On Eliminants, and Associated Roots: E. B. Elliott, M.A.
On Five Properties of Certain Solutions of a Differential Equation of the Second Order: Dr. Routh, F.R.S.
On the Arguments of Points on a Surface: R. A. Roberts, M.A.
On Congruences of the Third Order and Class: Dr. Hirst, F.R.S.

The following presents were received:—

"The Common Sense of the Exact Sciences," by the late W. K. Clifford, 8vo; London, 1885.

"Curve Tracing in Cartesian Coordinates," by Prof. W. Woolsey Johnson, 8vo; New York, 1884.

"Johns Hopkins University Circulars," Vol. IV., Nos. 37 and 38, March, 1885.

"Memoirs of the National Academy of Sciences," Vol. II., 1883, 4to; Washington, 1884.

"Archives Néerlandaises des Sciences Exactes et Naturelles," T. XIX., L. 4 and 5.

"Bulletin de la Société Mathématique de France," Tome XIII., No. 2.

"Tidsskrift for Mathematik," V. Raekke, 1 Aargang, Nos. 1—6.

"Bulletin des Sciences Mathématiques et Astronomiques," T. IX., March, 1885.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. ix., St. 3.

"Teorema per la Quadratura del Circolo," di Ugo Amerighi, 8vo; Firenze, 1885.

"Acta Mathematica," v., 2, 3, and 4.

"Journal für Mathematik," Bd. XCVIII., Heft 2; March, 1885.

"Atti della R. Accademia dei Lincei,—Rendiconti," Vol. I., Fasc. 7; Roma, March, 1885.

"Annales de l'Ecole Polytechnique de Delft," 2 liv.; Leide, 1885.

On Eliminants, and Associated Roots. *By* E. B. Elliott, M.A.

[*Read April 2nd,* 1885.]

1. The object of the following paper is to obtain some more general results akin to and including the well-known simple and elegant theorems due to Professor Sylvester and others, which connect the common root of two quantics with the differential coefficients of their resultant, the repeated root of a single quantic with those of the discriminant whose vanishing expresses its existence, &c. The method adopted is that of Salmon's *Higher Algebra*, § 92.

Let $$u_x \equiv a_1\psi_1(x)+a_2\psi_2(x)+\dots+a_m\psi_m(x),$$

and $$v_y \equiv b_1\chi_1(y)+b_2\chi_2(y)+\dots+b_n\chi_n(y),$$

and let the condition in the coefficients, which expresses that a value of x satisfying $u = 0$, and one of y which makes $v = 0$, are connected by a given relation
$$\phi(x, y) = 0,$$
be
$$F(a_1, a_2, \dots a_m, b_1, b_2, \dots b_n) = 0.$$

Also let δx, δy, $\delta a_1, \dots \delta b_n$ be infinitesimal increments given to the variables and coefficients, restricted only by the conditions that the equations $u = 0$, $v = 0$, $\phi = 0$ still hold. It is a consequence that the relation $F = 0$ still holds equally. In other words, retaining only first powers of the increments, the three limitations of arbitrariness,

$$\psi_1(x)\,\delta a_1 + \dots + \psi_m(x)\,\delta a_m + \frac{du_x}{dx}\,\delta x = 0,$$

$$\chi_1(y)\,\delta b_1 + \dots + \chi_n(y)\,\delta b_n + \frac{dv_y}{dy}\,\delta y = 0,$$

$$\frac{d\phi}{dx}\,\delta x + \frac{d\phi}{dy}\,\delta y = 0,$$

must produce as a consequence

$$\frac{dF}{da_1}\,\delta a_1 + \dots + \frac{dF}{da_m}\,\delta a_m + \frac{dF}{db_1}\,\delta b_1 + \dots + \frac{dF}{db_n}\,\delta b_n = 0.$$

Consequently, for certain values of three undetermined multipliers λ, μ, ν, we must have simultaneously

$$\frac{\frac{dF}{da_1}}{\psi_1(x)} = \frac{\frac{dF}{da_2}}{\psi_2(x)} = \dots = \frac{\frac{dF}{da_m}}{\psi_m(x)} = \lambda,$$

$$\frac{\frac{dF}{db_1}}{\chi_1(y)} = \frac{\frac{dF}{db_2}}{\chi_2(y)} = \dots = \frac{\frac{dF}{db_n}}{\chi_n(y)} = \mu,$$

$$\nu\,\frac{d\phi}{dx} - \lambda\,\frac{du_x}{dx} = 0,$$

$$\nu\,\frac{d\phi}{dy} - \mu\,\frac{dv_y}{dy} = 0,$$

altogether $m+n+2$ relations, of which the last two tell us that the value of the ratio $\lambda : \mu$ of the common values of the first and second sets of equal fractions is

