GENERALISATION OF A THEOREM IN THE THEORY OF DIVERGENT SERIES

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[Received July 23rd, 1907.—Read November 14th, 1907.]

1. In a paper recently printed in these *Proceedings** I proved the following theorem† :—*If*

(1) Σa_n *is a series summable by Césaro's method of mean values, i.e., if*

$$(s_0+s_1+\ldots+s_n)/(n+1),$$

where
$$s_n = a_0+a_1+\ldots+a_n,$$

tends to a finite limit as n tends to infinity;

(2) f_n *is a function of n which, together with its first and second differences*

$$f_n-f_{n+1}, \qquad f_n-2f_{n+1}+f_{n+2},$$

is positive for all values of n;

then the series $\Sigma a_n f_n$ *is also summable.*

Further, if f_n *is also a function of a variable x, and the condition* (2) *is satisfied throughout a certain interval of values of x, say* (0, 1), *and* f_0 *has a finite upper limit throughout this interval,*‡ *then the series* $\Sigma a_n f_n$ *is* **uniformly** *summable throughout the interval: and if every* f_n *is a continuous function of x, the sum of the series is also a continuous function of x.*

I also stated (*l.c.*, p. 267) that I had no doubt of the truth of an obvious generalisation of this theorem. Suppose that the first of the quantities

$$s_n^1 = \frac{s_0+s_1+\ldots+s_n}{n+1},$$

$$s_n^2 = \frac{s_0^1+s_1^1+\ldots+s_n^1}{n+1},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 247.

† *L.c.*, p. 256.

‡ It is obvious that the same is true of f_n.

which tends to a limit as n tends to infinity, is s_n^k. Then the series Σa_n may be said to be *summable* (Hk).*

Then it is natural to suppose that the theorem may be generalised by supposing Σa_n to be summable (Hk), and the $k+1$ sets of differences

$$\Delta f_n,\ \Delta^2 f_n,\ \ldots,\ \Delta^{k+1} f_n,$$

to be positive. But when I wrote my former paper I had not been able to overcome the considerable algebraical difficulties which appeared to be involved in the proof of this theorem.

On the other hand, the theorem which I had proved was not sufficient to deal with all the interesting particular cases which actually arise when we try to make applications of it (*v.* p. 264 of my former paper). I was therefore led to consider in greater detail the most interesting particular case, viz., that in which the f_n's are such that $\Sigma a_n f_n$ is *convergent* for all points of (0, 1) except $x = 0$, and

$$\lim_{x \to 0} f_n = 1,$$

for all values of n; and I obtained three theorems† which were sufficiently general for the purposes of the applications which I had in view. Mr. Bromwich then proved a more general theorem which included all these theorems and also some very similar theorems arrived at independently, for the case of $k = 1$, by Dr. C. N. Moore.‡

It is mainly owing to suggestions derived from these latter investigations that I have since been able to prove a theorem which, so far as I know, includes all the theorems which have been referred to. This theorem stands to the generalisation contemplated in my former paper in the same relation which Mr. Bromwich's theorem bears to the first of the theorems which I proved in the *Math. Annalen*: that is to say, the condition

$$f_n,\ \Delta f_n,\ \Delta^2 f_n,\ \ldots,\ \Delta^{k+1} f_n \geqslant 0$$

is replaced by the more general condition that

$$\Sigma n^k \mid \Delta^{k+1} f_n \mid$$

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* This extension of Césaro's method is due (implicitly) to Hölder, *Math. Annalen*, Bd. XX., p. 535.

† "Some Theorems concerning Infinite Series," *Math. Annalen*, Bd. LXIV., p. 77.

‡ Moore, *Trans. Amer. Math. Soc.*, Vol. VIII., p. 299; Bromwich, *Math. Annalen*, Bd. LXV., pp. 359 and 362.

is convergent, or (when f_n is a function of x) that

$$\sum_{\nu=0}^{n} \nu^k \,|\Delta^{k+1} f_\nu| < K$$

for all values of x and n.

