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$\mathbb{Z}/p$-equivariant maps between lens spaces and spheres

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1 Introduction and statement of results

In this article we prove a generalization of a theorem by Stephan Stolz [S], in which he gives a complete description of the level of real projective spaces $\mathbb{R}\mathrm{P}^{2m-1}$. Let X be any topological space with a free $\mathbb{Z}/2$-action. Then the level of X is defined to be the number

$$s(X) := \min\{n : \exists\ f : X \xrightarrow{\mathbb{Z}/2} S^{n-1}\}\ ,$$

where S^{n-1} is equipped with the antipodal action of $\mathbb{Z}/2$ and $f : X \xrightarrow{\mathbb{Z}/2} S^{n-1}$ means that f is $\mathbb{Z}/2$-equivariant. For $X = \mathbb{R}\mathrm{P}^{2m-1}$ we

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abbreviate $s(\mathbb{R}\mathrm{P}^{2m-1})$ by $s(m)$; here the $\mathbb{Z}/2$-action on $\mathbb{R}\mathrm{P}^{2m-1}$ is induced by the $\mathbb{Z}/4$ action on S^{2m-1} given by multiplication with the complex number i. Stolz achieved the following result:

Theorem ([S]). *$s(1) = 2$, and for $m \geq 2$:*

$$s(m) = \begin{cases} m+1 & \textit{if} \quad m \equiv 0, 2 \bmod 8 \\ m+2 & \textit{if} \quad m \equiv 1, 3, 4, 5, 7 \bmod 8 \\ m+3 & \textit{if} \quad m \equiv 6 \bmod 8 \end{cases}$$

□

The level of $\mathbb{R}\mathrm{P}^{2m-1}$ can also be described as $s(m) := \min\{n : \exists\ f : S^{2m-1} \xrightarrow{\mathbb{Z}/4} S^{n-1}\}$, and more generally we define

$$\begin{aligned} s_k(m) &:= \min\{n : \exists\ f : S^{2m-1} \xrightarrow{\mathbb{Z}/2^k} S^{n-1}\} \\ &= \min\{n : \exists\ f : \mathbf{L}^{2m-1}(2^{k-1}) = S^{2m-1}/\ (\mathbb{Z}/2^{k-1}) \xrightarrow{\mathbb{Z}/2} S^{n-1}\}\ , \end{aligned}$$

where $\mathbb{Z}/2^k$ acts on S^{2m-1} via multiplication with a primitive 2^kth root of unity ζ and on S^{n-1} via $\zeta \mapsto \zeta^{2^{k-1}}$. Hence $s_2(m) = s(m)$.

The aim of this article is to prove an analogue of the results of [S] for odd primes p: In this case we define

$$\begin{aligned} v_{p,k}(m) &:= \min\{n : \exists\, f\colon S^{2m-1} \xrightarrow{\mathbb{Z}/p^k} S^{2n-1}\} \\ &= \min\{n : \exists\, f\colon \mathbf{L}^{2m-1}(p^{k-1}) = S^{2m-1}/\ (\mathbb{Z}/p^{k-1}) \xrightarrow{\mathbb{Z}/p} S^{2n-1}\}. \end{aligned}$$

The following lower estimates for $s_k(m)$ and $v_{p,k}(m)$ have been proved by Thomas Bartsch in [B1, Proof of Thm. 1.2]:

$$s_k(m) \geq \left\langle \frac{2(m-1)}{2^{k-1}} \right\rangle + 1$$

$$v_{p,k}(m) \geq \left\langle \frac{m-1}{p^{k-1}} \right\rangle + 1$$

(Here $\langle x \rangle$ denotes the smallest integer bigger or equal to x, i.e. $\langle x \rangle = -[-x]$.)

Our main result is the following:

Theorem. Let p be an odd prime. Then $v_{p,2}(1) = 1$, and for $m \geq 2$ we have

$$\begin{aligned} \left\langle \tfrac{m-2}{p} \right\rangle + 1 \leq v_{p,2}(m) &\leq \left\langle \tfrac{m-2}{p} \right\rangle + 2 \quad \text{for } m \not\equiv 2 \bmod p \\ v_{p,2}(m) &= \quad \tfrac{m-2}{p} + 2 \quad \text{for } m \equiv 2 \bmod p\ . \end{aligned}$$

□

So the upper and lower estimates differ by exactly one, except for the cases $m = 1$ or $m \equiv 2 \bmod p$, where the result is sharp.

To achieve this result we investigate a question, equivalent to the original problem, about the existence of sections of certain vector bundles ξ over lens spaces, analogous to Stolz's approach in the case $p = 2$. In a certain metastable range the vanishing of the stable cohomotopy Euler class $\gamma(\xi)$ of such a bundle is a sufficient condition for the existence of a section. However, in order to get a useful dimension range in the case $p > 2$ we generalize this criterion for the specific bundles under consideration: Before we try to extend any section that is given on a subcomplex of the base space to the whole space, we compose it with certain self maps of the bundle. In this way we can achieve that effectively we are in a p-local situation and this improves the dimension range in the desired way. (I would like to thank Michael Crabb for suggesting this method to me. It significantly simplifies the argument which originally used p-localized vector bundels instead. It also leads to a slight improvement of Cor. 3.12 and thus of the main Theorem 4.8.)

Subsequently we analyse the Adams spectral sequence converging to the p-primary part of $\pi^*(M(-\xi))$, where $M(-\xi)$ is the Thom spectrum of a bundle inverse to the bundle ξ mentioned above, as in [S]. Since $\gamma(\xi)$ can be interpreted as an element in $\pi^*(M(-\xi))$, this gives us the necessary dimensional conditions for the vanishing of $\gamma(\xi)_{(p)}$. Again, for $p > 2$ this analysis becomes quite a bit more involved: In order to determine the E_2-term we use an inductive argument, but since this does not yield enough information, we have to turn to the E_3-term applying a well-known relationship between the Adams and the Bockstein spectral sequences.

As an application of our results we investigate Borsuk-Ulam functions b_G for $G = \mathbb{Z}/p^k$. Here the Borsuk-Ulam function $b_G : \mathbb{N} \to \mathbb{N}$ is defined to be the greatest function such that the existence of a G-equivariant map $SV \to SW$ between fixed-point free representation spheres implies $\dim W \geq b_G(\dim V)$. We show that there is a simple relationship between the quantities $s_k(m)$ resp. $v_{p,k}(m)$ and the Borsuk-Ulam functions $b_{\mathbb{Z}/p^k}$, which allows us to draw conclusions about the growth of $b_{\mathbb{Z}/p^k}$. More precisely, we show that $\lim_{m\to\infty} \frac{b_G(m)}{m} = \frac{1}{p^{k-1}}$ for all primes p and $k \geq 2$. For p-groups other than cyclic or elementary abelian the calculation of the Borsuk-Ulam functions or even only this limit is still an open question.

The paper is organized in the following way:

2. Reformulation of the problem
3. The cohomotopy Euler class and sections of vector bundles

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2 Reformulation of the problem

In analogy to [S] we will use the following description of the problem: For $p = 2$ consider the real line bundle

$$L_{\mathbb{R},k} : \mathbf{L}^{2m-1}(2^{k-1}) \times_{\mathbb{Z}/2} \mathbb{R} \to \mathbf{L}^{2m-1}(2^k) \ .$$

Then any $\mathbb{Z}/2$-equivariant map $f : \mathbf{L}^{2m-1}(2^{k-1}) \to S^{n-1}$ corresponds to a nowhere vanishing section of $nL_{\mathbb{R},k}$ and vice versa (here $nL_{\mathbb{R},k}$ is the n-fold Whitney sum of $L_{\mathbb{R},k}$). Analogously, for $p \neq 2$ consider the complex line bundle

$$L_{\mathbb{C},k} : \mathbf{L}^{2m-1}(p^{k-1}) \times_{\mathbb{Z}/p} \mathbb{C} \to \mathbf{L}^{2m-1}(p^k) \ .$$

Here $\mathbb{Z}/p$-equivariant maps $f : \mathbf{L}^{2m-1}(p^{k-1}) \longrightarrow S^{2n-1}$ correspond to nowhere vanishing sections of $nL_{\mathbb{C},k}$ and vice versa. Thus we have

$$s_k(m) = \min\{n : \exists \text{ nowhere vanishing section of } nL_{\mathbb{R},k}\}$$

and

$$v_{p,k}(m) = \min\{n : \exists \text{ nowhere vanishing section of } nL_{\mathbb{C},k}\} \ .$$

The big advantage of this description is that we can now tackle the question of the existence of equivariant maps by investigating an equivalent question about the existence of sections of the above bundles.

