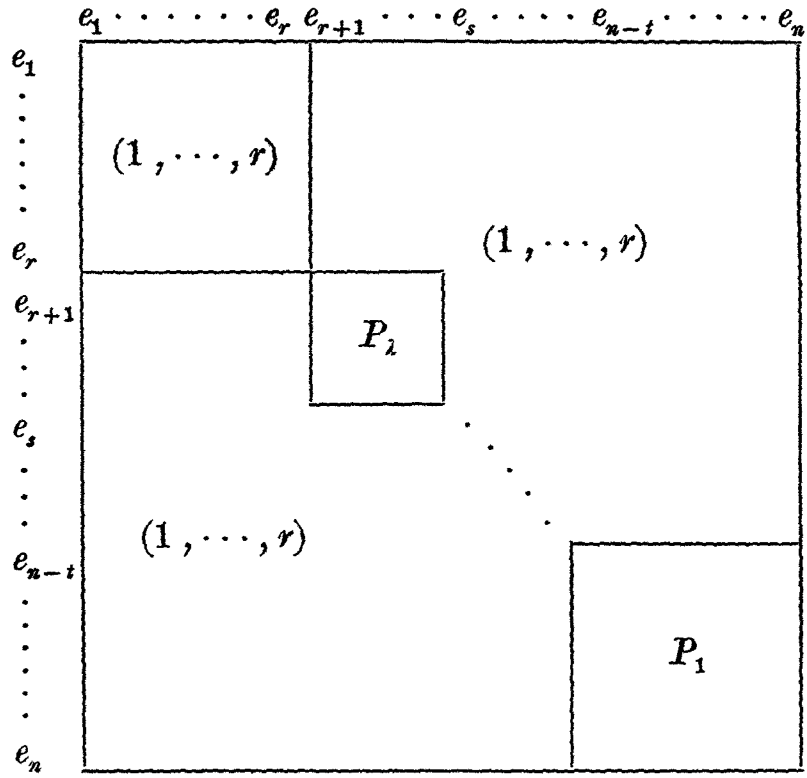
$$P_1, P_2, \cdots, P_\lambda$$

are independent of each other. The product of units one or more of which is not a unit of a primitive system can contain no unit of a primitive system.

*) loc cit.

**) loc. cit.

These theorems we may display in a table as follows.



where $(1, \cdots, r)$ indicates a number involving the units $e_1, \cdots, e_r$ and all the subsystems $P_1, \cdots, P_\lambda$ contain a square number of units whose multiplication table is given on page 439.

The order of these primitive systems along the principal diagonal is immaterial. It is convenient to assume that the primitive systems in one unit appear first, followed by those of higher order. When a system is thrown in any way into normal form (M) the same set of primitive systems appears, showing that they are an invariant of the system. We can now see that the necessary and sufficient condition that a system is a quaternion system is that at least one of its primitive systems is of order $\geqq 2$. The necessity of the condition follows since any four units $e_{\lambda\lambda}, e_{\lambda\tau}, e_{\tau\lambda}, e_{\tau\tau}$ of a primitive system have (H) for a multiplication table.

2. Normal form (P).

Def. 6. A number α is called *idempotent* if $\alpha^2 = \alpha$.

Def. 7. A number α is called *nilpotent* if $\alpha^2 = 0$.

If a system contains an idempotent number, this number may be taken as the unit e_n and the other units of the system so chosen as to fall into the following groups: —

Group I contains only units e_k such that

$$e_k e_n = e_n e_k = e_k.$$

Group II contains only units e_k such that

$$e_k e_n = 0;\ e_n e_k = e_k.$$

Group III contains only units e_k such that

$$e_k e_n = e_k;\ e_n e_k = 0.$$

Group IV contains only units e_k such that

$$e_k e_n = e_n e_k = 0.*)$$

When the transformation bringing the system into this form has been performed, the system is called regular with respect to e_n. The four groups are symbolized respectively by (dd), (dn), (nd), (nn). Evidently e_n itself is in group I. The following multiplication table shows the group to which the non-vanishing product of units of any two groups must belong.

