On a Certain Class of Plane Quartics. *By* Prof. G. B. MATHEWS.

[*Read Jan. 8th*, 1891.]

A quartic curve of the most general kind may always be obtained as the envelope of the first polars of points on a conic with respect to a fixed cubic. The determination of the cubic involves nine conditions, and that of the conic five more; in general, therefore, when the quartic is given, the problem of finding a suitable conic and cubic admits of a finite number of solutions; the number is, in fact, sixty-three. Suppose, however, that the S invariant of the cubic vanishes; then the canonical equation of the cubic may be taken to be

$$x^3+y^3+z^3=0 \quad\ldots\ldots\ldots\ldots(1);$$

and if
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad\ldots\ldots\ldots\ldots(2)$$

is the conic, then the quartic is

$$\begin{vmatrix} a & h & g & x^2 \\ h & b & f & y^2 \\ g & f & c & z^2 \\ x^2 & y^2 & z^2 & 0 \end{vmatrix} = 0,$$

or
$$\phi \equiv Ax^4+By^4+Cz^4+2Fy^2z^2+2Gz^2x^2+2Hx^2y^2=0 \quad\ldots\ldots\ldots(3).$$

Now, if λ, μ, ν are any three constants, none of which is zero, the equation of the quartic may be expressed in the form

$$\begin{vmatrix} a\lambda^2 & h\lambda\mu & g\nu\lambda & \lambda x^2 \\ h\lambda\mu & b\mu^2 & f\mu\nu & \mu y^2 \\ g\nu\lambda & f\mu\nu & c\nu^2 & \nu z^2 \\ \lambda x^2 & \mu y^2 & \nu z^2 & 0 \end{vmatrix} = 0,$$

and hence there is a doubly infinite system of conics and cubics from which the quartic may be derived, namely,

$$a\lambda^2x^2+b\mu^2y^2+c\nu^2z^2+2f\mu\nu yz+2g\nu\lambda zx+2h\lambda\mu xy=0 \quad\ldots\ldots\ldots(4),$$

$$\lambda x^3+\mu y^3+\nu z^3=0 \quad\ldots\ldots\ldots\ldots(5).$$

All the cubics (5) have the same Hessian,

$$xyz = 0.$$

The line $x = 0$ cuts (4) in two points, given by

$$b\mu^2 y_1^2 + 2f\mu\nu y_1 z_1 + c\nu^2 z_1^2 = 0,$$

and, combining this with $\mu y_1 y^2 + \nu z_1 z^2 = 0$,

we get $$bz^4 - 2fy^2z^2 + cy^4 = 0 \quad\text{.........................(6)}$$

for the equation of the two pairs of double tangents to the quartic which go through the point $(y = 0,\ z = 0)$.

Similarly, $$cx^4 - 2gz^2x^2 + az^4 = 0,$$

$$ay^4 - 2hx^2y^2 + bz^4 = 0$$

represent four other pairs of double tangents.

Observing that

$$A\phi = (Ax^2 + Hy^2 + Gz^2)^2 + \Delta\,(cy^4 - 2fy^2z^2 + bz^4),$$

where $$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix};$$

we see that the eight points of contact of the four tangents

$$cy^4 - 2fy^2z^2 + bz^4 = 0,$$

lie on the conic $$Ax^2 + Hy^2 + Gz^2 = 0 \quad\text{.........................(7).}$$

