

Hence $I \frac{d^2 s}{dt^2} = -gh \left\{ \sin(z-r) \cos n \frac{\sin \rho \sin \rho'}{\sin(\rho+\rho')} - \sin r \sin z \right\} s$;

therefore, if L be the length of the equivalent pendulum,

$$\frac{I}{hL} = \sin(z-r) \cos n \frac{\sin \rho \sin \rho'}{\sin(\rho+\rho')} - \sin r \sin z.$$

For the next approximation, let us suppose the rolling body to be a right cone on the summit of another right cone. We then require to include the *cubes* of small quantities. By an easy process of differentiation we find the equation becomes

$$\frac{I}{gh} \frac{d^2 s}{dt^2} = - \frac{\sin^2 \rho \sin z \sin(\rho'+z)}{\sin(\rho+\rho')} s + P \cdot \frac{\sin \rho \sin(\rho'+z)}{\sin^2 z \sin^2 \rho' \sin(\rho+\rho')} \cdot \frac{s^3}{6},$$

where $P = \sin \rho \sin^2(\rho'+z) + \sin \rho \sin^2 \rho' - 3 \sin(\rho'+z) \cos \rho \sin \rho' \sin z$.

The effect of this latter term on the period of the principal term may be easily found.

Inversion, with special reference to the Inversion of an Anchor Ring or Torus. By H. M. TAYLOR, M.A.

[Read April 9th, 1874.]

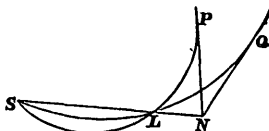
We premise that a straight line inverts into a circle passing through the pole, and *vice versa*; that a circle inverts into a circle, the two circles being subcontrary sections of a cone of the second degree passing through the pole; and that the angles between lines and surfaces at their points of intersection are the same as the angles between the inverse lines and surfaces at the inverse points.

A normal is a straight line cutting a curve or a surface at right angles; it will therefore invert into a circle through the pole cutting the inverse curve or surface at the inverse point at right angles. Such a circle we will call a normal circle.

We will now prove that, if two normals at any two points of a surface intersect and be equal, the normals at the inverse points of the inverse surface also will intersect and be equal.

Let S be the pole, and let PN, QN , the normals to a surface at P, Q , be equal and intersect in N .

If we draw a circle to touch PN at P and pass through S , this will cut SN in a point L such that $NL \cdot NS = NP^2$; and because NP, NQ are equal, the circle touching QN at Q and passing through S will pass



through the same point L. These circles invert into the normals of the inverse surface, which must therefore intersect in the inverse point of L.

Also, in the inverse figure, L and N merely interchange; hence the normals in the inverse figure are equal.

We may state this proposition in other words, thus: If a sphere be described to touch any surface at a number of points, the normals to the inverse surface at the inverse points will all intersect and be equal. This is seen at once by noticing that the touching sphere inverts into a touching sphere.

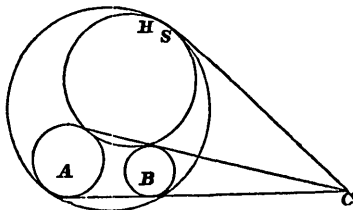
If P, Q, two such points, be very near together on a surface, the line PQ is an element of a line of curvature on the surface. If P', Q' be the inverse points of P, Q, then P'Q' will be an element of a line of curvature on the inverse surface. Hence the known theorem, lines of curvature invert into lines of curvature.

If we draw the four common tangents to two given circles, the intersection of the two external tangents, the intersection of the two internal tangents, and the two centres, are collinear. The other four intersections lie on a circle whose centre is collinear with the four points. By inversion we see that, if we take any two given circles, and draw through a fixed point the four circles which touch each of the given circles, the intersection of the two whose contacts are of opposite kinds, and the intersection of the two whose contacts are of the same kind, lie on the circle through the fixed point orthogonal to the given circles; also that the other four intersections lie on a circle orthogonal to the orthogonal circle.

The locus of a point, the inverses with respect to which of two given circles are two equal circles, is the circle with respect to which the two given circles are inverses of each other.

An analytical proof of this proposition was given in this paper as first presented to the Mathematical Society; but the following proof is more in accordance with the general character of the paper:

If one circle be drawn touching each of two given circles A and B externally, and another touching each internally, these two circles in general will intersect in two points, S and H say, which are collinear with the external centre of similitude of A and B. If the figure be



inverted from S as pole, the inverse of H will be the external centre of similitude of the inverses of A and B. If these inverse circles are equal, the inverse of H must go off to infinity, or, in other words, S

and H must coincide ; and therefore CS^2 must be equal to the product of the tangents from C to A and B.