$$\frac{d\phi}{dx}\cdot\frac{dv_y}{dy} : \frac{d\phi}{dy}\cdot\frac{du_x}{dx}.$$

2. It becomes desirable to interpret special cases of these general results. Firstly, then, let us take the case of direct elimination of a single variable x between two equations

$$u_x \equiv a_1\psi_1(x)+a_2\psi_2(x)+\ldots+a_m\psi_m(x)=0,$$

and $$v_x \equiv b_1\chi_1(x)+b_2\chi_2(x)+\ldots+b_n\chi_n(x)=0;$$

that is to say, let us take

$$\phi(x, y) \equiv x-y,$$

and let us now write the result of eliminating

$$E(a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_n)=0.$$

Our conclusion is that the common x which satisfies $u_x=0$ and $v_x=0$ satisfies also the $m+n-1$ equations

$$\frac{dv_x}{dx}\cdot\frac{\psi_1(x)}{\frac{dE}{da_1}}=\frac{dv_x}{dx}\cdot\frac{\psi_2(x)}{\frac{dE}{da_2}}=\ldots=\frac{dv_x}{dx}\cdot\frac{\psi_m(x)}{\frac{dE}{da_m}}$$

$$=-\frac{du_x}{dx}\cdot\frac{\chi_1(x)}{\frac{dE}{db_1}}=-\frac{du_x}{dx}\cdot\frac{\chi_2(x)}{\frac{dE}{db_2}}=\ldots=-\frac{du_x}{dx}\cdot\frac{\chi_n(x)}{\frac{dE}{db_n}}.$$

To particularise still further, let $\psi_1(x), \ldots \psi_m(x), \chi_1(x), \ldots \chi_n(x)$ be powers of x. The two lines of equalities then give $m-1$ and $n-1$, immediate expressions for the common value of x, as ratios of differential coefficients of the eliminant or as roots of such ratios. When the powers of x are two sets of successive powers, so that u_x and v_x are complete binary quantics expressed in their natural form, the values thus given are those of Salmon's *Higher Algebra*, § 92.

3. Another course that may be taken by way of deducing more special results from the general ones of § 1 will be, keeping $\phi(x, y)$ general, to restrict the forms of the $m+n$ functions $\psi(x)$ and $\chi(y)$. For instance, we obtain immediately that, if

$$K=0$$

is the condition in the coefficients that x, a root of

$$U_x \equiv a_1x^{r_1}+a_2x^{r_2}+\ldots+a_mx^{r_m}=0,$$

and y, one of $$V_y \equiv b_1y^{s_1}+b_2y^{s_2}+\ldots+b_ny^{s_n}=0,$$

are connected by a relation

$$\phi(x, y)=0,$$

then equivalent values of these roots are given by pairs of the $m+n$

equations $$\frac{dK}{da_1} = \lambda x^{r_1},\quad \frac{dK}{da_2} = \lambda x^{r_2},\quad \ldots\ldots,\quad \frac{dK}{da_m} = \lambda x^{r_m},$$

$$\frac{dK}{db_1} = \mu y^{s_1},\quad \frac{dK}{db_2} = \mu y^{s_2},\quad \ldots\ldots,\quad \frac{dK}{db_n} = \mu y^{s_n},$$

where $$\lambda \frac{d\phi}{dy} \cdot \frac{dU_x}{dx} = \mu \frac{d\phi}{dx} \cdot \frac{dV_y}{dy}.$$

For the case expressed by

$$r_1 = r_2+1 = r_3+2 = \ldots = r_m+m-1,$$

$$s_1 = s_2+1 = s_3+2 = \ldots = s_n+n-1,$$

this, again, is another immediate generalisation of Salmon, § 92.

4. Returning to the general theorems of § 1, a case of special interest and exceptional nature is that in which the two sets of coefficients $a_1, a_2, \ldots, b_1, b_2, \ldots$ are identical. In such a case the numbers of functions $\psi(x)$ and $\chi(y)$ will generally be equal; but other cases will be included.