Similarly, $$Hx^2 + By^2 + Fz^2 = 0,$$

$$Gx^2 + Fy^2 + Cz^2 = 0,$$

pass each through eight points of contact of four double tangents. In order to find the other double tangents, put

$$cy^4 - 2fy^2z^2 + bz^4 = c\,(y^2 - m_1^2 z^2)(y^2 - m_2^2 z^2) \quad\text{............(8),}$$

then the conic

$$\lambda^2\,(y - m_1 z)(y - m_2 z) + 2\lambda\,(Ax^2 + Hy^2 + Gz^2)$$

$$-\Delta c\,(y + m_1 z)(y + m_2 z) = 0 \quad\text{............(9)}$$

touches the quartic in four points, and will represent a pair of double tangents if

$$\begin{vmatrix} 2\lambda A & 0 & 0 \\ 0 & \lambda^2+2H\lambda-\Delta c & \tfrac{1}{2}(\lambda^2+\Delta c)(m_1+m_2) \\ 0 & \tfrac{1}{2}(\lambda^2+\Delta c)(m_1+m_2) & (\lambda^2-\Delta c)\,m_1m_2+2\lambda G \end{vmatrix} = 0.$$

The roots $\lambda = 0$ and $\lambda = \infty$ give

$$(y+m_1z)(y+m_2z) = 0 \quad \text{and} \quad (y-m_1z)(y-m_2z) = 0.$$

The other roots satisfy the equation

$$(m_1-m_2)^2\lambda^4-8(G+Hm_1m_2)\lambda^3+\{2\Delta c(m_1+m_2)^2+8\Delta cm_1m_2-16GH\}\lambda^2$$
$$+8\Delta c\,(G+Hm_1m_2)\,\lambda+(m_1-m_2)^2\Delta^2c^2 = 0.$$

Now $$m_1^2+m_2^2 = \frac{2f}{c},$$

$$m_1^2m_2^2 = \frac{b}{c};$$

therefore $$(m_1-m_2)^2 = \frac{2\,(f-\sqrt{bc})}{c},$$

$$G+Hm_1m_2 = \frac{(hf-bg)\,c+(fg-ch)\sqrt{bc}}{c}$$

$$= \frac{(f-\sqrt{bc})(ch+g\sqrt{bc})}{c};$$

and $$2\Delta c\,(m_1+m_2)^2+8\Delta cm_1m_2-16GH$$
$$= 4\Delta f+12\Delta\sqrt{bc}-16GH$$
$$= -12\Delta\,(f-\sqrt{bc})+16\,(\Delta f-GH)$$
$$= -12\Delta\,(f-\sqrt{bc})-16AF$$
$$= 4\,(f-\sqrt{bc})\,\{4F\,(f+\sqrt{bc})-3\Delta\}.$$

Thus the equation for λ becomes, on reduction,

$$\lambda^4-4\,(ch+g\sqrt{bc})\,\lambda^3+2c\,\{4F\,(f+\sqrt{bc})-3\Delta\}\,\lambda^2$$
$$+4\Delta c\,(ch+g\sqrt{bc})\,\lambda+\Delta^2c^2 = 0 \quad \text{..............(10).}$$

Putting $$\lambda = \mu\sqrt{\Delta c},$$

we get
$$\mu^4-\frac{4\,(ch+g\sqrt{bc})}{\sqrt{\Delta c}}\,\mu^3+2\,\frac{4F\,(f+\sqrt{bc})-3\Delta}{\Delta}\,\mu^2$$

$$+\frac{4\,(ch+g\sqrt{bc})}{\sqrt{\Delta c}}\,\mu+1=0;$$

that is,
$$\left(\mu-\frac{1}{\mu}\right)^2-\frac{4\,(ch+g\sqrt{bc})}{\sqrt{\Delta c}}\left(\mu-\frac{1}{\mu}\right)$$

$$-\frac{4\,\{\Delta-2F\,(f+\sqrt{bc})\}}{\Delta}=0.$$

Hence
$$\mu-\frac{1}{\mu}=\frac{2\,(f\sqrt{a}+g\sqrt{b}+h\sqrt{c}+\sqrt{abc})}{\sqrt{\Delta}}\quad\ldots\ldots\ldots\ldots(11),$$

where the ambiguities are to be taken so that
$$\sqrt{a}\,.\,\sqrt{b}\,.\,\sqrt{c}=\sqrt{abc}.$$