	(dd)	(dn)	(nd)	(nn)
(dd)	(dd)	(dn)	0	0
(dn)	0	0	(dd)	(dn)
(nd)	(nd)	(nn)	0	0
(nn)	0	0	(nd)	(nn)

If an idempotent number independent of e_n remains in any group it may be táken as a unit, say e_{n-1}, and the system made regular with respect to it without disturbing the regularity with respect to e_n. This process may be continued until no two idempotent numbers occur in the same group with respect to any unit. A system in which every unit is in one of the four groups with respect to each and every idempotent unit is called *regular*, or in normal form (P). The modulus of a system in form (P) is the sum of its idempotent units.

Theorem I. *Normal forms (M) and (P) are compatible.*

Assume that the system S is in form (M). We must show that without destroying form (M) the system may be thrown simultaneously into form (P). The idempotent units in the primitive systems of (M) are all in group IV with respect to each other, and no idempotent unit independent of these units exists else it would appear as a primitive system of order 1. The nilpotent units of the primitive subsystems are already in either group II, III or IV with respect to each idempotent unit. It only remains to regularize the units $e_1, \cdots, e_r$ which are not in

*) For proofs of this and the following theorems see my paper in Transactions, loc. cit.

any primitive subsystem. That this can be accomplished without affecting the form (M), but by linear transformations involving only the units $e_1, \cdots, e_r$ appears from the method of regularizing a system.*)

3. Normal Form (S).

For a non-quaternion system which, as we have seen, is a system containing no primitive subsystem of order greater than 1, Scheffers has given a normal form, and the explicit enumeration of distinct systems for $n \leqq 5$ has been given by him. The enumeration of systems in one idempotent unit has been carried out for the general case by Starkweather.**) Explicit enumeration for the case where the system contains more than one idempotent unit will be found for $n = 6$ in vol. 58 of these Annalen, page 370, and for $n = 7$ in the American Journal of Mathematics vol. 26.

Theorem II. *Normal form (M) and normal form (S) are compatible.*

If in any quaternion system the nilpotent units of the primitive sub-systems are deleted we have an associative non-quaternion system which may by a transformation T which involves only the remaining nilpotent units, be thrown into normal form (S), all the distinct types of which for a given order are known. Since such a transformation does not involve the idempotent units it might properly have been applied to the undeleted system thus throwing the non primitive portion of the system together with the corresponding idempotent units into form (S). Since a non quaternion system in form (S) is also in form (P) this transformation does not affect the regularity.

We can now restate the results as follows: —

If a system is in form (M) it may be transformed so as to fall simultaneously in form (P). The units exclusive of the nilpotent units in the primitive subsystems form a non-quaternion system and may be assumed in form (S). A system in this form is said to be in form (N).

§ 3.

Principles of Classification.

1. Equivalence.

Theorem III. *If two systems in form (N) have different numbers of idempotent units they are inequivalent.*

This is evident from the invariance of the primitive sub-systems.

The significance of this theorem is that when we are seeking all types of inequivalent systems of order n, we may make our enumeration

*) Transactions, loc. cit. page 314.

**) American Journal of Mathematics, Vols. 21, 23.

for different numbers of idempotent units separately without possibility of repetition.

Lemma. *If the systems S and S' are in form (N) and are equivalent, and if e' is an idempotent unit in S' then the equation of transformation*

$$e' = \sum_{i=1}^{n} a_i e_i \tag{1}$$

reduces to

$$e' = e$$

where e is an idempotent unit of S.