Suppose, in § 1, that n is equal to m. We are told that, for all positive integral values of r not exceeding m,

$$\frac{dF}{da_r} = \lambda\psi_r(x), \quad \text{and} \quad \frac{dF}{db_r} = \mu\chi_r(y),$$

where $$\lambda \frac{d\phi}{dy} \cdot \frac{du_x}{dx} = \mu \frac{d\phi}{dx} \cdot \frac{dv_y}{dy}.$$

Now suppose that, on replacing $b_1, b_2, \ldots b_m$ by $a_1, a_2, \ldots a_m$ respectively,

the function $$F(a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m),$$

becomes $$f(a_1, a_2, \ldots a_m),$$

then, for each value of r,

$$\frac{dF}{da_r} + \frac{dF}{db_r} \text{ becomes } \frac{df}{da_r}.$$

Consequently, we obtain that, if

$$f(a_1, a_2, \ldots a_m) = 0$$

be the condition in the coefficients that

$$u_x \equiv a_1\psi_1(x) + a_2\psi_2(x) + \ldots + a_m\psi_m(x) = 0,$$

and $$u'_y \equiv a_1\chi_1(y) + a_2\chi_2(y) + \ldots + a_m\chi_m(y) = 0$$

are satisfied by an x and a y connected by

$$\phi(x, y) = 0,$$

it follows that $$\frac{df}{da_1} = \lambda\psi_1(x) + \mu\chi_1(y),$$

$$\frac{df}{da_2} = \lambda\psi_2(x) + \mu\chi_2(y),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

and $$\frac{df}{da_m} = \lambda\psi_m(x) + \mu\chi_m(y);$$

where λ and μ are in the ratio

$$\frac{d\phi}{dx} \cdot \frac{du'_y}{dy} : \frac{d\phi}{dy} \cdot \frac{du_x}{dx}.$$

No modification is necessary if m and n be unequal. If, for instance, $m > n$ we have only to consider in these results that $\chi_{n+1}(y), \ldots \chi_m(y)$ are identically zero.

5. Let now, in this last, u_x and u'_y be the same functions of x and y respectively. We conclude at once that, if x and y be two roots of the equation

$$u_x \equiv a_1\psi_1(x) + a_2\psi_2(x) + \ldots + a_m\psi_m(x) = 0,$$

which are connected by $\phi(x, y) = 0,$

and, if $\mathbf{F}(a_1, a_2, \ldots a_m) = 0$

be the condition in the coefficients expressive of the fact, then

$$\frac{d\mathbf{F}}{da_1} = \lambda\psi_1(x) + \mu\psi_1(y),$$

$$\frac{d\mathbf{F}}{da_2} = \lambda\psi_2(x) + \mu\psi_2(y),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{d\mathbf{F}}{da_m} = \lambda\psi_m(x) + \mu\psi_m(y),$$

where
$$\frac{\lambda}{\mu} = \frac{\frac{d\phi}{dx} \cdot \frac{du_y}{dy}}{\frac{d\phi}{dy} \cdot \frac{du_x}{dx}},$$

which ratio may be written $\quad -\frac{du_y}{du_x}$.

In particular, give u_x the form

$$U_x \equiv a_1 x^{r_1} + a_2 x^{r_2} + \ldots + a_m x^{r_m};$$

and write L for the form then assumed by $F(a_1, a_2, \ldots a_m)$.

We deduce m equations, of which the first

$$\frac{dL}{da_1} = \lambda x^{r_1} + \mu y^{r_1}$$

is representative, and conclude that, if only $r_1, r_2, \ldots r_n$ are commensurable quantities, $\frac{dL}{da_1}, \frac{dL}{da_2}, \ldots \frac{dL}{da_m}$ are terms of a recurring series. In particular, if $r_1, r_2, \ldots r_m$ be positive integers, the scale of relation of this recurring series is $1-(x+y)+xy$; and the various differential coefficients of L, arranged in order as above, are the $r_1^{\text{th}}, r_2^{\text{th}}, \ldots r_m^{\text{th}}$ terms of the series. More particularly still, if U_x be the ordinary complete binary $(m-1)$-ic, $\qquad a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_m$,

those differential coefficients are the first m terms of a recurring series whose scale of relation is

$$xy-(x+y)+1 = 0.$$

Since the ratio $\lambda : \mu$ is given, one known term will determine our recurring series completely in terms of x and y.

6. Desiring to proceed to the case of two equal roots of the same equation $u_x = 0$, two courses are open to us. We may take the results of the last article, specialise them by assuming

$$\phi(x, y) \equiv x - y + \delta,$$

and proceed to the limit by making δ infinitesimal. [Merely to replace $\phi(x, y)$ by $x-y$ would, for several apparent reasons, be nugatory.] Or, remembering that equal roots of $u_x = 0$ are roots also of $\frac{du_x}{dx} = 0$, we may apply § 4, taking $u_y' \equiv \frac{du_y}{dy}$, and $\phi(x, y) \equiv x-y$. The results obtained by the two processes are, as they should be, identical. The latter is the one here chosen for exhibition. Briefly stated, it tells us that, $\Delta(a_1, a_2, \ldots a_m) = 0$