For future reference we make the following observation: The twisted line bundle $L_{\mathbb{C},k}$ over $\mathbf{L}^{2m-1}(p^k)$ is isomorphic to $f^*(\tau^{\otimes p^{k-1}})$, where τ is the canonical line bundle over $\mathbb{C}\mathrm{P}^{m-1}$ and $f : \mathbf{L}^{2m-1}(p^k) \cong S^{2m-1}/(\mathbb{Z}/p^k) \to \mathbb{C}\mathrm{P}^{m-1} \cong S^{2m-1}/S^1$ is the canonical projection. Hence for the Euler class $e(L_{\mathbb{C},k})$ in integral cohomology we get $e(L_{\mathbb{C},k}) = e(f^*\tau^{\otimes p^{k-1}}) = p^{k-1} \cdot f^*e(\tau)$. In particular, $e(L_{\mathbb{C},k})$ is an element of order p in $H^2(\mathbf{L}^{2m-1}(p^k);\mathbb{Z}) \cong \mathbb{Z}/p^k$.

3 The cohomotopy Euler class and sections of vector bundles

To start this section, we will give a short review of the Euler class in stable cohomotopy theory. A detailed account of the material in the first part of this section can be found in [CS] and in [C1].

Let $Q_i \to X$, $i = 0, 1$, be two locally trivial bundles of finite pointed CW-spaces over a fixed finite CW-space X and denote by $[Q_0; Q_1]_X$ the set of homotopy classes of fibrewise pointed maps over X. The fibrewise smash product with the identity map of the trivial bundle $X \times \mathbb{R}^+ \to X$ induces a map $[Q_0; Q_1]_X \to [\mathbb{R}^+ \wedge_X Q_0; \mathbb{R}^+ \wedge_X Q_1]_X$. We define the group of pointed stable bundle maps from $Q_1 \to X$ to the j-fold fibrewise suspension of $Q_2 \to X$ which are trivial over the subcomplex Y of X by

$$\omega^j_{(X,Y)}\{Q_0; Q_1\} := \operatorname*{colim}_{k\to\infty} \left[(\mathbb{R}^k)^+ \wedge_X Q_0; (\mathbb{R}^{k+j})^+ \wedge_X Q_1\right]_{(X,Y)} ,$$

where the subscript (X, Y) means that we only consider maps that send fibres in Q_0 over points in Y to the base point in the appropriate fibres in Q_1.

If $(X \supset Y \supset Z)$ then in the usual way we get a long exact sequence

$$\cdots \longrightarrow \omega^{j-1}_{(X,Z)}\{Q_0; Q_1\} \longrightarrow \omega^{j-1}_{(Y,Z)}\{Q_0|Y; Q_1|Y\} \xrightarrow{\Delta} \omega^j_{(X,Y)}\{Q_0; Q_1\} \longrightarrow \cdots .$$

In the special case that X is a point, the definitions reduce to ordinary stable homotopy theory, and the index X will be dropped. Using the same notation as in [CS], in this case we write

$$\omega^j\{Q_0; S^0\} =: \tilde{\omega}^j(Q_0) \text{ and } \omega^j\{S^0; Q_1\} =: \tilde{\omega}_{-j}(Q_1) .$$

For the purpose of this paper it suffices to consider the case where the bundles in question are fibrewise one-point compactifications of finite dimensional real vector bundles which we shall denote by α_i^+, $i = 0, 1$. In this case the group $\omega^0_X\{\alpha_0^+; \alpha_1^+\}$ can be viewed as stable cohomotopy group of the Thom space of the virtual bundle $\alpha_0 - \alpha_1$ over X in the following way: Let X^ξ be the Thom space of the bundle ξ over X; for a subcomplex $Y \subset X$ denote by $(X, Y)^\xi$ the quotient $X^\xi / Y^{\xi|Y}$. A virtual bundle α over X is nothing more but an ordered pair (α_0, α_1) of real vector bundles over X denoted by $\alpha = \alpha_0 - \alpha_1$. In order to define $\tilde{\omega}^j(X^\alpha)$, choose a bundle σ over X inverse to α_1

together with a trivialization $\phi : \alpha_1 \oplus \sigma \longrightarrow X \times \mathbb{R}^m$, and define $\tilde{\omega}^j(X^\alpha) := \tilde{\omega}^{j+m}(X^{\alpha_0 \oplus \sigma})$. Then we get

$$\omega^j_X\{\alpha_0^+;\alpha_1^+\} \cong \omega^j_X\{(\alpha_0 \oplus \sigma)^+; X \times \mathbb{R}^{m+}\} \cong \tilde{\omega}^{j+m}(X^{\alpha_0 \oplus \sigma}) =: \tilde{\omega}^j(X^\alpha),$$

and analogously $\omega^j_{(X,Y)}\{\alpha_0^+;\alpha_1^+\} \cong \tilde{\omega}^j((X,Y)^\alpha)$.

Remarks. (i) The groups $\tilde{\omega}^j(X^\alpha)$ can be viewed as *twisted* stable cohomotopy groups of the base space X. If α is a trivial n dimensional vector bundle, then indeed $\tilde{\omega}^j(X^\alpha) \cong \tilde{\omega}^{j-n}(X^+)$. This twisting by the bundle α is often expressed by using local coefficients in α:

$$\tilde{\omega}^j(X^\alpha) =: \omega^j(X;\alpha) \text{ and } \tilde{\omega}^j\left((X,Y)^\alpha\right) =: \omega^j(X,Y;\alpha)$$

(ii) If we define the Thom spectrum of α as $M(\alpha) := \Sigma^{-m}\Sigma^\infty X^{\alpha_0 \oplus \sigma}$, then $\tilde{\omega}^j(X^\alpha)$ is just the jth cohomotopy group of this spectrum, $\pi^j(M(\alpha))$.

We define the cohomotopy Euler class of the real finite dimensional vector bundle ξ as follows:

Definition 3.1 *The cohomotopy Euler class of ξ is the class $\gamma(\xi) \in \omega^0_X\{0^+;\xi^+\} = \omega^0(X;-\xi)$ induced by the inclusion of the zero section.*

Now let s be a section of the sphere bundle $S\xi$ over the subcomplex Y of X. In the disk bundle $D\xi$ over the whole space X we choose an extension $\bar{s}$ of the section s. There is a standard bundle map $c : D\xi \to \xi^+$ which fiberwise maps $S\xi \subset D\xi$ to the base point $+$ in ξ^+ and the homotopy class of the map $c \circ \bar{s}$ is independent of the choice of extension $\bar{s}$. Thus the following definition makes sense:

Definition 3.2 *The relative cohomotopy Euler class of the bundle ξ relative to s is defined as $\gamma(\xi;s) = E(c \circ \bar{s}) \in \omega^0_{(X,Y)}\{0^+;\xi^+\} \cong \omega^0(X,Y;-\xi)$. Here E denotes the stabilization map.*

We also want to compare two sections t_0 and t_1 of $S\xi$ over X that agree on a subcomplex Y. Let $p : X \times I \to X$ be the projection. Over $X \times \partial I \cup Y \times I \subset X \times I$ we define the section t of the induced bundle $S(p^*\xi)$ in such a way that on $X \times \{0\}$ it agrees with t_0, on $X \times \{1\}$ with t_1, and on $Y \times I$ with the common value of t_0 and t_1.