2. There are two alternative definitions of the sum of a divergent series on mean value lines when Césaro's original definition fails. One is Hölder's definition stated above, which defines *summability* (Hk). But Césaro himself gave a somewhat similar definition.* Let

$$A_n^k = \frac{(n+1)(n+2)\dots(n+k)}{k!}$$

—which we may, in the ordinary continental notation, write in the form

$$A_n^k = \binom{n+k}{k},$$

—and let $$S_n^k = A_n^k a_0 + A_{n-1}^k a_1 + \dots + A_0^k a_n.$$

And suppose that, as n tends to infinity,

$$S_n^k / A_n^k$$

tends to a limit. Then we shall say that Σa_n is *summable* (Ck).

For $k = 1$ Hölder's and Césaro's definitions are identical. That this is so for $k = 2$ has been proved by Mr. Bromwich.† In all ordinary cases (as applied, *e.g.*, to the series $1^s - 2^s + 3^s - \dots$) the two definitions lead to the same result: and it has been proved by K. Knopp‡ that Césaro's definition *includes* Hölder's—*i.e.*, that *if* a series is summable (Hk) it is also summable (Ck), and the sums agree. It is not unlikely that Césaro's definition is more general: it is conceivable that the two always cover the same ground. But Césaro's definition should certainly be adopted as the standard one; for it is *at least* equally general, and is far more easy to work with in practice, owing to the fact that the expression of a_n in terms of the sums S_n^k is as simple as the reverse equation, whereas the expression of a_n in terms of s_n^k is complicated and clumsy. The contrast appears very clearly when Mr. Bromwich's work, with Césaro's definition, is contrasted with his own, or mine, with Hölder's.

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* Bromwich, *Infinite Series*, pp. 311 *et seq.*

† See pp. 363–5 of his paper in the *Math. Annalen* already quoted.

‡ *Grenzwerte von Reihen u. s. w.*, Inaugural Dissertation, Berlin, 1907, p. 19.

3. The first part of the theorem is as follows :—

THEOREM A.—*If* Σa_n *is summable* (Ck) *and*

$$\Sigma n^k \mid \Delta^{k+1} f_n \mid$$

is convergent, then $\Sigma a_n f_n$ *is summable* (Ck). *Further, its sum is equal to that of the series*

$$\Sigma S_n^k \Delta^{k+1} f_n,$$

which is absolutely convergent.

We note as a matter of minor detail that, if Σa_n is summable (Hk), it is also summable (Ck), and so $\Sigma a_n f_n$ is summable (Ck) : but we cannot affirm that the latter series is summable (Hk), except for $k = 1, 2$.

That $\Sigma S_n^k \Delta^{k+1} f_n$ is absolutely convergent follows at once from the fact that S_n^k / n^k tends to a limit as $n \to \infty$.

Some Algebraical Preliminaries.

4. We denote the sum

$$A_n^k a_0 f_0 + A_{n-1}^k a_1 f_1 + \ldots + A_0^k a_n f_n$$

formed from $\Sigma a_n f_n$, as S_n^k is formed from Σa_n, by T_n^k: and we proceed to express T_n^k in terms of

$$S_0^k,\ S_1^k,\ \ldots,\ S_n^k,$$

and the differences of the functions f_n. We have

$$(1) \qquad a_n = S_n^k - \binom{k+1}{1} S_{n-1}^k + \binom{k+1}{2} S_{n-2}^k - \ldots + (-)^{k+1} S_{n-k-1}^k.*$$

Thus

$$(2) \qquad T_n^k = \sum_{\nu=0}^{n} A_{n-\nu}^k f_\nu \sum_{r=0}^{k+1} (-)^r \binom{k+1}{r} S_{\nu-r}^k.$$

This expression, as it stands, involves a certain number of terms S_j^k with negative suffixes j : these must be considered to be defined as being equal to zero. In this formula for T_n^k the coefficient of S_j^k is

$$\sum_{\nu=j}^{j+k+1} (-)^{\nu-j} \binom{k+1}{\nu-j} A_{n-\nu}^k f_\nu,$$

or

$$\sum_{i=0}^{k+1} (-)^i \binom{k+1}{i} A_{n-j-i}^k f_{j+i}.$$

* It is easy to see (Bromwich, *Infinite Series*, *l.c.*) that

$$\Sigma S_n^k x^n \equiv (1-x)^{-(k+1)} \Sigma a_n x^n,$$

and

$$\Sigma a_n x^n \equiv (1-x)^{k+1} \Sigma S_n^k x^n.$$

If this expression contains any terms for which $j+i > n$, they may simply be omitted. Thus, with this proviso, (2) may be written in the form

$$T_n^k = \sum_{j=0}^{n} S_j^k \sum_{i=0}^{k+1} (-)^i \binom{k+1}{i} A_{n-j-i}^k f_{j+i} = \sum_{j=0}^{n} a_j S_j^k, \tag{3}$$

say.