being the condition that an x simultaneously satisfies

$$u_x \equiv a_1\psi_1(x) + a_2\psi_2(x) + \ldots + a_m\psi_m(x) = 0,$$

and $$\frac{du_x}{dx} \equiv a_1\psi_1'(x) + a_2\psi_2'(x) + \ldots + a_m\psi_m'(x) = 0,$$

then $$\frac{d\Delta}{da_1} = \lambda\psi_1(x) + \mu\psi_1'(x),$$

$$\frac{d\Delta}{da_2} = \lambda\psi_2(x) + \mu\psi_2'(x),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{d\Delta}{da_m} = \lambda\psi_m(x) + \mu\psi_m'(x),$$

simultaneously, where λ and μ satisfy

$$\lambda\frac{du_x}{dx} + \mu\frac{d^2u_x}{dx^2} = 0;$$

which last expresses, since for the particular value x in question $\frac{du_x}{dx} = 0$, that, in general, $\frac{\mu}{\lambda} = 0$. We can neglect, therefore, μ in comparison with λ, and so arrive at the equalities

$$\frac{\frac{d\Delta}{da_1}}{\psi_1(x)} = \frac{\frac{d\Delta}{da_2}}{\psi_2(x)} = \ldots = \frac{\frac{d\Delta}{da_m}}{\psi_m(x)} = \lambda.$$

Of this general theorem with regard to equal roots, the case in which the functions $\psi(x)$ are successive powers of x is, of course, a familiar one. (Salmon's *Higher Algebra*, § 109.)

7. Similar theories to all the above, though more cumbrous in form, can be developed with regard to functions of higher numbers of variables. Let it suffice here to state the results corresponding to the comprehensive ones of the first article for the case when u and v involve each two variables.

Suppose $$u \equiv a_1\psi_1(x, x') + a_2\psi_2(x, x') + \ldots + a_m\psi_m(x, x'),$$

and $$v \equiv b_1\chi_1(y, y') + b_2\chi_2(y, y') + \ldots + b_n\chi_n(y, y'),$$

and let the condition in the coefficients, which expresses that certain values of x, x', y, y', which make $u = 0$ and $v = 0$, are connected by three given relations

$$\phi_1(x, x', y, y') = 0,$$
$$\phi_2(x, x', y, y') = 0,$$
$$\phi_3(x, x', y, y') = 0,$$

be

$$F(a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_n) = 0;$$

then shall

$$\frac{\frac{dF}{da_1}}{\psi_1(x, x')} = \frac{\frac{dF}{da_2}}{\psi_2(x, x')} = \ldots = \frac{\frac{dF}{da_m}}{\psi_m(x, x')} = \lambda,$$

and

$$\frac{\frac{dF}{db_1}}{\chi_1(y, y')} = \frac{\frac{dF}{db_2}}{\chi_2(y, y')} = \ldots = \frac{\frac{dF}{db_n}}{\chi_n(y, y')} = \mu,$$

where λ and μ are connected by the result of eliminating ν_1, ν_2, ν_3 from the four equations

$$\lambda \frac{du}{dx} = \nu_1 \frac{d\phi_1}{dx} + \nu_2 \frac{d\phi_2}{dx} + \nu_3 \frac{d\phi_3}{dx},$$

$$\lambda \frac{du}{dx'} = \nu_1 \frac{d\phi_1}{dx'} + \nu_2 \frac{d\phi_2}{dx'} + \nu_3 \frac{d\phi_3}{dx'},$$

$$\mu \frac{dv}{dy} = \nu_1 \frac{d\phi_1}{dy} + \nu_2 \frac{d\phi_2}{dy} + \nu_3 \frac{d\phi_3}{dy},$$

$$\mu \frac{dv}{dy'} = \nu_1 \frac{d\phi_1}{dy'} + \nu_2 \frac{d\phi_2}{dy'} + \nu_3 \frac{d\phi_3}{dy'},$$

that is to say, by the relation

$$\lambda \left\{ \frac{du}{dx} \cdot \frac{d(\phi_1, \phi_2, \phi_3)}{d(x', y, y')} - \frac{du}{dx'} \cdot \frac{d(\phi_1, \phi_2, \phi_3)}{d(x, y, y')} \right\}$$
$$+ \mu \left\{ \frac{dv}{dy} \cdot \frac{d(\phi_1, \phi_2, \phi_3)}{d(x, x', y')} - \frac{dv}{dy'} \cdot \frac{d(\phi_1, \phi_2, \phi_3)}{d(x, x', y)} \right\} = 0.$$