Definition 3.3 *The difference class of t_0 and t_1 is the element $\delta(t_0,t_1) \in \omega^{-1}_{(X,Y)}\{0^+;\xi^+\} \cong \omega^{-1}(X,Y;-\xi)$ which maps to $\gamma(p^*\xi;t) \in \omega^0(X \times I, X \times \partial I \cup Y \times I; -p^*\xi)$ under the suspension isomorphism.*

We will need the following basic lemma:

Lemma 3.4 ([C2, Lemma 1.2]) *Let t_0, t_1 and t_2 be sections of $S\xi$ over $Y \subset X$ that agree on the subcomplex $Z \subset Y$. Then we have:*

1. $\delta(t_0, t_2) = \delta(t_0, t_1) + \delta(t_1, t_2) \in \omega^{-1}(Y, Z; -\xi|Y)$
2. The connecting homomorphism of the triple (X, Y, Z)

$$\Delta : \omega^{-1}(Y, Z; -\xi|Y) \to \omega^0(X, Y; -\xi)$$

maps $\delta(t_0, t_1)$ *to* $\gamma(\xi; t_0) - \gamma(\xi; t_1)$. □

Finally we consider the situation where a fixed section s_0 of the sphere bundle $S\xi$ over X is given. In this case the section s_0 defines a splitting $\xi = \zeta \oplus \mathbb{R}$ of the bundle ξ, and $S\xi$ can be identified with ζ^+ in a canonical way, where the section s_0 of $S\xi$ corresponds to the base point s^+ of ζ^+. Now let s be a section of $S\xi$ which over Y coincides with s_0, or equivalently a section of ζ^+ which over Y is equal to s^+. Then the homotopy class of s is an element of $[0^+; \zeta^+]_{(X,Y)}$, whose stabilization lies in $\omega^0_{(X,Y)}\{0^+; \zeta^+\} = \omega^0(X, Y; -\zeta)$. More precisely:

Proposition 3.5 ([CS]) *The two groups* $\omega^0(X, Y; -\zeta)$ *and* $\omega^{-1}(X, Y; -\xi)$ *can be identified in a canonical way, and under this identification* $E(s) = \delta(s, s_0)$. □

From the definition of the cohomotopy Euler class it is clear that the existence of a section of $S\xi$ over the whole space X implies the vanishing of $\gamma(\xi)$, as is the case for the Euler class in integral cohomology if ξ is oriented. In fact, it is not hard to see that under the stable Hurewicz homomorphism

$$h : \omega^k(X; -\xi) = \pi^k(M(-\xi)) \to H^k(M(-\xi); \mathbb{Z}) =: H^k(X; -\xi)$$

the Euler class in stable cohomotopy $\gamma(\xi)$ is mapped to a *twisted* Euler class $h(\gamma(\xi)) \in H^0(X; -\xi)$, which (in case of an oriented bundle ξ) corresponds to the ordinary Euler class $e(\xi) \in H^{\dim \xi}(X; \mathbb{Z})$ under the Thom isomorphism. The big advantage of the cohomotopy Euler class is that in a certain "metastable" range the vanishing of $\gamma(\xi)$ is sufficient for the existence of a section of $S\xi$:

Proposition 3.6 ([C1, Prop. 2.4(ii)]) *Let* ξ *be an* n*-dimensional real vector bundle over the finite CW-space* X *with* $\dim X < 2n - 2$. *Let* v *be a section of* $S\xi|Y$, *where* Y *is a subcomplex of* X. *Then* v *can be extended to a section of* $S\xi$ *over the whole space* X *if and only if* $\gamma(\xi; v) = 0 \in \omega^0(X, Y; -\xi)$. *In particular, there is a section of* $S\xi$ *over* X *if and only if* $\gamma(\xi) \in \omega^0(X; -\xi)$ *vanishes.*

Proof The proposition follows by induction on the subcomplexes $X_k = Y \cup X^{(k)}$, where $X^{(k)}$ denotes the k-skeleton of X, with $X_{-1} := Y$. The proof is a cell-by-cell application of the Freudenthal suspension theorem and uses the following lemma. □

Lemma 3.7 ([C1, Prop. 2.4(i)]) *Let v be a section of $S\xi|Y$ and c an element of $\omega^{-1}(Y, Z; -\xi|Y)$, where $Z \subset Y \subset X$ with $\dim X < 2n-2$. Then there is a section w of $S\xi|Y$ that over Z agrees with v and such that*

$$\delta(v, w) = c \in \omega^{-1}(Y, Z; -\xi|Y).$$

Proof The lemma is proved by induction on the subcomplexes $Y_k := Z \cup Y^{(k)}$ of Y, with $Y_{-1} := Z$. Again, one uses the Freudenthal suspension theorem, classical obstruction theory, Prop. 3.5 and Lemma 3.4 $\square$

It turns out that the restriction on the dimension of X makes this proposition as it stands useless for our specific problem. However, for certain bundles this dimension range can be improved significantly:

Definition 3.8 *The finite dimensional real vector bundle ξ over the finite CW-space X is said to have property (†) if for every integer $d \equiv 1 \bmod p$ there exists a map of sphere bundles $S\phi_d : S\xi \to S\xi$ with degree d in each fibre. In that case the bundle map $\xi \to \xi$ induced by $S\phi_d$ will be denoted by ϕ_d.*

Note that the bundles $L_{\mathbb{C},2}$ that we are primarily interested in do posess property (†); we can take the map given by the complex dth power in each fibre. More precisely, taking dth powers is a map $L_{\mathbb{C},2} \to L_{\mathbb{C},2}^{\otimes d}$, and if $d \equiv 1 \bmod p$, then $L_{\mathbb{C},2}^{\otimes d}$ can be identified with $L_{\mathbb{C},2}$ since the (ordinary) Euler classes of the two bundles agree in this case. Indeed, by Section 2 the Euler class of $L_{\mathbb{C},2}$ is an element of order p in $H^2(\mathbf{L}^{2m-1}(p^2); \mathbb{Z}) \cong \mathbb{Z}/p^2$, and hence for $d \equiv 1 \bmod p$ we obtain $e(L_{\mathbb{C},2}^{\otimes d}) = d \cdot e(L_{\mathbb{C},2}) = e(L_{\mathbb{C},2})$. – Any multiple $nL_{\mathbb{C},2}$ of $L_{\mathbb{C},2}$ also has property (†): $S(nL_{\mathbb{C},2}) \cong S(L_{\mathbb{C},2}) * S((n-1)L_{\mathbb{C},2})$, and so we can take the fibrewise join of the complex dth power on the first factor with the identity on the other.

The idea for improving Prop. 3.6 now is the following: The map $\phi_d^* : \omega^j(X; -\xi) \to \omega^j(X; -\xi)$ induced by composition with the map $\phi_d : \xi \to \xi$ is multiplication with an element $\{\phi_d\}$ in $\omega^0(X; \xi - \xi) = \omega^0(X)$. But the inclusion of a base point $*$ in X induces a splitting $\omega^0(X) \cong \ker(\omega^0(X) \to \omega^0(*)) \oplus \omega^0(*)$, where $\omega^0(*) = \mathbb{Z}$, and it follows from Nishida's nilpotency theorem that the elements in $\ker(\omega^0(X) \to \omega^0(*))$ are nilpotent and of finite order. It follows that $\{\phi_d\} = d + x$, with $x \in \omega^0(X)$ nilpotent and of finite order. It is elementary to show that for an appropriately chosen large k the class $\{\phi_d\}^k$ is just d^k which is still congruent to 1 modulo p. Now if we want to extend a section v of $S\xi$ that is given over the subcomplex Y of X, then by composing v with a suitable sequence $\phi_{d_1}^{k_1} \circ \cdots \circ \phi_{d_r}^{k_r} =: \phi_{\mathbf{d}}$

we can achieve that effectively we are in a p-local situation - all the obstructions to extension of $\phi_{\mathbf{d}} \circ v$ are p-primary. So all we have to do is to replace the Freudenthal suspension theorem in the proof of Prop. 3.6 and Lemma 3.7 by the p-primary version of it (see e.g. [Sp, Thm. 11,Chap. 9.7]) and thus obtain the following result:

Proposition 3.9 *Let ξ be an n-dimensional real vector bundle over the finite CW-space X with $\dim X < pn - 2$. Assume that $n = 2l$ and that ξ has property $(\dagger)$. Let v be a section of $S\xi|Y$, where Y is a subcomplex of X. Then there exists $d \equiv 1 \bmod p$ and an extension of the section $\phi_d \circ v$ of $S\xi$ over the whole space X if and only if $\gamma(\xi; v)_{(p)} = 0 \in \omega^0(X, Y; -\xi)_{(p)}$. In particular, there is a section of $S\xi$ over X if and only if $\gamma(\xi)_{(p)} \in \omega^0(X; -\xi)_{(p)}$ vanishes.* □