Symbolize by s any nilpotent units of S not contained in a primitive sub-system, and by p nilpotent units that are contained in some primitive sub-system. The number $e' = s + p$ is then not idempotent. For products of the form s^2, sp, ps involve only units s by the table on page 440, while an idempotent number of the primitive subsystems consisting of nilpotent units does not exist. Thus the transformation equation (1) must contain at least one idempotent unit of S in its right hand member. Since the primitive sub-systems are an invariant of the system there are the same number of idempotent units in S as in S'. If then two or more idempotent units of S occur in the right hand member of (1), one of them must appear in the equation of transformation of a second idempotent unit of S', say e''. But since $e'^2 = e'$, $e''^2 = e''$ and $e'e'' = 0$ the right hand members of the equations in question cannot contain the same idempotent unit of S. Thus each idempotent equation contains one and only one idempotent unit of S. It remains to prove that (1) can contain no nilpotent unit in its right hand member. The equation (1) is then in form

$$e' = s + p + e_i$$

where e_i is an idempotent unit of S. Let now p_{ji}, p_{ik}, p_{lm}, represent those units in p which are in groups III, II and IV respectively with respect to e_i. Thus

$$e' = s + p_{lm} + p_{ji} + p_{ik} + e_i \qquad (l, m, j, k \neq i).$$

Squaring we get

$$p_{lm} = p_{lm}^2 + p_{lm}p_{ji} + p_{lm}p_{ik} + p_{ik}p_{lm} + p_{ik}p_{ji} + p_{ji}p_{lm} + p_{ji}p_{ik}.$$

But by the table on page 441 this reduces to

$$p_{lm} = p_{lm}^2 + p_{ji}p_{ik}.$$

One observes that p_{lm}^2 is idempotent, which is however impossible since it is expressed in terms of nilpotent units. Thus (1) is reduced to the form

$$e' = s + p_{ji} + p_{ik} + e_i. \qquad (j, k \neq i) \tag{2}$$

Let now e_k be the idempotent unit of S with respect to which some units of p_{ik} are in group III, and let

$$e'' = s + p'_{lk} + p'_{km} + e_k \qquad (l, m \neq k) \tag{3}$$

be the equation of transformation containing e_k. Since $e'e'' = 0$ we have

$$p_{ji}p'_{lk} + p_{ik}p'_{km} + e_i p'_{lk} + p_{ik}e_k = 0$$

Since $k \neq m$, $p_{ik}p'_{km} = 0$; since $j \neq i$, $p_{ji}p'_{lk} = 0$ i. e. $i \neq l$; since $i \neq l$, $e_i p'_{lk} = 0$. Thus $p_{ik}e_k = 0$ which is contrary to the hypothesis. Consequently $p_{ik} = 0$. We can show similarly that $p_{ji} = 0$, and our equation (1) is reduced to the form

$$e' = s + e.$$

Since all units s are in normal form (S), there can be no s units in the right hand member of (1)*), and our equation (1) is reduced to the form

$$e' = e.$$

Theorem IV. *If S and S' are equivalent there is a one to one correspondance between the idempotent units of the two systems such that the number of units in the groups I, II, III, IV, with respect to corresponding units is the same.*

The preceding lemma shows the existence of the one to one correspondence between the idempotent units of S and S'. That the theorem is true so far as the units of the primitive subsystems are concerned follows from the invariance of these subsystems in equivalent systems. The remaining units are in normal form (S) and the complete validity of the present theorem is established by the corresponding theorem on non-quaternion systems.**) This theorem puts us in a position to write down all possible combinations of groups with respect to idempotent units into which the remaining non-idempotent units may fall, and assures us that no two systems in different combinations can be equivalent.

2. Reducibility.

Evidently any system all of whose units fall in primitive subsystems is reducible. This is a special case of the theorem proved in general in my paper in the Annalen***), that the necessary and sufficient condition that a system is reducible is that its modulus falls into parts, each of which is the modulus of a certain subsystem. Thus if we start with an idempotent unit and find it connected with every other idem-

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*) Scheffers, loc. cit., page 329.

**) Math. Annalen, vol. 58, page 365.

***) loc. cit. page 366.

potent unit by a chain of non-vanishing multiplicative relations with other units, then the whole modulus lies wholly in one subsystem and the system is irreducible. If on the other hand not all idempotent units are thus connected the modulus falls apart and the system is reducible.

3. Reciprocity.

Reciprocal systems evidently have the same number of idempotent units. Let S and S^{-1} be systems with their units so chosen that the table for S passes into that of S^{-1} by an interchange of rows and columns. We may assume that both systems are in normal form (N).