This proves the proposition for the locus circle whose centre is the *external* centre of similitude. The proof can easily be adapted to the circle whose centre is the *internal* centre of similitude.

If the two given circles intersect, the two locus circles are real and pass through the points of intersection. If the two circles do not intersect, one only of the locus circles is real. The two given circles and the two locus circles have the same radical axis in all cases.

It can easily be deduced, from the results already obtained, that a pair of circles and a locus circle invert, with respect to any pole in their plane, into a pair of circles and a locus circle.

It is clear that the above propositions hold also for spheres, thus : The locus of points, the inverses with respect to which of two given spheres are two equal spheres, is a sphere (the locus sphere) with respect to which the two given spheres are each other's inverses ; and if we invert two spheres and their locus sphere, we obtain two other spheres and their locus sphere.

It follows that it is possible to invert, in plane geometry, any three circles into three equal circles ; and, in solid, any four spheres into four equal spheres.

Any two circles on a sphere can be inverted into two equal circles in a plane through the pole by taking the pole on a definite circle on the sphere.

This can be seen in the following manner : Draw through the two circles two spheres orthogonal to the given sphere. The intersection of the sphere with respect to which these two spheres are each other's inverses with the given sphere will be the locus of the pole required.

From this we can see that any three circles on a sphere can be inverted into three equal circles in a plane through the pole, by taking one of two definite points on the sphere for pole.

An anchor ring is the surface generated by the revolution of a circle about a straight line in its plane.

This straight line is called the *axis* of the anchor ring.

Any plane through the axis cuts the surface in two equal circles of constant radius. Such a plane or section is called *axial*.

Any plane at right angles to the axis cuts the surface in two concentric circles. Such a plane or section is called *transverse*.

All the normals of the anchor ring intersect the axis, and also the circle which is the locus of the centre of the generating circle of the ring. The plane of this last circle is called the principal plane of the ring. A circular section of either system is a line of curvature.

There is a third system of circular sections of the anchor ring. It can be shown that a plane which has double contact with the ring cuts it in two equal circles which intersect at the points of contact of the plane. We will call these the *inclined* circular sections.

We can therefore make the statement that any plane which cuts the ring in a circle cuts it also in a second circle.

The same is true of spheres, as we shall afterwards prove.

The inverse of a surface which has circular lines of curvature will invert into one which has the same property.

The anchor ring has all its lines of curvature circles; therefore its inverse with respect to any pole must have the same property.

A surface which has all its lines of curvature circles is in this paper called a *cyclide*.*

We may remark that the inverse surfaces of all cones, cylinders, and surfaces of revolution must have one set of their lines of curvature circles. The inverses of the right cone and circular cylinder must have all their lines of curvature circles; but as the right cone and right circular cylinder are special cases of the anchor ring, we will restrict ourselves to the anchor ring at present.

Any sphere through one circular section of an anchor ring must meet the ring in another circular section.

If the first section be *transverse*, then the sphere must have its centre in the axis, and therefore will cut the ring in another transverse section.

If the first section be *axial*, then the sphere must have its centre in the principal plane of the ring, and therefore will cut the ring in another axial section.

If the first section be *inclined*, we see that the centre of the sphere must lie on a definite straight line at right angles to the section; and, no matter what point we take in it for the centre of the sphere, we can easily show that there is a second inclined section through which the sphere must pass.

The proposition is therefore true that a sphere through *any* circular section must pass through a second.

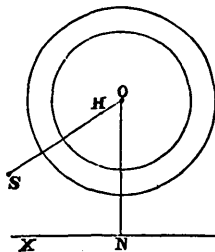
Any plane which cuts the anchor ring in a circle, cuts it also in another circle.

Therefore any sphere or plane which cuts the cyclide in one circle, cuts it also in another circle.

* See note at the end of the paper by Prof. Cayley.

Suppose the plane of the paper to be the principal plane of the anchor ring, O its centre. Let S be any point not in the principal plane which we will take for the pole of inversion.