The improved version of Lemma 3.7 is the following:

Lemma 3.10 *Let X and ξ be as in Prop. 3.9. Suppose that over the subcomplex Y of X we are given a section v of the sphere bundle $S\xi|Y$. Let c be an element of $\omega^{-1}(Y, Z; -\xi|Y)$, where $Z \subset Y \subset X$. Then there exists an integer $a \equiv 1 \bmod p$ and a section w of $S\xi|Y$ that over Z agrees with v and such that*

$$\delta(\phi_a \circ v, w) = a \cdot c \in \omega^{-1}(Y, Z; -\xi|Y) .$$

□

It immediately follows from Prop. 3.9 that for bundles that fulfill the conditions of that proposition the cohomotopy Euler class is p-primary, i.e. $\gamma(\xi) = \gamma(\xi)_{(p)}$. We can even say something more general:

Lemma 3.11 *For any finite dimensional real vector bundle ξ over a finite CW-space X that has property $(\dagger)$ the cohomotopy Euler class $\gamma(\xi)$ is p-torsion.*

Proof By the definition of $\gamma(\xi)$ we have $\{\phi_d\}\gamma(\xi) = \gamma(\xi)$ for any $d \equiv 1 \bmod p$, in particular for $d = 1 + p$. Thus by our earlier remarks there exists some element x in $\omega^0(X)$ which is nilpotent and of finite order such that $\{\phi_d\} = d + x$. So $(p + x)\gamma(\xi) = 0$, and since for some k we have $(p + x)^k = p^k$, the claim follows. □

As we already pointed out the bundles $nL_{\mathbb{C},2}$ do have property $(\dagger)$, so we immediately get the following:

Corollary 3.12 *Consider the twisted line bundle $nL_{\mathbb{C},2}$ over $\mathbf{L}^{2m-1}(p^2)$. Suppose that:*

1. $2m - 1 < pn - 2$
2. $\gamma(nL_{\mathbb{C},2})_{(p)} = 0 \in \omega^0(\mathbf{L}^{2m-1}(p^2); -nL_{\mathbb{C},2})_{(p)}$

Then there exists a section of the sphere bundle $S(nL_{\mathbb{C},2})$ over the whole space $\mathbf{L}^{2m-1}(p^2)$. □

4 The upper estimate for $v_{p,2}(m)$

We will now apply the Adams spectral sequence $\{E_r^{s,t}\left(M(-kL_{\mathbb{C},2}), \Sigma^\infty S^0\right)\}$ with E_2-term

$$\operatorname{Ext}^{s,t}_{\mathcal{A}_p}\left(H^*(\Sigma^\infty S^0), H^*(M(-L_{\mathbb{C},2}))\right) = \operatorname{Ext}^{s,t}_{\mathcal{A}_p}\left(\mathbb{Z}/p, H^*(M(-kL_{\mathbb{C},2}))\right)$$

converging to the p-primary part of $\pi^*(M(-kL_{\mathbb{C},2})) = \omega^*(\mathbf{L}^{2m-1}(p^2), -kL_{\mathbb{C},2})$ in order to get information on the cohomotopy Euler class $\gamma(kL_{\mathbb{C},2})$. All cohomology modules will be with $\mathbb{Z}/p$ as coefficients which we leave out of the notation. We shall abbreviate $M(-kL_{\mathbb{C},2})$ as M.

Let L denote the lens space $\mathbf{L}^{2m-1}(p^2)$, $L^{(j)}$ the j-skeleton and $i_{j-1} : L^{(j-1)} \hookrightarrow L^{(j)}$ the inclusion ($0 < j \leq 2m-1$). Considering the cellular chain complex with coefficients in $\mathbb{Z}/p$ associated to L one easily deduces that $H^q(L^{(j)}) = \mathbb{Z}/p$ for $0 \leq q \leq j$ and 0 otherwise and that $i_{j-1}^* : H^q(L^{(j)}) \to H^q(L^{(j-1)})$ is an isomorphism for $0 \leq q < j$.

Let σ be any orientable l-dimensional bundle over L and denote the Thom space of the induced bundle $i_j^*(\sigma)$ by $L^{i_j^*(\sigma)}$. Then from the Thom isomorphism $H^*(L^{(j)}) \to \tilde{H}^{*+l}(L^{i_j^*(\sigma)})$ it follows that $\tilde{i}_{j-1}^* : \tilde{H}^{q+l}(L^{i_j^*(\sigma)}) \to \tilde{H}^{q+l}(L^{i_{j-1}^*(\sigma)})$ also is an isomorphism for $0 \leq q < j$. On the other hand, the $(l+j)$-skeleton of L^σ is just $L^{i_j^*(\sigma)}$ (cf. e.g. [Sw, Chap. 12]), so that actually for $l < s \leq l + 2m - 1$ the inclusions $\tilde{i}_{s-1} : (L^\sigma)^{(s)} \hookrightarrow (L^\sigma)^{(s-1)}$ induce isomorphisms

$$\tilde{i}_{s-1}^* : \tilde{H}^r\left((L^\sigma)^{(s)}\right) \longrightarrow \tilde{H}^r\left((L^\sigma)^{(s-1)}\right)$$

for $l \leq r < s$.

Next consider the cofibrations $(L^\sigma)^{(l+r)} \hookrightarrow (L^\sigma)^{(l+r+1)} \to S^{l+r+1}$ for $0 \leq r < 2m-1$ and note that $(L^\sigma)^{(l)} \cong S^l$. If we take the special case where σ is an l-dimensional inverse bundle to $kL_{\mathbb{C},2}$ and remember our definition $M := M(-kL_{\mathbb{C},2}) := \Sigma^{-(2k+l)}\Sigma^\infty L^\sigma$, then we can translate this to the appropriate spectra: For $-2k \leq i < -2k + 2m - 1$ we get a cofibration $M^{(i)} \hookrightarrow M^{(i+1)} \longrightarrow \Sigma^{i+1}\Sigma^\infty S^0$ of spectra, where $M^{(i)}$ denotes the i-skeleton of the spectrum M (cf. [Sw, 8.7]). Here we note that $M^{(-2k)} = \Sigma^{-2k}\Sigma^\infty S^0$. This cofibration will be one of the main ingredients in our investigation of the Adams spectral sequence we are considering.

There is the following result of Liulevicius:

Proposition 4.1 ([L, Cor. 1]) *For $0 < t - s < (2p-2)s - 2$ and $s \geq 1$ we have $Ext^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, \mathbb{Z}/p) = 0$.* □

Obviously, $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p,\mathbb{Z}/p) = 0$ for $t-s<0$. Also, for any generator $a_0 \in \mathrm{Ext}^{1,1}_{\mathcal{A}_p}(\mathbb{Z}/p,\mathbb{Z}/p) = \mathbb{Z}/p$, for all $s \geq 1$ the term $\mathrm{Ext}^{s,s}_{\mathcal{A}_p}(\mathbb{Z}/p,\mathbb{Z}/p) = \mathbb{Z}/p$ is generated by a_0^s (cf. [Sw, Ch. 19]).

