Coherent Vortex Simulation (CVS) of 2D bluff body flows using an adaptive wavelet method with penalisation

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Summary

In this paper we present an adaptive wavelet method to integrate the velocity-vorticity formulation of the two-dimensional Navier-Stokes equations coupled with a penalisation technique to handle easily solid boundaries of arbitrary shape. The validity of this method, called Coherent Vortex Simulation (CVS), is demonstrated by computing flows past different bluff bodies. Firstly, we show the computation of a flow around an impulsively started cylinder at Reynolds number 3000. The results are compared with those of a DNS using a spectral method and with others computed with two different vortex methods. Secondly, we also present simulations of a flow around a NACA airfoil profile at Reynold number 1000.

1 Introduction

Recently, we proposed a new CFD method, called Coherent Vortex Simulation (CVS) [6], [8], [21] for computing fully developed turbulent flows. It results from the observation that turbulent flows contain both an organized part (the coherent vortices) and a random part (the incoherent background flow) [6], [7]. The CVS method is based on the wavelet filtered Navier-Stokes equations, which corresponds to the coherent flow whose evolution is computed deterministically in an adaptive wavelet basis [10], [11], [17]. The influence of the incoherent background flow onto the coherent flow is statistically modelled.

Wavelets have been used so far for analyzing, modeling and computing turbulent flows, for a review we refer e.g. to [5], [15], [16]. Different adaptive wavelet methods have been developed to solve the two-dimensional Navier-Stokes equations [10], [4], [11], [18], but all of them are limited to simple geometries, i.e. squares

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or rectangles, mostly assuming periodic boundary conditions. To take into account complex geometries, we propose to couple the CVS method we have developed with the penalisation technique, introduced by Arquis and Caltagirone [2]. Therewith walls or solid obstacles, even if their shape varies in time, are modelled as a porous medium whose porosity η tends to zero. A mathematical theory proving convergence of this physically based approach has been given by Angot et al. [1]. This technique has been applied in the context of low order methods (finite difference/volume schemes, e.g. [13]) and also with spectral methods e.g. [12], [9]. The motivation to couple the penalisation technique with an adaptive wavelet solver comes from the fact that adaptive wavelet methods dynamically refine the grid in regions of strong gradients. Hence, we expect the solver to adapt automatically, not only to the evolution of the flow, but also to the geometry of walls or bluff bodies.

After a short presentation of the governing equations together with the penalisation method, we present the adaptive wavelet scheme for the penalized equations. For more details on the numerical scheme we refer the reader to [20]. To validate this penalised CVS method we compute a flow past an impulsively started cylinder at high Reynolds number ($Re = 3000$). We illustrate the self-adaptive grid evolution and compare the results with the ones obtained with two different vortex methods [14]. We also present first results of a flow past a NACA 23012 airfoil at $Re = 1000$ computed with the penalisation technique. Finally, we give some conclusions and perspectives for future work.

2 Governing equations and numerical scheme

This section presents the governing equations together with the penalisation technique and introduces its coupling with the adaptive wavelet scheme.

2.1 Physical problem

We consider a viscous incompressible fluid governed by the Navier-Stokes equations. In velocity-vorticity formulation the transport equation for a two-dimensional flow reads

$$\partial_t \omega + \boldsymbol{v} \cdot \nabla \omega - \nu \nabla^2 \omega = 0 \tag{1}$$

where $\boldsymbol{v}(\boldsymbol{x}, t) = (u(x, y, t), v(x, y, t))$ is the velocity, $\omega = \nabla \times \boldsymbol{v}$ the vorticity and ν the kinematic viscosity. The incompressibility, i.e. $\nabla \cdot \boldsymbol{v} = 0$, implies that $\boldsymbol{v}$ is related to ω by the Biot-Savart relation:

$$\nabla^2 \boldsymbol{v} = \nabla^{\perp} \omega \tag{2}$$

with $\nabla^{\perp} = (-\partial y, \partial x)$. Considering a flow around a solid obstacle Ω_s, translating with velocity $\boldsymbol{V}_0$, the velocity of the fluid is equal to the velocity of the obstacle at its boundary, i.e. $\boldsymbol{v}|_{\partial \Omega_s} = \boldsymbol{V}_0$. The free-stream velocity $\boldsymbol{V}_\infty$ is defined as $\lim_{|\boldsymbol{x}| \longrightarrow \infty} \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{V}_\infty$.