Now all the spheres which pass through S and cut the anchor ring in axial circles, must pass through a point H in SO such that $OH \cdot OS$ is equal to the square of the tangent from O to the ring. All these spheres must have their centres on NX , the intersection of the principal plane with the plane bisecting SH at right angles; and N , the foot of the perpendicular from O on NX , will be the centre of the section of any one of these spheres by the plane SON ; in other words, all these spheres have a common circular section, the circle whose centre is N and radius NS or NH .



Therefore, in the cyclide, all the planes which give circular sections of one system pass through a straight line (which will be in the plane SNO , and at right angles to SN).

Because N is the centre of this circle, and $OH \cdot OS$ is equal to the square of the tangent from O to it, the axis of the ring is the common radical axis of this circle and the two axial circles in the plane SON . Any circle, therefore, which cuts these axial circles orthogonally will also cut the third circle orthogonally.

Now the orthogonal sphere of the anchor ring through S will invert into a plane cutting the cyclide orthogonally at every point, which we will call one of the principal planes of the cyclide. The straight line through which we have shown the planes of one set of the circular sections must pass will therefore be at right angles to this plane.

Now we have seen that one sphere can be described through S to cut the ring orthogonally, and that its section by the axial plane through S will give a circle passing through S and cutting the two axial circles in that plane orthogonally. Now if we describe two circles to touch each of the equal circles symmetrically and pass through S , they must also pass through K , where K is such that the axis bisects SK at right angles. The inverse, therefore, of K will be the external centre of similitude of the circles, the inverses of the equal circles. Therefore all the planes of the circular sections of one system of the cyclide pass through the centre of similitude of the two circular sections of the principal perpendicular plane.

All spheres which cut the ring in one of the transverse circular sections, and pass through S , must also pass through K . Furthermore, since all their centres lie on the axis, they must have a common circular section (the circle on SK as diameter, coaxial with the anchor ring).

In the cyclide, therefore, all planes which cut the surface in one of the second set of circular sections must cut it also in a second circle, and all these planes must pass through the straight line the inverse of the circle on SK as diameter, coaxial with the ring.

It is also true that a plane or sphere passing through one of the circles which are the inverses of the inclined sections of the anchor ring, which are not lines of curvature, passes through another circular section of the same kind.

In the case of the anchor ring, it is always possible to draw through a given point a plane and a sphere, orthogonal at once to the ring and to each other, each of which intersects the ring in two circles. In the cyclide, therefore, through any point, it is possible to draw two spheres which cut the cyclide and each other orthogonally. If we invert the cyclide with respect to any point, these two spheres through the point will invert into two planes which will be the principal planes of the cyclide so obtained. They will be at right angles to each other and to the surface which they cut in two pairs of circles; and it may be shown that this second inverse is exactly of the same nature—i. e., is generated in the same way—as the first inverse, or that all the cyclides may be obtained from the anchor ring by one inversion.

It is possible to draw through any point two spheres which will touch the anchor ring along circles. Every cyclide, therefore, must have a pair of tangent planes which touch the surface along circles.

Let C be the centre of an axial circle, and P any point on it; the normal circle to the anchor ring at P lies in the plane SCP, and must therefore cut CS again in some point T such that

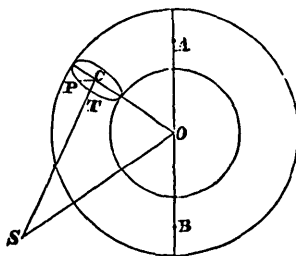
$$CT \cdot CS = CP^2.$$

Therefore T is a fixed point for all positions of P on the axial circle.

In the cyclide, therefore, all the normals along the inverse circle will pass through the inverse of T; and all the normals will intersect the curve the inverse of the locus of T.

Now T must be a point on the sphere through S, orthogonal to the anchor ring, since $CT \cdot CS = CP^2$; and it also lies on the cone of the second degree, vertex S, and base the circle the locus of C. The inverse of the locus of T is therefore a plane conic in the principal section of the cyclide.

The nature of this conic can be determined in the following manner: The extremities of its axis must lie in the lines SA, SB, where A and B are the centres of the axial sections whose plane passes through S; and they will be the inverse points of D and E in which



the circle through S orthogonal to these axial circles meets SA , SB .

If D , E be on the same side of S on the cone, the conic must be an ellipse, and so on.

Whence we see that the conic will be an ellipse or hyperbola according as S is outside or inside both of the circles, or outside only one of the circles; *i. e.*, according as S is without or within the substance of the ring.

The normal to the anchor ring at P cuts the axis in some point N .

