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# III. On the summation of slowly converging and diverging infinite series

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or $$\log s = 0{\cdot}42616.$$

A calculation made with this value would have given the required value with perfect accuracy.

As soon as $s$ and consequently $u$, $\rho$ and $\rho''$ are found, the elements of the parabola may be derived in various ways from the two extreme observations. It will, however, be interesting previously to investigate the degree of approximation obtained by the assumption in (15) and the applicability of the formulæ (16).

[To be continued.]

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## III. *On the Summation of slowly converging and diverging Infinite Series.* By J. R. Young, *Professor of Mathematics in Belfast College.*

[Continued from vol. vi. p. 354.]

THE foregoing examples have been chosen of the form

$$S = b - cx + dx^2 - ex^3 + \&c.$$

because it is only for slow series of this form that the transformation furnished by the differential theorem offers any practical facilities. When the series is of the form

$$b + cx + dx^2 + ex + \&c. \quad \ldots \quad \ldots \quad \text{(D.)}$$

we may indeed convert it into

$$\frac{1}{1-x}\left\{ b - \frac{\Delta x}{1-x} + \frac{\Delta^2 x^2}{(1-x)^2} - \frac{\Delta^3 x^3}{(1-x)^3} + \&c. \right\} \ldots \text{(E.)}$$

by simply substituting $-x$ for $x$, in the formula (B.), and this is a form to which the foregoing arithmetical process may be applied. But it is easy to see that that process would conduct us to the original series, and would terminate in the actual summation of its successive terms; for by the formula (B.) it appears that the new series, into which (E.) would be transformed by the process alluded to, would proceed according to the powers of $\frac{x}{1-x} \div \left(\frac{x}{1-x} + 1\right)$, instead of according to the powers of $\frac{x}{1-x}$, as at present, that is, it would proceed according to the powers of $x$; and, agreeably to this change, the factor $\frac{1}{1-x}$, which multiplies the series (E.), would become simply 1; so that (E.) would thus be converted into a series of the form

$$b + c'x + d'x^2 + e'x^3 + \&c.$$

which being equal to (D.), independently of particular values of $x$, must be identical with it; hence the transformation (E.) does not assist us in the summation of (D.). Slow series, however, of the form (D.) may, when $b$, $c$, $d$, &c. are a series of divisors in arithmetical progression, be easily changed into others, of quicker convergency, by a method which a single example will fully illustrate. Suppose, for instance, the summation of $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} +$ &c. be required. Put

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \text{\&c.} = S_1$$

$$\because \frac{x}{3} + \frac{x^3}{5} + \frac{x^5}{7} + \frac{x^7}{9} + \text{\&c.} = \frac{S_1 - x}{x^2}.$$

Subtracting

$$\frac{x}{1.3} + \frac{x^3}{3.5} + \frac{x^5}{5.7} + \frac{x^7}{7.9} + \text{\&c.} = \frac{1}{2}\left\{S_1 - \frac{S_1 - x}{x^2}\right\} = S_3.$$

Hence

$$S_1 = \frac{2x^2 S_3 - x}{x^2 - 1} = \frac{x}{x^2 - 1}\left\{2x S_3 - 1\right\},$$

and consequently $S_1$ will be obtained by summing the more rapid series $S_3$. Again,

$$\frac{x}{1.3} + \frac{x^3}{3.5} + \frac{x}{5.7} + \frac{x^7}{7.9} + \text{\&c.} = S_3$$

$$\therefore \frac{x}{3.5} + \frac{x^3}{5.7} + \frac{x^5}{7.9} + \frac{x^7}{9.11} + \text{\&c.} = \frac{S_3 - \frac{x}{1.3}}{x^3}.$$

Subtracting

$$\frac{x}{1.3.5} + \frac{x^3}{3.5.7} + \frac{x^5}{5.7.9} + \frac{x^7}{7.9.11} + \text{\&c.} = \frac{1}{4}\left\{S_3 - \frac{S_3 - \frac{x}{1.3}}{x^2}\right\} = S_5.$$

Hence $$S_3 = \frac{4x^2 S_5 - \frac{x}{1.3}}{x^2 - 1} = \frac{x}{x^2 - 1}\left\{4x S_5 - \frac{1}{1.3}\right\};$$

and thus $S_3$, and therefore $S_1$, may be obtained by summing the more converging series $S_5$. And generally

$$S_{n-2} = \frac{x}{x^2 - 1}\left\{(n-1)x S_n - \frac{1}{1.3 \ldots n-2}\right\};$$

so that by summing a few terms of $S_n$ we may, by this formula, obtain near approximate values to $S_{n-2}$, $S_{n-4}$, &c. in succession, and finally, to $S_1$, each step being deduced from the preceding by simple arithmetical operations.