Solving the equation for μ, we obtain
$$\mu=\frac{f\sqrt{a}+g\sqrt{b}+h\sqrt{c}+\sqrt{abc}\pm\sqrt{2\,(\sqrt{bc}+f)(\sqrt{ca}+g)(\sqrt{ab}+h)}}{\sqrt{\Delta}}$$
$$\ldots\ldots\ldots\ldots\ldots(12).$$

It is found, without much difficulty, that
$$\lambda^2+2H\lambda-\Delta c=2\lambda\,(\sqrt{bc}+f)(\sqrt{ca}+g),$$

$$(\lambda^2-\Delta c)\,m_1m_2+2\lambda G=2\lambda\,(\sqrt{bc}+f)(\sqrt{ab}+h);$$

and finally, we obtain the equations of the sixteen remaining double tangents in the form
$$\pm\sqrt{\sqrt{bc}+f}\,.\,x\pm\sqrt{\sqrt{ca}+g}\,.\,y\pm\sqrt{\sqrt{ab}+h}\,.\,z=0,$$

where the ambiguities of the inner radicals are to be taken so that
$$\sqrt{bc}\,.\,\sqrt{ca}\,.\,\sqrt{ab}=-abc.$$

February 12th, 1891.

Prof. GREENHILL, F.R.S., President, in the Chair.

The President, having informed the members present of Dr. Casey's recent death, called upon Mr. Tucker to read a biographical note which had been drawn up by an intimate friend of the deceased. Dr. Larmor added a few personal reminiscences.

Mr. Tucker communicated two notes on "Isoscelians," and Mr. Heppel read a paper on "Quartic Equations interpreted by the Parabola." The Chairman read a note from Mr. W. E. Heal, of Indiana, on "the Equation of the Bitangential of the Quintic" (communicated by Prof. Cayley). Mr. Tucker read an abstract of a paper by Mr. J. Buchanan on "the Oscillations of a Spheroid in a Viscous Liquid."

The following presents were received :—

Cabinet likeness of Dr. Forsyth, F.R.S., for the Society's Album.

"Proceedings of the Royal Society," Nos. 295 and 296.

"Educational Times," for February.

"Journal of the Institute of Actuaries," Vol. xxix., Part i.

"Transactions of the Royal Irish Academy," Vol. xxix., Part xiv.

"Proceedings of the Royal Irish Academy," 3rd Ser., Vol. i., No. 4 ; Jan., 1891.

"Annals of Mathematics," Vol. v., No. 4 ; University of Virginia, June, 1890.

"Annual Report of the Smithsonian Institution to July, 1888," 8vo ; Washington, 1890.

"Report of the Superintendent of the U.S. Naval Observatory for the year ending June 30, 1890," 8vo; Washington, 1890.

"Beiblätter zu den Annalen der Physik und Chemie," Band xiv., Stück 12 ; Band xv., Stück 1.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. vi., Fasc. 10, 11, e 12, e Indico del Volume.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," No. 122, and Index and Tavola Sinottica to ditto.

"Journal für die reine und angewandte Mathematik," Band cvii., Heft 3 u. 4.

"Nyt Tiddskrift for Mathematik," A., Vol. i., No. 8 ; B., Vol. i., No. 4 ; Kjobenhavn, 1890.

"Rivista di Matematica," Fasc. i., Jan., 1891 ; Torino, 1891.

"American Journal of Mathematics," Vol. xii., No. 2.

"The Mathematical Magazine," Vol. ii., No. 3 ; Washington, D.C., 1891.

"An Introduction to the Logic of Algebra, with Illustrative Exercises," by Ellery W. Davis, Ph.D. (New York, Wiley & Sons) ; presented by Mr. Tucker.

"Elliptic Functions, an Elementary Text-book for Students of Mathematics," by A. L. Baker, Ph.D. (New York, Wiley & Sons) ; presented by Mr. Tucker.