5. Now

$$a_j = \sum_{i=0}^{k+1} (-)^i \beta_{j+i} f_{j+i}, \tag{4}$$

where
$$\beta_{j+i} = \binom{k+1}{i} A_{n-j-i}^k.$$

From this it follows that

$$a_j = \sum_{i=0}^{k+1} (-)^i \gamma_{j+i} \Delta^{k+1-i} f_{j+i}, \tag{5}$$

where

$$\gamma_{j+i} = \beta_{j+i} - \binom{k-i+2}{1} \beta_{j+i-1} + \binom{k-i+3}{2} \beta_{j+i-2} - \dots \tag{6}$$

$$= \sum_{\nu=0}^{i} (-)^\nu \binom{k-i+1+\nu}{\nu} \beta_{j+i-\nu}.$$

To verify this result substitute for γ_{j+i} in the expression (5), and pick out the coefficient of $\beta_{j+\lambda}$. We find this coefficient to be $(-1)^\lambda$ times

$$\binom{k-\lambda+1}{0} \Delta^{k-\lambda+1} f_{j+\lambda} + \binom{k-\lambda+1}{1} \Delta^{k-\lambda} f_{j+\lambda+1}$$
$$+ \binom{k-\lambda+1}{2} \Delta^{k-\lambda-1} f_{j+\lambda+2} + \dots = \sum_{i=\lambda}^{k+1} \binom{k-\lambda+1}{i-\lambda} \Delta^{k-i+1} f_{j+i},$$

and it is easy to see that this reduces to $f_{j+\lambda}$.* Thus

$$\gamma_{j+i} = \sum_{\nu=0}^{i} (-)^\nu \binom{k-i+1+\nu}{\nu} \binom{k+1}{i-\nu} A_{n-j-i+\nu}^k. \tag{8}$$

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* The simplest proof is probably by means of symbolical operators. Let E denote the operation which, when performed on f_n, changes it into f_{n+1}. The expression above, on writing $i = \lambda + \mu$, becomes

$$\sum_{\mu=0}^{k+1-\lambda} \binom{k+1-\lambda}{\mu} \Delta^{k+1-\lambda-\mu} E^\mu f_{j+\lambda} = \left(1 + \frac{E}{\Delta}\right)^{k+1-\lambda} \Delta^{k+1-\lambda} f_{j+\lambda} = (\Delta + E)^{k+1-\lambda} f_{j+\lambda}.$$

But
$$(\Delta + E) f_n = f_n - f_{n+1} + f_{n+1} = f_n,$$

whence the result.

But this expression may be simplified considerably. For

$$\binom{k-i+1+\nu}{\nu}\binom{k+1}{i-\nu}A^k_{n-j-i+\nu}$$

$$= \frac{(k-i+1+\nu)!}{(k-i+1)!\,\nu!}\,\frac{(k+1)!}{(i-\nu)!\,(k-i+1+\nu)!}\,\frac{(n-j-i+\nu+k)!}{k!\,(n-j-i+\nu)!}$$

$$= \binom{k+1}{i}\binom{i}{\nu}\binom{n-j-i+\nu+k}{k},$$

and so

(9) $$\gamma_{j+i} = \binom{k+1}{i}\sum_{\nu=0}^{i}(-)^\nu\binom{i}{\nu}\binom{n-j-i+\nu+k}{k}.$$