Based on the charactistic size of the obstacle D (e.g. the diameter of the cylinder) and the mean velocity $\boldsymbol{V} = \boldsymbol{V}_0 - \boldsymbol{V}_\infty$ we introduce a Reynolds number $Re = |\boldsymbol{V}| D/\nu$ and a non-dimensional time $T = |\boldsymbol{V}| t/(2D)$.

2.2 The penalisation technique

The penalisation technique is based on the physical idea to model solid walls or obstacles as porous media whose porosity η tends to zero [2]. The complex geometry is described by a mask function $\chi(\boldsymbol{x})$ which is 1 inside the solid regions and 0 elsewhere. Hence, the penalisation method can also take into account obstacles with time-varying shapes by simply introducing a mask function which varies in time accordingly. The Navier–Stokes equations are modified by adding a supplementary term containing the mask function. For $\eta \to 0$ the flow evolution is governed by Navier–Stokes equations in the fluid regions, and by Darcy law (velocity proportional to pressure gradient) in solid regions where the obstacles or walls are.

For an impulsively started obstacle with veloctiy $\boldsymbol{V}_0$ and free-stream velocity $\boldsymbol{V}_\infty$ we obtain in vorticity-velocity formulation:

$$\partial_t \omega \;+\; (\boldsymbol{v} + \boldsymbol{V}_\infty) \cdot \nabla \omega - \nu \, \nabla^2 \, \omega \;+\; \nabla \times \left(\frac{1}{\eta} \, \chi \, (\boldsymbol{v} - \boldsymbol{V}_0)\right) \;=\; 0 \tag{3}$$

with the mask function

$$\chi_{\Omega_s}(\boldsymbol{x}) \;=\; \begin{cases} 1 & \text{for} \quad \boldsymbol{x} \in \bar{\Omega}_s \\ 0 & \text{elsewhere.} \end{cases} \tag{4}$$

It has been shown rigorously that the above equations converge towards the Navier–Stokes equations with no-slip boundary conditions, with order $\eta^{3/4}$ inside the obstacle and with order $\eta^{1/4}$ elsewhere [1], in the limit when η tends to zero. In numerical simulations an improved convergence of order η has been reported [1], [12].

The resulting forces $\boldsymbol{F}$ on the obstacle are computed by integrating the penalized velocity over the obstacle's volume [1]:

$$\boldsymbol{F} \;=\; \lim_{\eta \to 0} \int_{\Omega_s} \nabla P_\eta \, d\boldsymbol{x} \;=\; -\lim_{\eta \to 0} \frac{1}{\eta} \int_{\Omega_s} \boldsymbol{V}_\eta \, d\boldsymbol{x} \;=\; \int_{\partial \Omega_s} S(\boldsymbol{V}, P) \cdot \boldsymbol{n} \, d\gamma \tag{5}$$

where Ω_s denotes the volume of the obstacle, $\partial\Omega_s$ its boundary, $\boldsymbol{n}$ its outer normal and $S(\boldsymbol{V}, P) \;=\; \frac{1}{2\nu} \left(\nabla \boldsymbol{V} + (\nabla \boldsymbol{V})^T\right) - PI$ is the stress tensor. Therefore the drag and the lift (forces parallel and perpendicular to $\boldsymbol{V}_0$, respectively) induced by the obstacle are easy to compute as volume integrals instead of contour integrals.

2.3 Adaptive wavelet method

For the numerical solution of (3) we first discretize the equations in time using semi-implicit finite differences, i.e. Euler–backwards for the viscous term and Adams–Bashforth extrapolation for the nonlinear term, which are both of second order.

This yields an elliptic problem in each time step:

$$(\gamma I - \nu \nabla^2) \omega^{n+1} \;=\; \frac{4}{3} \gamma \omega^n - \frac{1}{3} \gamma \omega^{n-1} - \nabla \cdot (\omega^\star \boldsymbol{v}^\star + \boldsymbol{V}_\infty) + \nabla \times \left(\frac{1}{\eta} \, \chi \, (\boldsymbol{v}^\star - \boldsymbol{V}_0)\right) \tag{6}$$

where

$$\omega^\star = 2\,\omega^n - \omega^{n-1} \qquad\qquad \boldsymbol{v}^\star = 2\,\boldsymbol{v}^n - \boldsymbol{v}^{n-1} \tag{7}$$

with time step Δt, $\gamma = 3/(2\Delta t)$ and I representing the identity.