The E_2-term of the Adams spectral sequence $\{E_r^{s,t}(\Sigma^\infty S^0, \Sigma^\infty S^0)\}$ thus has a vanishing line with slope $1/(2p-2)$, which for $t-s=0$ is broken by an infinite spike. Generally, we define a spike as follows (cf. [MM]):

Definition 4.2 *Let F and G be $\mathcal{A}_p$-modules. We say that $x \in Ext^{s,t}_{\mathcal{A}_p}(F,\mathbb{Z}/p)$ resp. $y \in Ext^{u,v}_{\mathcal{A}_p}(\mathbb{Z}/p, G)$ generate a* spike*, if x is not of the form a_0x' and $a_0^i x \neq 0$ for all $i \geq 0$ resp. if y is not of the form $y'a_0$ and $ya_0^i \neq 0$ for all $i \geq 0$. We also use the analogous expressions for all terms $E_r^{s,t}(\Sigma^\infty S^0, X)$ resp. $E_r^{s,t}(Y, \Sigma^\infty S^0)$ for spectra X and Y.*

Proposition 4.3 *For the E_2-term of $\{E_r^{s,t}(M, \Sigma^\infty S^0)\}$ the following is true ($s \geq 1$):*

1. *$Ext^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M)) = 0$ for $t-s < 2k-(2m-1)$ and for $2k < t-s < [(2p-2)s-2] + [2k-(2m-1)]$.*
2. *For $2k-(2m-1) \leq t-s =: -j \leq 2k$ and $t-s < [(2p-2)s-2]+[2k-(2m-1)]$ we have that $Ext^{s,s-j}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M))$ is a spike generated by an element $b_j \in Ext^{s_j, s_j-j}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M))$ with $s_{(2m-1)-2k} = 1$.*

Proof By induction on the skeleta $M^{(i)}$ of M for $i = -2k, \ldots, [-2k+(2m-1)]$. Thus the claim is ($s \geq 1$):

I. $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)})) = 0$ for $t-s<-i$ and for $2k < t-s < [(2p-2)s-2]-i$.

II. For $-i \leq t-s = -j \leq 2k$ and $t-s < [(2p-2)s-2]-i$ we have that $\mathrm{Ext}^{s,s-j}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)}))$ is a spike generated by an element $b_j^{(i)} \in \mathrm{Ext}^{s_j^{(i)}, s_j^{(i)}-j}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)}))$ with $s_i^{(i)} = 1$.

We start the induction with $i=-2k$. In this case the claim is true since $M^{(-2k)} \cong \Sigma^{-2k}\Sigma^\infty S^0$ and because of Prop. 4.1. Suppose that I. and II. are true for all skeleta of M up to dimension i inclusive. We consider the above mentioned cofibration which leads to a short exact sequence of $\mathcal{A}_p$-modules

$$0 \longrightarrow H^*(\Sigma^{i+1}\Sigma^\infty S^0) \longrightarrow H^*(M^{(i+1)}) \longrightarrow H^*(M^{(i)}) \longrightarrow 0$$

which in turn gives a long exact sequence of the appropriate Ext-terms:

$$\cdots \xrightarrow{\delta_{s-1}} \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(\Sigma^{i+1}\Sigma^\infty S^0))$$
$$\xrightarrow{j_\#} \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i+1)})) \xrightarrow{i_\#} \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)}))$$
$$\xrightarrow{\delta_s} \mathrm{Ext}^{s+1,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(\Sigma^{i+1}\Sigma^\infty S^0)) \xrightarrow{j_\#} \cdots$$

To analize this sequence we need the following

Lemma 4.4 *The connecting homomorphism δ_s of the long exact sequence of Ext-terms is zero for $t-s=-i$, $s \geq 0$.*

Proof of Lemma 4.4: Since the bundle $kL_{\mathbb{C},2}$ is orientable, there is a Thom isomorphism $H^*(\mathbf{L}^{2m-1}(p^2)) \xrightarrow{\cong} H^{*-2k}(M)$, from which it follows that the Bockstein β_p^* acts trivially on $H^*(M^{(i+1)})$. Thus the inclusion $i : M^{(i)} \hookrightarrow M^{(i+1)}$ induces an isomorphism

$$i_\# : \mathrm{Hom}^{-i}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i+1)})) \to \mathrm{Hom}^{-i}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)}))\ ,$$

and δ_0 is 0.

For $s=1$ the claim is that

$$\mathrm{Ext}^{1,1-i}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)})) \xrightarrow{\delta_1} \mathrm{Ext}^{2,1-i}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(\Sigma^{i+1}\Sigma^\infty S^0))$$

is the zero map. This can be seen by explicitly considering the definition of the connecting homomorphism δ_1.

For $s>1$ we use the fact that $\mathrm{Ext}^{s,s-i}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)}))$ is a spike generated by an element $b_i^{(i)} \in \mathrm{Ext}^{1,1-i}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)}))$ (inductive assumption II of the theorem). Consider the multiplication

$$\begin{aligned} \mathrm{Ext}^{1,1-i}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)})) \otimes \mathrm{Ext}^{s-1,s-1}_{\mathcal{A}_p}(\mathbb{Z}/p, \mathbb{Z}/p) & \\ \xrightarrow{\cdot} \mathrm{Ext}^{s,s-i}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)})) & \\ b_i^{(i)} \otimes a_0^{s-1} \quad \longmapsto \quad b_i^{(i)} \cdot a_0^{s-1}\ , & \end{aligned}$$

then $\delta_s(b_i^{(i)} \cdot a^{s-1}) = \delta_1(b_i^{(i)}) \cdot a^{s-1} = 0$, and so the claim of the lemma follows. $\square$

We thus get the following result for $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i+1)}))$ with $s \geq 1$:

(i) For $t-s < -(i+1)$ and for $2k < t-s < [(2p-2)s-2]-(i+1)$,we have $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(\Sigma^{i+1}\Sigma^\infty S^0)) = \mathrm{Ext}^{s,t+i+1}_{\mathcal{A}_p}(\mathbb{Z}/p,\mathbb{Z}/p) = 0$ by Prop. 4.1, and $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)})) = 0$ by the inductive hypothesis I. This implies $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i+1)})) = 0$.

(ii) For $t-s=-i$ we have $\delta_s = 0$ as well as

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(\Sigma^{i+1}\Sigma^\infty S^0)) = 0\ ,$$

and for $-i < t-s < [(2p-2)s-2]-(i+1)$, $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(\Sigma^{i+1}\Sigma^\infty S^0))$ and $\mathrm{Ext}^{s+1,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(\Sigma^{i+1}\Sigma^\infty S^0))$ are equal to 0. Hence for $-(i+1) < t-s < [(2p-2)s-2]-(i+1)$ we get that

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i+1)})) \xrightarrow{i_\#} \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)}))$$

is an isomorphism.

(iii) For $t-s=-(i+1)$ we have $\delta_{s-1}=0$ as well as

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)})) = 0\ ,$$

i.e. in this case

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(\Sigma^{i+1}\Sigma^\infty S^0)) \xrightarrow{j_\#} \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i+1)}))$$

is an isomorphism.

This gives us the following result:

(I) For $t-s < -(i+1)$ and for $2k < t-s < [(2p-2)s-2]-(i+1)$ we deduce that $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i+1)})) = 0$ (by (i)).

(II) For $t-s=-(i+1)$ we get a new spike which comes from the spike in $\mathrm{Ext}^{s,s-(i+1)}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(\Sigma^{i+1}\Sigma^\infty S^0))$ and which is generated by an element in $\mathrm{Ext}^{1,1-(i+1)}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i+1)}))$ (by (iv)).
For $-(i+1) < t-s =: j \leq 2k$ and $t-s < [(2p-2)s-2]-(i+1)$, we get the spikes from $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}/p, H^*(M^{(i)}))$ (by (ii)), except maybe for elements in low filtrations.

This concludes the induction. □

We want to use the Adams spectral sequence to obtain information on the cohomotopy Euler class $\gamma(kL_{\mathbb{C},2}) \in \pi^0(M(-kL_{\mathbb{C},2}))$. Since the E_2-term of this spectral sequence has a spike for $t-s=0$ (in the interesting cases with $2k < 2m-1$), we will use a well known relation between the Adams spectral sequence and the Bockstein spectral sequence (c.f. [Br]) in order to "get rid of" this spike. We will use the following

Theorem 4.5 ([MM, Thm. 5]) *Let X be an $(l-1)$-connected spectrum with integral homology of finite type. Let C_r, $r \geq 1$, be a basis of the r-th term $\tilde{E}^r(X)$ in the (mod p) homology Bockstein spectral sequence of X and assume the C_r chosen so that*

$$C_r = D_r \cup \beta_*^r D_r \cup C_{r+1},$$

where D_r, $\beta_*^r D_r$ and C_{r+1} are disjoint linearly independent subsets of $\tilde{E}^r X$ such that $\beta_*^r D_r = \{\beta_*^r d \mid d \in D_r\}$ and C_{r+1} is a set of cycles under β_*^r, which projects to the chosen basis for $\tilde{E}^{r+1}(X)$. Then the following is true:

1. *There is a 1-1 correspondence between the set of spikes in the E_r-term of the Adams spectral sequence $\{E_r^{s,t}(\Sigma^\infty S^0, X)\}$ and C_r; if $c \in C_r$ has degree q (i.e. $c \in \tilde{E}_q^r(X)$) and $\gamma \in E_r^{s,t}(X)$ generates the corresponding spike, then*
$$(2p-2)s + l \leq q = t - s.$$
2. *If $d \in D_r$ and if $\delta \in E_r^{s,t}(X)$ and $\varepsilon \in E_r^{u,v}(X)$ with $v-u = t-s-1$ generate the spikes corresponding to d and to $\beta_*^r d$, then*
$$d_r(a_0^i \delta) = a_0^{i+r+s-u} \varepsilon$$
provided that $l + (2p-2)(i+s) \geq t - s$.