But $$\sum_{\nu=0}^{i}(-)^\nu\binom{i}{\nu}\binom{n-j-i+\nu+k}{k}$$

is the coefficient of t^k in

$$\sum_{\nu=0}^{i}(-)^\nu\binom{i}{\nu}(1+t)^{n-j-i+\nu+k},$$

or $$(1+t)^{n-j-i+k}\{1-(1+t)\}^i,$$

or $$(-)^i\,t^i\,(1+t)^{n-j-i+k};$$

and is therefore equal to $(-)^i\binom{n-j-i+k}{k-i}$,

if $0 \leqslant i \leqslant k$, and to zero if $i = k+1$. Thus

(10) $$\gamma_{j+i} = (-)^i\binom{k+1}{i}\binom{n-j-i+k}{k-i}.$$

Hence

(11) $$a_j = \sum_{i=0}^{k}\binom{k+1}{i}\binom{n-j-i+k}{k-i}\Delta^{k+1-i}f_{j+i},$$

and

(12) $$T^k_n = \sum_{j=0}^{n} S^k_j \sum_{i=0}^{k}\binom{k+1}{i}\binom{n-j-i+k}{k-i}\Delta^{k+1-i}f_{j+i},$$

with the proviso, we may repeat, that if $j+i > n$ we must write 0 for f_{j+i}. This formula is the end of our algebraical transformations.*

* It has been suggested to me that these transformations should be capable of being simplified, and I do not doubt that this is so; but I have not been able to effect any appreciable simplification.

6. Suppose, *e.g.*, that $k = 1$. Then (12) becomes

$$(12_1) \qquad T_n^1 = \sum_{j=0}^{n} S_j^1 \{(n-j+1)\,\Delta^2 f_j + 2\Delta f_{j+1}\},$$

which is easily verified. If $k = 2$, (12) becomes

$$(12_2) \quad T_n^2 = \sum_{j=0}^{n} S_j^2 \left\{\frac{(n-j+2)(n-j+1)}{2}\,\Delta^3 f_j + 3(n-j+1)\,\Delta^2 f_{j+1} + 3\Delta f_{j+2}\right\},$$

and so on.

7. We can now proceed to the proof of our theorem. We suppose that

$$\Sigma n^k \,|\, \Delta^{k+1} f_n \,|$$

is convergent. If this is so the same is true, as has been shown by Mr. Bromwich,* of all the series

$$\Sigma n^{k-\lambda} \,|\, \Delta^{k+1-\lambda} f_n \,| \quad (\lambda = 0, 1, \ldots, k).$$

We have to show that in these circumstances

$$\lim (T_n^k / A_n^k) = \sum_{0}^{\infty} S_n^k \Delta^{k+1} f_n.$$

8. We consider first the terms in T_n^k for which $i = 0$. These give

$${}_0T_n^k = \sum_{j=0}^{n} \binom{n-j+k}{k} S_j^k \Delta^{k+1} f_j.$$

Now $\binom{n-j+k}{k} - A_n^k = \frac{1}{k!}\{(n-j+1)(n-j+2)\ldots(n-j+k)$

$$- (n+1)(n+2)\ldots(n+k)\},$$

which is negative and numerically less than

$$Kn^{k-1},$$

where K is a constant. Thus

$$\frac{{}_0T_n^k}{A_n^k} = \sum_{j=0}^{n} S_j^k \Delta^{k+1} f_j + R_0,$$

where

$$|R_0| < \frac{K}{n} \sum_{j=0}^{n} |\Delta^{k+1} f_j|;$$

and so

$$(13) \qquad \lim_{n\to\infty} \left(\frac{{}_0T_n^k}{A_n^k}\right) = \sum_{j=0}^{\infty} S_n^k \Delta^{k+1} f_j.$$

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* *Math. Annalen*, *l.c.*, p. 361.

9. Next we consider

$$ {}_iT_n^k = \binom{k+1}{i} \sum_{j=0}^{n} \binom{n-j-i+k}{k-i} S_j^k \Delta^{k+1-i} f_{j+i}. $$

Since

$$ \binom{n-j-i+k}{k-i} = (n-j+1)(n-j+2)\dots(n-j+k-i)/(k-i)! < Kn^{k-i}, $$

it follows that
$$ \left| \frac{{}_iT_n^k}{A_n^k} \right| < \frac{K}{n^i} \sum_{j=0}^{n} \mid \Delta^{k+1-i} f_{j+1} \mid ; $$

and therefore

(14) $$ \lim_{n \to \infty} \left(\frac{{}_iT_n^k}{A_n^k} \right) = 0. $$

From (13) and (14) it follows that

(15) $$ \lim (T_n^k / A_n^k) = \sum_{j=0}^{\infty} S_j^k \Delta^{k+1} f_j, $$

which establishes the theorem.