For discretizing the resulting system in space, we use a Petrov–Galerkin scheme. Therefore the vorticity is developed into a set of trial functions and the minimization of the weighted residual of (6) requires that the projection onto a space of test functions vanishes. As space of trial functions we employ a two-dimensional multiresolution analysis (MRA) and develop ω^n at time step n into an orthonormal wavelet series

$$\omega^n(x,y) \;=\; \sum_j \sum_{k_x=0}^{2^j-1} \sum_{k_y=0}^{2^j-1} \sum_{\mu=1,2,3} \langle \omega^n \,,\, \psi^\mu_{j,k_x,k_y} \rangle \; \psi^\mu_{j,k_x,k_y}(x,y) \quad . \tag{8}$$

The test functions θ^μ_{j,i_x,i_y} are defined as solutions of the linear part of eq. (6):

$$(\gamma I - \nu \nabla^2)\theta^\mu_{j,i_x,i_y} \;=\; \psi^\mu_{j,i_x,i_y} \quad . \tag{9}$$

This avoids assembling the stiffness matrix and solving a linear equation at each time step. The functions θ, called vaguelettes, are explicitely calculated in Fourier space and have similar localization properties as wavelets [11]. The solution of (6) therewith reduces to a simple change of basis:

$$\begin{aligned} \tilde\omega^{\mu,n+1}_{j,i_x,i_y} \;&=\; \langle \omega^{n+1} \,,\, \psi^\mu_{j,i_x,i_y} \rangle \\ &= \langle (\frac{4}{3}\gamma\omega^n - \frac{1}{3}\gamma\omega^{n-1} - \nabla\cdot(\omega^\star \boldsymbol{v}^\star + \boldsymbol{V}_\infty) + \nabla\times(\frac{1}{\eta}\,\chi\,(\boldsymbol{v}^\star - \boldsymbol{V}_0))), \theta^\mu_{j,i_x,i_y} \rangle \quad . \end{aligned} \tag{10}$$

Applying at each time step a nonlinear wavelet thresholding technique we obtain an adaptive discretization by retaining only those wavelet coefficients $\tilde\omega^{\mu,n}_{j,i_x,i_y}$ with absolute value above a given threshold $\epsilon = \epsilon_0\sqrt{Z}$, where ϵ_0 is a constant and $Z = \frac{1}{2}\int |\omega(\boldsymbol{x})|^2 d\boldsymbol{x}$ is the enstrophy. For the next time step the index coefficient set (which addresses each coefficient in wavelet space) is determined by adding neighbours to the retained wavelet coefficients. Consequently, only those coefficients $\tilde\omega$ in (10) belonging to this extrapolated index set are computed using the adaptive vaguelette decomposition [11]. The nonlinear term $-\nabla\cdot(\omega^\star \boldsymbol{v} - {\boldsymbol{V}_\infty}^\star) + \nabla\times(\frac{1}{\eta}\,\chi\,(\boldsymbol{v}^\star - \boldsymbol{V}_0))$ is evaluated by partial collocation on a locally refined grid. The vorticity $\omega^\star$ is reconstructed in physical space on an adaptive grid from its wavelet coefficients $\tilde\omega^\star$ using the adaptive wavelet reconstruction algorithm [11]. From the adaptive vaguelette decomposition with $\theta = (\nabla^2)^{-1}\psi$, we solve $\nabla^2\Psi^\star = \omega^\star$ to get the stream function $\tilde\Psi^\star$ and reconstruct $\Psi^\star$ on a locally refined grid. By means of centered finite differences of 4th order we compute $\nabla\omega^\star$, $\boldsymbol{v}^\star = (-\partial_y\Psi^\star, \partial_x\Psi^\star)$ and $\nabla\times(\frac{1}{\eta}\,\chi\,(\boldsymbol{v}^\star - \boldsymbol{V}_0))$ on the adaptive grid. Subsequently, the nonlinear term is summed up pointwise and finally (10) is solved using the adaptive vaguelette decomposition.