The normal circle at P must therefore cut NS in some point Q such that

$$NQ \cdot NS = NP^2.$$

Therefore Q is a fixed point for all positions of P on a transverse circle.

In the cyclide all the normals along one of the circular sections pass through a fixed point, and all the normals pass through a fixed plane curve the inverse of the locus of Q , which is clearly a plane curve.

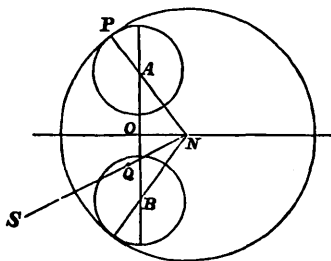
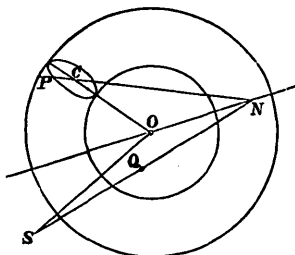
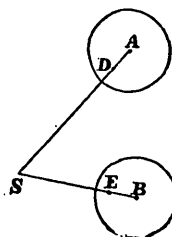
We may consider the locus of Q as obtained by this same property from points P on the axial circles in the axial plane through S .

The circle whose centre is N and radius NP must touch not only the circle (centre A), but the equal circle (centre B), and Q will invert into the centre of the circle inverse of the circle (centre N).

All these circles touch two given equal circles, the normals at the points of contact intersecting in a straight line.

In the inverse figure, the circles will touch two given circles, and therefore the normals at the points of contact will intersect on a conic which will be an ellipse or hyperbola according as the pole of inversion is inside only one, or outside or inside both of the equal circles; *i. e.*, according as the pole is within or without the substance of the anchor ring. Of course both of these conics are parabolas, if the pole be on the surface of the ring.

We have proved that all the normals of a cyclide pass through two plane conics in planes at right angles; and since the inverse of a cyclide with respect to any pole is a cyclide, it follows that all the normal circles of a cyclide drawn through a given point pass through



two spherical curves, which would invert into plane conics; each curve being the intersection of a cone of the second degree whose vertex is the point and a sphere drawn through the point, the two spheres also being orthogonal to one another.

If the point be one such that we can invert the cyclide with respect to it into an anchor ring, these two curves are two circles in planes at right angles, one of the circles passing through the point.

Suppose a system of anchor rings generated by a series of concentric circles revolving round a fixed straight line in their plane; they of course form a series of parallel surfaces. Their orthogonal surfaces are a series of planes passing through the common axis of the rings, and a series of right circular cones coaxial with the rings and passing through the circle traced out by the revolution of the centre of the revolving circle.

If we invert the system of surfaces, we get a second system which cut each other along lines of curvature. They will be a series of cyclides the inverses of the anchor rings, a second series of cyclides the inverses of the cones, and a series of spheres the inverses of the planes.

The cyclides have been investigated before; but I do not know that they have been examined from this point of view, as the inverse figures of the anchor ring, many of whose geometrical properties are as easily seen as those of the circle. Some of the properties I have given have been given by Maxwell, *Quarterly Journal of Mathematics*, vol. ix., where excellent stereoscopic views of four species of cyclides are given; and by Cayley in the same Journal, vol. xii.; and in a paper in the *Phil. Trans.*, vol. clxi., by Casey.

[NOTE BY PROF. CAYLEY.—The inverse of the anchor ring (in the foregoing paper called the cyclide) is in fact the general binodal cyclide or binodal bicircular quartic; viz., assuming it to be a cyclide (bicircular quartic), to see that it is binodal, it need only be observed that the anchor ring is binodal (has two real or imaginary conic points, viz., these are the intersections of the circles in the several axial planes); and to see that it is the general binodal cyclide, we have only to count the constants; viz., general cyclide or surface

$$(x^2 + y^2 + z^2)^2 + (x^2 + y^2 + z^2)(\alpha x + \beta y + \gamma z) + (\alpha, b, c, d, f, g, h, l, m, n)(x, y, z, 1)^3 = 0$$

contains 13 constants, and therefore binodal cyclide $13 - 2 = 11$ constants. But anchor ring, irrespective of position, contains 2 constants; centre of inversion, taken in given axial plane, has 2 constants; radius of inversion, 1 constant; in all $2 + 2 + 1 = 5$ constants; or taking the inverse surface in an arbitrary position, the number of constants is $5 + 6 = 11$.]