10. THEOREM B.—*If, in addition,*

$$ \sum_{0}^{n} \nu^k \mid \Delta^{k+1} f_\nu \mid < K^* $$

for all values of n and x, then the series

$$ \Sigma S_j^k \Delta^{k+1} f_j $$

is **uniformly** *convergent.*

Let S be the sum (Ck) of the series Σa_n: and let $\Sigma a_n'$ be the series for which

$$ a_0' = a_0 - S, \quad a_n' = a_n \quad (n > 0), $$

so that $S' = 0$. Then

$$ \sum_{j=m}^{m'} S_j^k \Delta^{k+1} f_j = S \sum_{m}^{m'} \Delta^{k+1} f_j + \sum_{m}^{m'} S_j'^k \Delta^{k+1} f_j = \sigma_1 + \sigma_2, $$

say. Choose m so that for $j \geqslant m$,

$$ \mid S_j'^k / A_j^k \mid < \epsilon. $$

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* Mr. Bromwich (*l.c.*, p. 361) has proved that the same is then true of

$$ \sum_{0}^{n} \nu^{k-\lambda} \mid \Delta^{k+1-\lambda} f_\nu \mid \quad (\lambda = 0, 1, \dots, k). $$

Then

(16) $$|\sigma_2| < 2\epsilon K.$$

Also

(17) $$\left|\sum_{m}^{m'} \Delta^{k+1} f_j\right| < \frac{1}{m^k}\sum_{m}^{m'} j^k \mid \Delta^{k+1} f_j \mid < \frac{K}{m^k},$$

and from (16) and (17) the theorem follows.

COROLLARIES.—(α) *If every f_n is continuous, the sum of the series $\Sigma a_n f^n$ is continuous.*

(β) *If all the differences*

$$f_n,\ \Delta f_n,\ \ldots,\ \Delta^{k+1} f_n$$

are positive, the condition $\sum_{0}^{n} n^k \Delta^{k+1} f_n < K$

is certainly satisfied, and the conclusions of the theorem apply.

The proof of this will be found in Lemma A of my paper in the *Math. Annalen* quoted above.

11. *Applications.*—I have already stated that the very general theorems proved by Messrs. Fejér, Moore, and Bromwich, and myself, with especial reference to a particular case, enable us to deal effectively enough with the majority of interesting special applications which occur naturally in analysis. It would therefore be futile to give any considerable number of illustrations here. In the paper cited above* I pointed out the kind of case in which a more general theorem of the kind here proved is necessary. A simple example is given by supposing

$$f_n = \frac{1}{(a+nx)^s} \quad (s > 0).$$

If the series Σa_n is summable (Ck) it follows that

$$\Sigma \frac{a_n}{(a+nx)^s}$$

is uniformly summable (Ck) in any interval $(0, \xi)$. Thus, *e.g.*,

$$\Sigma \frac{(-)^n n^t}{(a+nx)^s},$$

* *L.c.*, p. 85.

where $0 < t < k$, is uniformly summable (Ck) and is a continuous function of x for $x = +0$. In order to deal with this by my former theorems it was necessary to suppose $s > k+1$, while Mr. Bromwich's theorem required $s > k$—the series being then *convergent* except for $s = 0$.

Even in the theorem here proved, however, it must be observed that f_n is what Dr. Moore has called a *convergence factor:* its introduction into the series Σa_n makes that series, if not convergent, at any rate *more summable*. The series

$$\Sigma (-)^n n^t (a+nx)^s \quad (s > 0),$$

in which f_n is a *divergence* factor, and $\Sigma a_n f_n$ *less* summable than Σa_n, falls outside the scope of any theorem hitherto proved, though, of course, it may be dealt with easily enough by special devices.

The Theorems A, B, however, seem to me interesting less on account of any of their applications than as a contribution to the abstract theory of divergent series, and as marking something like the limit of what may reasonably be expected to be proved concerning the introduction of convergence factors into series summable by the method of mean values.