3 Numerical results

3.1 Impulsively started cylinder at $Re = 3000$

As application of the CVS method we compute a typical unsteady separated flow, i.e. the flow past an impulsively started cylinder at $Re = 3000$ proposed in [14]. The numerical difficulty comes from the fact that, due to the impulsive start, a thin boundary layer develops and thus the drag coefficient exhibits a $t^{-1/2}$ singularity. The free-stream velocity V_∞ is set to zero and the obstacle's velocity $\boldsymbol{V}_0$ is set to $(1, 0)$ at $t = 0^+$. The computational domain is $[0, 4D]^2$ where $D = 1$ is the diameter of the cylinder being centered in the domain. We use a spatial resolution of 512^2, with a time step $\Delta t = 5 \cdot 10^{-4}$, a threshold parameter $\epsilon_0 = 10^{-5}$ and a penalisation parameter $\eta = 10^{-3}$. In Fig.1 (left) we show isolines of the vorticity field for three instants together with the corresponding locally refined grid shown in Fig.1 (right). We observe that the grid automatically adapts to the obstacle and follows the flow evolution, since it is refined in regions of strong vorticity gradients. A comparison of the time evolution of the drag coefficient, computed using the CVS method, direct numerical simulation (DNS) with penalisation [20] and two different vortex methods [14] shows the validity of the adaptive wavelet method. Note that compared with a spectral method (DNS) only about 8% of the total number of modes are used. To illustrate the stiffness of the problem we plot in Fig. 4 an horizontal cut of vorticity at position $y = -0.29$ for $T = 5$. We observe very steep gradients of vorticity in front of the cylinder which are well resolved by the adaptive wavelet discretization.

3.2 Impulsively started flow around an airfoil at $Re = 1000$

To demonstrate the ability of the penalisation method to adapt to arbitrary geometries we present the computation of a flow around a NACA 23012 airfoil at $Re = 1000$, which has been impulsively started with an angle of incidence of $30°$. Fig. 5 shows the isolines of vorticity at 4 different time instants. We observe at early time the formation of a small vortex at the trailing edge which then detaches and is advected by the mean flow. In Fig. 6 we plot the time evolution of the enstrophy, which increases until the trailing vortex detaches and then continuously decreases.

4 Conclusions

In the paper we have presented an adaptive wavelet method, called Coherent Vortex Simulation (CVS), coupled with a penalisation technique, to compute two-dimensional turbulent flows in complex geometries. Computing a flow behind an impulsively started cylinder at Reynold number 3000, we have shown the validity of the CVS method by comparing the results with the one obtained with different methods. We illustrated the feature of automatic grid adaption and found a good prediction of the drag coefficient compared with a classical vortex method. In future work we