As we pass in this way from S to S^{-1} we note that the same units constitute the various groups I and IV in each system. The units of II with respect to a given unit pass into units of III with respect to the same unit and conversely, thus leaving the primitive subsystems unchanged. If then, from the totality of combinations of units into groups, we erase every combination which differs from another merely by an interchange of the number of units in the group II and III with respect to the various idempotent units, we shall erase all combinations which lead to systems reciprocal to those that remain, and only such.

4. Removal of Parameters.

The only parameters in normal form that remain to be removed are those found in the products of the non idempotent units in the primitive subsystems and those not in the subsystems. Application of the table on page 441 serves to remove most of them, while the remaining ones may be fixed by direct application of the associative law or by the principle of deletion.*)

§ 4.

Illustration; $n = 7$.

In the following tables of combinations of units into groups the primitive subsystem (H) contains the units e_4, e_5, e_6, e_7 of which e_4 and e_7 are idempotent, while e_5 and e_6 are in groups II and III respectively with respect to e_4, and in groups III and II respectively with regard to e_7. In the following tables the indices of the idempotent units are in the upper line of the table, while the indices of the non-idempotent numbers are in the left hand column. The group of e_i with respect to e_k is found at the intersection of the i^{th} row and k^{th} column.

*) Math. Annalen, vol. 58, page 367ff.

	4	7	4	7	4	7	4	7	4	7	4	7
1	I	IV	I	IV	I	IV	I	IV	II	III	II	III
2	I	IV	I	IV	II	III	II	III	II	III	III	II
3	I	IV	II	III	II	III	III	II	II	III	III	II
4	II	III	II	III	II	III	II	III	II	III	II	III
5	III	II	III	II	III	II	III	II	III	II	III	II

It appears that none of the six distinct tables of combination given yield associative systems. By the table on page 441 we see that for the first four combinations given

$$e_4 = \overleftarrow{e_1 e_5}\,\underrightarrow{e_6} = e_1$$

where the arrow indicates the order of multiplication. That is

$$e_4 = e_1 e_5 \cdot e_6 \quad \text{while} \quad e_1 \cdot e_5 e_6 = e_1 .$$

For the last two systems

$$e_5 = \overleftarrow{e_1 e_6}\,\underrightarrow{e_5} = e_1 .$$

Thus where $(\dot{H})$ is the only primitive subsystem we get no quaternion system for $n = 7$.

Suppose now we have in addition to the primitive subsystem (H) a primitive subsystem of order one; that is an idempotent unit e_3. We have then the following tables of combination.

	3	4	7	3	4	7	3	4	7	3	4	7	3	4	7
1	I	IV	IV	IV	I	IV	IV	IV	I	II	III	IV	II	III	IV
2	II	III	IV	II	III	IV	II	III	IV	II	III	IV	II	IV	III
5	IV	II	III	IV	II	III	IV	II	III	IV	II	III	IV	II	III
6	IV	III	II	IV	III	II	IV	III	II	IV	III	II	IV	III	II

3	4	7	3	4	7	3	4	7	3	4	7
II	III	IV	II	III	IV	II	III	IV	II	III	IV
III	II	IV	III	IV	II	IV	II	III	IV	III	II
IV	II	III	IV	II	III	IV	II	III	IV	II	III
IV	III	II	IV	III	II	IV	III	II	IV	III	II

Of these nine combinations the first four are seen to be non-associative by the product $e_2 e_5 e_6$, while the last four are also non-associative by the product $e_1 e_5 e_6$. The remaining combination affords the system

	1	2	3	4	5	6	7
1	0	0	0	1	2	0	0
2	0	0	0	0	0	1	2
3	1	2	3	0	0	0	0
4	0	0	0	4	5	0	0
5	0	0	0	0	0	4	5
6	0	0	0	6	7	0	0
7	0	0	0	0	0	6	7

which is the only quaternion system in seven units, as additional idempotent units would yield reducible systems.

Yale University, May 21, 1904.