□

We now apply this theorem to our case: From the general theory of the Adams spectral sequence (cf. e.g. [Ba]) it is clear that

$$\{E_r^{s,t}(M), \Sigma^\infty S^0)\} \text{ is isomorphic to } \{E_r^{s,t}(\Sigma^\infty S^0, DM)\},$$

where DM is a spectrum dual to M in the sense of Spanier-Whitehead-Duality. The cohomotopy groups $\pi^q(M)$ vanish for $q > (2m-1) - 2k$; thus for the homotopy groups of the dual spectrum DM we get $\pi_{-q}(DM)) \cong \pi^q(M) = 0$ for $-q < 2k - (2m-1)$, i.e. DM is $(2(k-m))$-connected.

Next we will have to think about the Bockstein spectral sequence associated to DM. Here we make use of two things: First, the homology Bockstein spectral sequence of DM is isomorphic to the cohomology Bockstein spectral sequence of M; second, because of the Thom isomorphism $H^q(\mathbf{L}^{2m-1}(p^2)) \xrightarrow{\cong} H^{q-2k}(M)$ the higher Bockstein homomorphisms for $H^{*-2k}(M)$ and $H^*(\mathbf{L}^{2m-1}(p^2))$ behave in exactly the same way. Thus it is enough to study the action of β_r^* in the (mod p)-cohomology Bockstein spectral sequence of $\mathbf{L}^{2m-1}(p^2)$. If we consider the cellular cochain complexes with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/p$, the coboundary operators are given by multiplication by p^2 and by 0 respectively. From this the operation of the Bockstein homomorphism can be seen directly; we have:

1. $\beta_1^* = 0$ in all dimensions, so $\tilde{E}_*^1(DM)) \cong \tilde{E}_*^2(DM)$.
2. $\beta_2^* : H^w(\mathbf{L}^{2m-1}(p^2)) \to H^{w+1}(\mathbf{L}^{2m-1}(p^2))$ is an isomorphism for $w = 2q-1$, $1 \leq q \leq m-1$, and 0 otherwise. $\beta_2^* : H^{w-2k}(M) \to H^{w+1-2k}(M)$ and $\beta_*^2 : H_{-w+2k}(DM) \to H_{-w-1+2k}(DM)$ behave in the same way.
3. $\beta_*^r = 0$ for all $r > 2$.

With respect to Thm. 4.5 this means:

(i) $C_2 = \{x_{2k}, x_{2k-1}, \ldots, x_{2k-(2m-1)}\}$, where the degree of x_i is equal to i. This corresponds exactly to the $2m$ spikes in the E_2-term of our Adams spectral sequence that we had already detected in Prop. 4.3.

(ii)

$$D_2 = \{x_{2k-1}, x_{2k-3}, \ldots, x_{2k-(2m-3)}\}$$

and

$$\beta_*^2 D_2 = \{x_{2k-2}, x_{2k-4}, \ldots, x_{2k-(2m-2)}\} .$$

(iii) $C_3 = \{x_{2k}, x_{2k-(2m-1)}\} = C_4 = \cdots = C_\infty$.

In particular only the two outermost spikes of the Adams spectral sequence survive the step from the E_2-term to the E_3-term (and all higher terms). We are especially interested in what filtration the spike in dimension 0 vanishes (given that $2k < 2m-1$):

From Thm. 4.5(i) it follows that the spike in dimension 1 is generated by an element $\delta \in E_2^{s,s+1}(\Sigma^\infty S^0, DM)$, where $(2p-2)s + 2k - (2m-1) \leq 1$, i.e.

$$s \leq \frac{m-k}{p-1},$$

while the spike in dimension 0 is generated by an element $\varepsilon \in E_2^{u,u}(\Sigma^\infty S^0, DM)$, where

$$u \leq \frac{m-k}{p-1} - \frac{1}{2p-2} .$$

From Thm. 4.5(ii) it follows that

$$d_2(a_0^i \delta) = a_0^{i+2+s-u} \varepsilon \in E_2^{i+2+s, i+2+s}(\Sigma^\infty S^0, DM) ,$$

provided that

$$i + 2 + s \geq 2 + \frac{m-k}{p-1} \ .$$

Thus, the spike in dimension 0 vanishes in the E_3-term as soon as the filtration is greater or equal to $2 + \frac{m-k}{p-1}$. In particular we know from Prop. 4.3 that for such filtrations there are no elements that do not come from a spike, so that

$$E_3^{r,r}(\Sigma^\infty S^0, DM) = E_\infty^{r,r}(\Sigma^\infty S^0, DM) = 0$$

for

$$r \geq 2 + \frac{m-k}{p-1} \quad \text{and} \quad 2k < 2m - 1 \ .$$

In order to be able to use this result to get information on the cohomotopy Euler class $\gamma(kL_{\mathbb{C},2})$, we must determine its Adams filtration.

Lemma 4.6 *$\gamma(kL_{\mathbb{C},2})$ has $\mathbb{Z}/p$-Adams filtration $\geq k$.*

Proof As observed in Section 2, the Euler class in integral cohomology $e(L_{\mathbb{C},2}) \in H^2(\mathbf{L}^{2m-1}(p^2);\mathbb{Z}) \cong \mathbb{Z}/p^2$ is an element of order p, so that under reduction mod p we obtain $e_{\mathbb{Z}/p}(L_{\mathbb{C},2}) = 0$. From this, however, it is clear that the $\mathbb{Z}/p$-Adams filtration of the cohomotopy Euler class $\gamma(L_{\mathbb{C},2})$ is ≥ 1, since it is trivial in $\mathbb{Z}/p$-cohomology: If $\gamma(L_{\mathbb{C},2})$ is represented by a map $f : M \longrightarrow \Sigma^\infty S^0$, then we consider

$$f^* : H^*(\Sigma^\infty S^0) \longrightarrow H^*(M) \xrightarrow{\cong} H^{*+2}(\mathbf{L}^{2m-1}(p^2))$$

where the latter map is the inverse of the Thom isomorphism. It is enough to show that $f^*\iota = 0$ for the generator ι of $H^*(\Sigma^\infty S^0) = \mathbb{Z}/p$. But

$$f^*\iota = h_{\mathbb{Z}/p}(\gamma(L_{\mathbb{C},2})) = e_{\mathbb{Z}/p}(L_{\mathbb{C},2}) = 0,$$

where $h_{\mathbb{Z}/p}$ is the stable mod p Hurewicz homomorphism. Now we use the general properties of the cohomotopy Euler class and conclude that $\gamma(kL_{\mathbb{C},2}) = \gamma(L_{\mathbb{C},2})^k$ has $\mathbb{Z}/p$-Adams filtration $\geq k$. □

Proposition 4.7 *If $m - 1 \geq k \geq \frac{m}{p} + \frac{2p-2}{p}$, then the cohomotopy Euler class $\gamma(kL_{\mathbb{C},2}) \in \omega^0(X; -kL_{\mathbb{C},2})$ vanishes.*

Proof We use the Adams spectral sequence

$$\{E_r^{s,t}(M; \Sigma^\infty S^0)\} \cong \{E_r^{s,t}(\Sigma^\infty S^0; DM)\}$$

converging to the p-primary part of $\pi^{s-t}(M)$. The assumption $m-1 \geq k \geq \frac{m}{p} + \frac{2p-2}{p}$ ensures that $E_3^{s,s}(\Sigma^\infty S^0, DM)$ vanishes for all $s \geq k$.