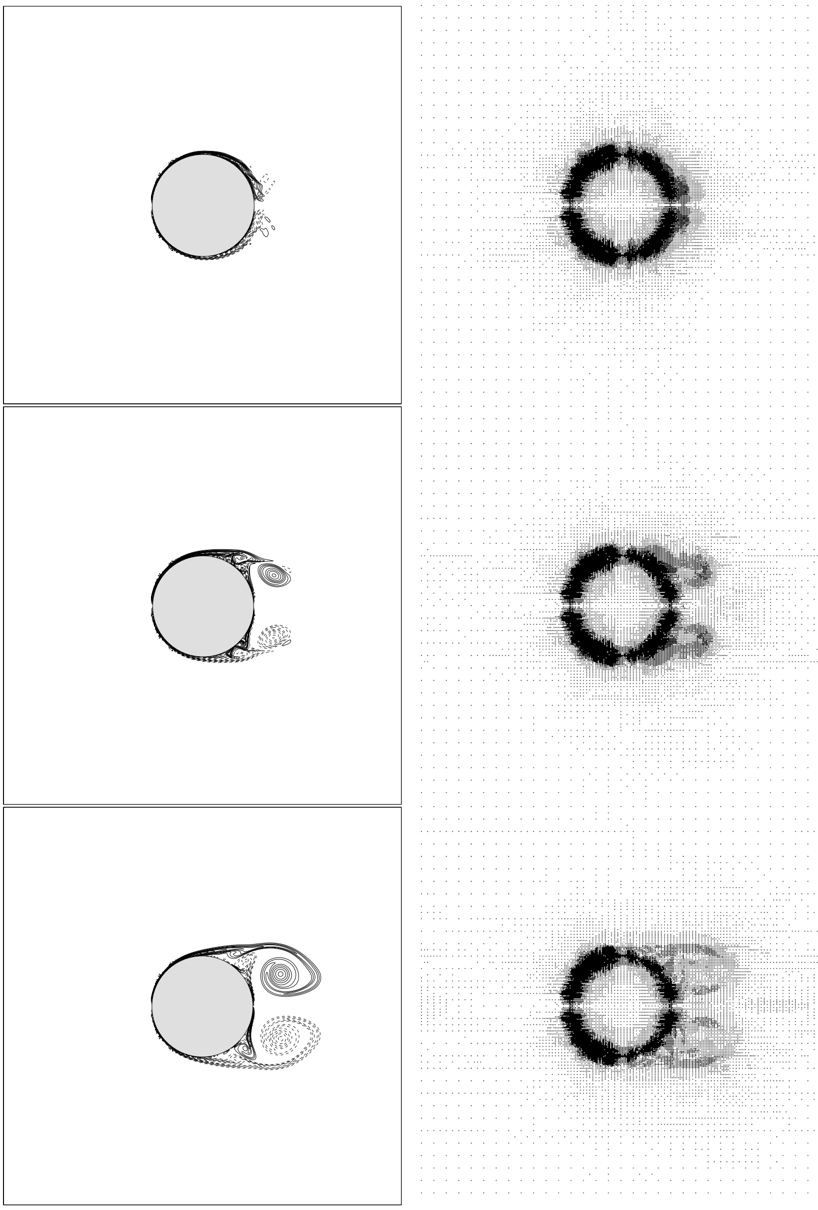

Figure 1 CVS of an impulsively started cylinder at $Re = 3000$. Left: isolines of vorticity at $T = 1, 3, 5$. Right: center of the active wavelet coefficients in physical space. Note that at $T = 1, 3, 5$ only $18781 (7.2\% N)$, $20089 (7.7\% N)$, $20764 (7.9\% N)$ out of $N = 512^2 = 262144$ wavelet modes are used.

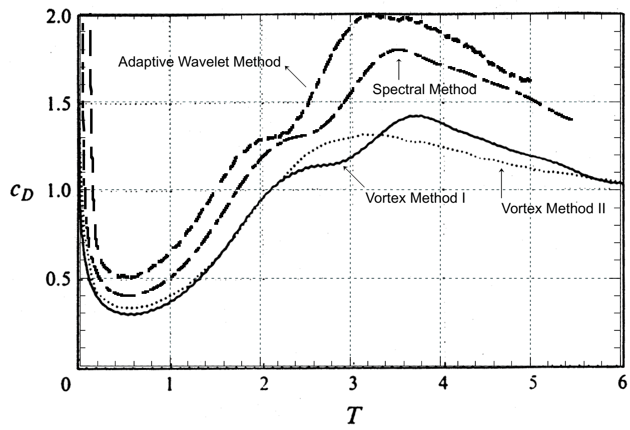

Figure 2 CVS of an impulsively started cylinder at $Re = 3000$. Comparison of the time evolution of the drag coefficient between the adaptive wavelet method (CVS), a spectral method with penalisation (DNS) and two different vortex methods [14].

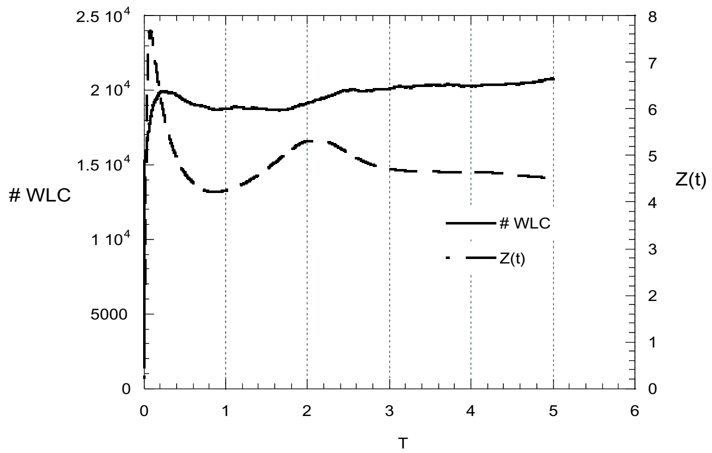

Figure 3 CVS of an impulsively started cylinder at $Re = 3000$. Time evolution of the number of active wavelet modes (solid lines) and of the total enstrophy Z (dashed lines).