The same is true for the corresponding E_∞-terms. Hence the filtration quotients

$$E_\infty^{s,s}(M;\Sigma^\infty S^0) = F_s\pi^0(M)/F_{s+1}\pi^0(M)$$

are also all trivial for $s \geq k$, and thus $\gamma(kL_{\mathbb{C},2})$ is in the intersection of all filtration groups (since its Adams filtration is $\geq k$). Thus, $\gamma(kL_{\mathbb{C},2})$ is a torsion element of order prime to p. However, we know from Lemma 3.11 that $\gamma(kL_{\mathbb{C},2})$ is p-torsion, so it must be zero. □

By Cor. 3.12 we now get a section of the bundle $kL_{\mathbb{C},2}$ as soon as $k \geq \frac{m}{p} + \frac{2p-2}{p}$, provided $m-1 \geq k$. This actually implies $m \geq 4$, so we have $v_{p,2}(m) \leq \langle\frac{m+2p-2}{p}\rangle$ for $m \geq 4$. It is easy to see that $v_{p,2}(1) = 1$ and that $v_{p,2}(2) = 2$, and for $m = 3$ we observe that $v_{p,2}(3) \leq v_{p,2}(4) \leq \langle\frac{2+2p}{p}\rangle = 3 = \langle\frac{1+2p}{p}\rangle$, so the above formula for $v_{p,2}(m)$ also holds for $m = 2$ or 3. If we combine this with the lower estimate in [B1, Proof of Thm. 1.2] and note that for $m \not\equiv 2 \bmod p$ the two expressions $\left\langle\frac{m-2}{p}\right\rangle$ and $\left\langle\frac{m-1}{p}\right\rangle$ are the same, while for $m \equiv 2 \bmod p$ they differ by one, we have proved the main theorem:

Theorem 4.8 $v_{p,2}(1) = 1$, *and for* $m \geq 2$ *we have*

$$\begin{aligned} \left\langle\frac{m-2}{p}\right\rangle + 1 \leq v_{p,2}(m) &\leq \left\langle\frac{m-2}{p}\right\rangle + 2 \quad \text{for } m \not\equiv 2 \bmod p \\ v_{p,2}(m) &= \frac{m-2}{p} + 2 \quad \text{for } m \equiv 2 \bmod p \end{aligned}$$

In particular $v_{p,2}(m) \leq 3$ *for* $p \geq m-2$, *i.e. for* p *big enough there always exists a* $\mathbb{Z}/p$*-equivariant map* $\mathbf{L}^{2m-1}(p) \longrightarrow S^5$. □

5 An application to Borsuk-Ulam functions

We now apply the results established so far to the investigation of Borsuk-Ulam functions:

Definition 5.1 *Let* G *be a compact Lie group, then the Borsuk-Ulam function* $b_G : \mathbb{N} \to \mathbb{N}$ *is defined as follows:* $b_G(n)$ *is the biggest natural number* k *such that if there exists a* G*-equivariant map* $SV \to SW$ *between representation spheres of* G *with* $\dim V \geq n$ *and* $W^G = 0$, *then* $\dim W \geq k$. *Here* W^G *is the fixed point set of the* G*-action on* W *(cf. [B2]).*

The connection between Borsuk-Ulam functions b_G and $s_k(m)$ resp. $v_{p,k}(m)$ is the following:

Proposition 5.2 (cf. [B2, Prop. 4.2]) *Let $k \geq 2$. For $G = \mathbb{Z}/p^k$ we have $b_G(2m) = s_k(m)$, and for $G = \mathbb{Z}/p^k$ with $p \neq 2$ we have $b_G(2m) = 2 \cdot v_{p,k}(m)$.*

Proof Assume given a G-map $SV \to SW$ with $(SW)^G = \emptyset$, $\dim_{\mathbb{R}} V = 2m$ and $\dim_{\mathbb{R}} W = b_G(2m) =: n$. For $p = 2$ we can decompose W into irreducible representations $W_1, W_2, \ldots, W_{2^{k-1}}$, where $W_i \cong \mathbb{R}$, and G acts via $\zeta \mapsto \zeta^i$. Then together with the obvious G-maps $W_i \to W_{2^{k-1}}$ we get a G-map $SV \to SW^n_{2^{k-1}}$. In case $p > 2$ we decompose W into irreducible complex representations $V_1, V_2, \ldots, V_{p^{k-1}}$, where $V_i \cong \mathbb{C}$, and G acts via $\zeta \to \zeta^i$, so that we get a G-map $SV \to SV^{n/2}_{p^{k-1}}$. Obviously, there is always a G-equivariant map $SV_1^m \to SV$, so composing the various maps we obtain a G-map $SV_1^m \to SW^n_{2^{k-1}}$ if $p = 2$ resp. $SV_1^m \to SV^{n/2}_{p^{k-1}}$ if $p > 2$. Thus, $b_G(2m) \geq s_k(m)$ resp. $b_G(2m) \geq 2 \cdot v_{p,k}(m)$.

Conversely, let $f : SV_1^m \to SW^n_{2^{k-1}}$ resp. $g : SV_1^m \to SV^{n/2}_{p^{k-1}}$ be G-equivariant maps. Since $\left(SV^{n/2}_{p^{k-1}}\right)^G = \emptyset = \left(SW^n_{2^{k-1}}\right)^G$, we get $b_G(2m) \leq s_k(m)$ for $p = 2$ and $b_G(2m) \leq 2 \cdot v_{p,k}(m)$ for $p > 2$. □

From the definition of $b_G(m)$ it is immediate that $b_G : \mathbb{N} \to \mathbb{N}$ is weakly increasing. Further:

Lemma 5.3 $b_G(m + n) \leq b_G(m) + b_G(n)$.

Proof For two G-maps $f : SV \to SW$ and $g : S\tilde{V} \to S\tilde{W}$ their join $f * g : S(V \oplus \tilde{V}) \cong SV * S\tilde{V} \to SW * S\tilde{W} \cong S(W \oplus \tilde{W})$ also is G-equivariant, and since $W^G = 0 = \tilde{W}^G$, we also have $(W \oplus \tilde{W})^G = 0$. □

Proposition 5.4 (cf. [B2, Prop. 4.2]) *Let $G = \mathbb{Z}/p^2$.*

1. *$p = 2$: $b_G(1) = 1$, $b_G(2) = s(1) = 2$ and for $m \geq 2$ the following holds:*

$$b_G(2m) = s(m) = \begin{cases} m+1 & \textit{for } m \equiv 0, 2 \bmod 8 \\ m+2 & \textit{for } m \equiv 1, 3, 4, 5, 7 \bmod 8 \\ m+3 & \textit{for } m \equiv 6 \bmod 8 \end{cases}$$

$$b_G(2m-1) = \begin{cases} m+1 & \textit{for } m \equiv 0, 1, 2, 3 \bmod 8 \\ m+2 & \textit{for } m \equiv 6, 7 \bmod 8 \end{cases}$$

For $m \equiv 4, 5 \bmod 8$ we have $m + 1 \leq b_G(2m - 1) \leq m + 2$.

2. $p > 2$*:* $\quad b_G(2) = 2 \cdot v_{p,2}(1) = 2$*, and for* $m \geq 2$ *the following holds:*
If $m \not\equiv 2 \bmod p$ *then*

$$2\left\langle \frac{m-2}{p} \right\rangle + 2 \leq b_G(2m) = 2 \cdot v_{p,2}(m) \leq 2\left\langle \frac{m-2}{p} \right\rangle + 4 \ ,$$

if $m \equiv 2 \bmod p$ *then*

$$b_G(2m) = 2 \cdot v_{p,2}(m) = 2\frac{m-2}{p} + 4 \ .$$

For all m *we have* $b_G(2m-1) = b_G(2m)$*, since there are no odd dimensional fixed point free representations of* $\mathbb{Z}/p^2$*.*

Proof 1. $p = 2$: $\quad b_G(1) = 1$ is obvious. The results for $b_G(2) = s(1)$ and for $b_G(2m) = s(m)$ (for $m \geq 2$) are the results of Stolz ([S]). For $b_G(2m-1)$ one uses the monotonicity of b_G and Lemma 5.3.
2. $p > 2$: $\quad b_G(1) = b_G(2) = 2$ is obvious. The inequalities for $b_G(2m) = 2 \cdot v_{p,2}(m)$ are immediate from Thm. 4.8. □

Hence, for $G = \mathbb{Z}/p^2$ and $p = 2$ there exist exact results for b_G in almost all cases (except for $b_G(2m-1)$ with $m \equiv 4, 5 \bmod 8$), whereas for $p > 2$ there only exist (quite good) estimates, except for $m \equiv 2 \bmod p$, where we have an exact result. However, if we are only interested in the growth behaviour of b_G, this does not matter: For $p = 2$ we have:

$$a_G := \lim_{m\to\infty} \frac{b_G(m)}{m} = \lim_{m\to\infty} \frac{s(m)}{2m} = \frac{1}{2}$$

and for $p > 2$

$$a_G := \lim_{m\to\infty} \frac{b_G(m)}{m} = \lim_{m\to\infty} \frac{v_{p,2}(m)}{m} = \frac{1}{p}.$$

This result can be generalized to $G = \mathbb{Z}/p^k$ for $k > 2$: From Thm. 1.2(a) in [B1] it follows that $s_k(m) \geq \frac{2(m-1)}{2^{k-1}} + 1$, so that for $G = \mathbb{Z}/2^k$

$$a_G := \lim_{m\to\infty} \frac{b_G(m)}{m} = \lim_{m\to\infty} \frac{s_k(m)}{2m} \geq \frac{1}{2^{k-1}}.$$

From the proof of the same theorem we get $v_{p,k}(m) \geq \frac{m-1}{p^{k-1}} + 1$, so that for $G = \mathbb{Z}/p^k$ with $p > 2$

$$a_G := \lim_{m\to\infty} \frac{b_G(m)}{m} = \lim_{m\to\infty} \frac{v_{p,k}(m)}{m} \geq \frac{1}{p^{k-1}}.$$

Theorem 5.5 *Let $G = \mathbb{Z}/p^k$ for some prime number p and $k \geq 2$. Then $a_G = \frac{1}{p^{k-1}}$.*

Proof The theorem is an immediate consequence of the following lemma. □

Lemma 5.6 *Let $a_k := a_{\mathbb{Z}/p^{k+1}}$. Then $a_0 = 1$, and $a_k \leq \frac{1}{p^k}$ for $k \geq 1$.*

Proof The assertion $a_0 = 1$ is the classical theorem by Borsuk-Ulam. For the general case we use induction on k: For $k = 1$ nothing needs to be shown. Let $k \geq 2$ and $H := \mathbb{Z}/p^k \subset \mathbb{Z}/p^{k+1} =: G$. Now we set $V := V_1^m$ and for $p = 2$ and $p > 2$ we set $W := W_{2^{k-1}}^n$ and $W := V_{p^{k-1}}^n$ respectively, where $V_i \cong \mathbb{C}$, $W_i \cong \mathbb{R}$, and H acts via $\zeta \mapsto \zeta^i$. By induction, for m and n big enough, for every $0 < \epsilon \leq 1$ there exists an H-equivariant map $Sf : SV \to SW$ such that

$$\frac{\dim_{\mathbb{R}} W}{\dim_{\mathbb{R}} V} \leq a_H + \epsilon \leq \frac{1}{p^{k-1}} + \epsilon \,.$$

H acts freely on SV, and the isotropy group of the H-action on SW is $\mathbb{Z}/p^{k-1} \subset \mathbb{Z}/p^k$. Consider the induced representations $\mathrm{ind}_H^G V$ resp. $\mathrm{ind}_H^G W$. In general, $\dim_{\mathbb{R}}(\mathrm{ind}_H^G Z) = |G/H| \cdot \dim_{\mathbb{R}} Z$, and the isotropy groups of the G-action on $\mathrm{ind}_H^G Z$ are the same as the isotropy groups of the H-action on Z. Thus, G acts freely on $S(\mathrm{ind}_H^G V)$, and the isotropy group of the induced G-action on $S(\mathrm{ind}_H^G W)$ is $\mathbb{Z}/p^{k-1} \subset \mathbb{Z}/p^{k+1}$. In particular, $\mathbb{Z}/p^2$ acts freely on $S(\mathrm{ind}_H^G W)$; thus we can apply the case $k = 1$. We use the notation that $\mathbb{Z}/p^2$-spaces are marked by a tilde. So there are $\tilde{n}$,$\tilde{l}$ and a $\mathbb{Z}/p^2$-map Sh,

$$S\left((\widetilde{\mathrm{ind}}_H^G W)^{2\tilde{n}}\right) \xrightarrow{\mathbb{Z}/p^2 \cong} S\left(\tilde{V}_1^{\tilde{n} \cdot p \cdot \dim_{\mathbb{R}} W}\right) \xrightarrow{Sh} S\tilde{U}_p^{\tilde{l}} \,,$$

with

$$\frac{\dim_{\mathbb{R}} \tilde{U}_p^l}{\dim_{\mathbb{R}} \left(\widetilde{\mathrm{ind}}_H^G W\right)^{2\tilde{n}}} \leq \frac{1}{p} + \epsilon \quad \text{and} \quad \left(\tilde{U}_p^{\tilde{l}}\right)^{\mathbb{Z}/p^2} = 0.$$

Here, $\tilde{U}_p = \tilde{V}_p \cong \mathbb{C}$ for $p > 2$ and $\tilde{U}_p = \tilde{W}_p \cong \mathbb{R}$ for $p = 2$, and $\mathbb{Z}/p^2$ acts via $\zeta \mapsto \zeta^p$. The $\mathbb{Z}/p^2$-map Sh gives us in a canonical way a $\mathbb{Z}/p^{k+1}$-map

$$S\tilde{h} : S\left((\mathrm{ind}_H^G W^{2\tilde{n}}\right) \longrightarrow SU_{p^k}^{\tilde{l}} \,.$$

Further, the induced map

$$S\left((\mathrm{ind}_H^G f)^{2\tilde{n}}\right) : S\left((\mathrm{ind}_H^G V)^{2\tilde{n}}\right) \longrightarrow S\left((\mathrm{ind}_H^G W)^{2\tilde{n}}\right)$$

is G-equivariant, and since G acts freely on $S\left((\mathrm{ind}_H^G V)^{2\tilde{n}}\right)$, there is a G-isomorphism

$$S(g) : S\left((\mathrm{ind}_H^G V)^{2\tilde{n}}\right) \xrightarrow{\cong} S\left(V_1^{p\cdot\dim_{\mathbb{R}} V\cdot\tilde{n}}\right).$$

Now the composition $S(g)^{-1} \circ S\left((\mathrm{ind}_H^G f)^{2\tilde{n}}\right) \circ S(\tilde{h})$ is the desired G-equivariant map from $S\left(V_1^{p\cdot\dim_{\mathbb{R}} V\cdot\tilde{n}}\right)$ to $SU_{p^k}^{\tilde{l}}$ and we have

$$\begin{aligned}\frac{\dim_{\mathbb{R}} U_{p^k}^{\tilde{l}}}{\dim_{\mathbb{R}}(V_1^{p\cdot\dim_{\mathbb{R}} V\cdot\tilde{n}})} &= \frac{\dim_{\mathbb{R}} U_{p^k}^{\tilde{l}}}{\dim_{\mathbb{R}}(\mathrm{ind}_H^G W)^{2\tilde{n}}} \cdot \frac{\dim_{\mathbb{R}}(\mathrm{ind}_H^G W)^{2\tilde{n}}}{\dim_{\mathbb{R}}(V_1^{p\cdot\dim_{\mathbb{R}} V\cdot\tilde{n}})} \\ &\leq (\frac{1}{p}+\epsilon)\cdot\frac{2\tilde{n}\cdot p\cdot\dim_{\mathbb{R}} W}{2\tilde{n}\cdot p\cdot\dim_{\mathbb{R}} V} \\ &\leq (\frac{1}{p}+\epsilon)(\frac{1}{p^{k-1}}+\epsilon) \leq \frac{1}{p^k}+3\epsilon.\end{aligned}$$

This concludes the induction. □

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