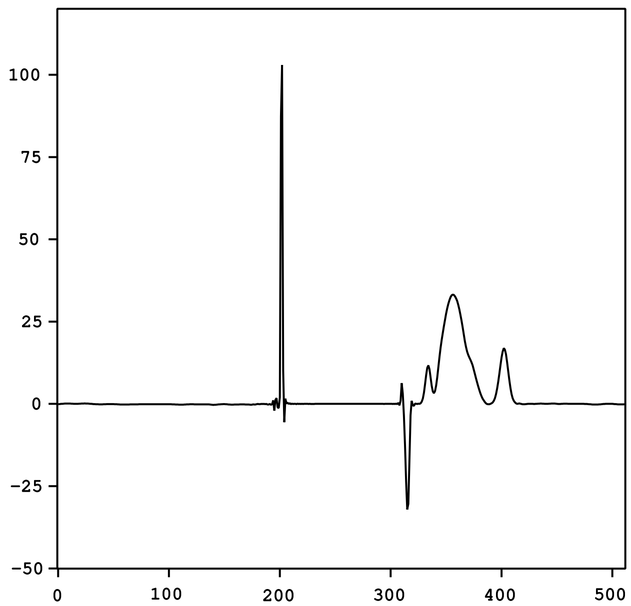

Figure 4 CVS of an impulsively started cylinder at $Re = 3000$. Horizontal cut of vorticity at location $y = -0.29$ for instant $T = 5$.

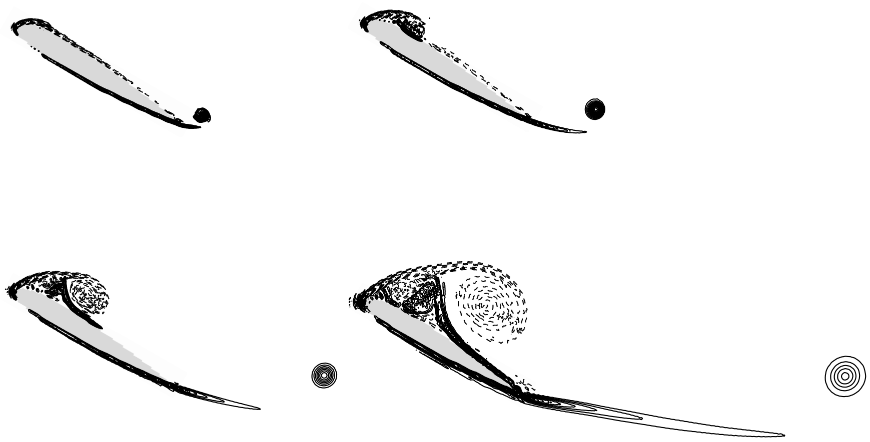

Figure 5 Airfoil NACA 23012, $\alpha = 30^\circ$, $Re = 1000$. Isolines of vorticity at instances $T = 0.2, 0.5, 1, 2$.

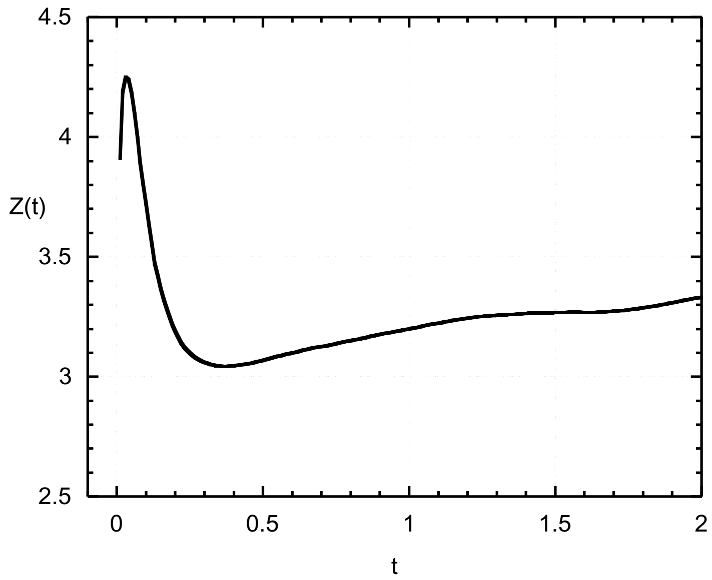

Figure 6 Airfoil NACA 23012, $\alpha = 30^\circ$, $Re = 1000$. Time evolution of total enstrophy.

will increase the threshold to get higher compression rates and develop a turbulence model to simulate the effect of the discarded wavelet modes onto the retained modes as shown in [8] for homogeneous flows. Work in progress is dealing with the extension of the CVS method to study the mixing of passive and reactive scalars [3] in complex geometries to study e.g. chemical reactors and other chemical engineering applications.

Acknowledgements

We thank Nicholas Kevlahan for fruitful discussion and for help to develop the spectral code with penalisation. We thankfully acknowledge financial support from the TMR project on 'Wavelets in numerical simulation' and the CNRS-DFG program on 'Numerical Flow Simulation', DFG contract No. Sch 649/1-1.

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