mathematicians, we like to tell or to hear some new thing. Philosophers may excusably hesitate, when the cry is "new lamps for old." Fortunately, we are all agreed that the old Geometry is accurate enough for all practical purposes. Lobatchewsky calculated that, at the time he wrote, in a right-angled isosceles triangle, whose sides equalled in length the distance from the earth to the sun, the sum of the angles could not differ from two right angles by more than 0″.0003 of a sexagesimal second. Euclid could not have had that satisfaction. Probably he never measured the angles of a triangle whose sides had the dimension of one foot. We may be sure he and his contemporaries could not measure a parasang without grievous error. He has, therefore, additional claims to our sympathy and admiration, since his ideal determinations bear the latest tests of science.

Note on Quartic Curves in Space. *By* WILLIAM SPOTTISWOODE, P.R.S.

[*Read Nov. 9th*, 1882.]

If we represent $\alpha x+\beta y+\gamma z+\delta t$ by $\alpha x+\ldots$, and put

$$\left.\begin{array}{ll} u=\alpha x+\ldots, & w=\alpha_2 x+\ldots \\ v=\alpha_1 x+\ldots, & s=\alpha_3 x+\ldots \end{array}\right\} \qquad \text{(1)},$$

then any quartic curves formed by the complete intersection of two quadric surfaces, viz., a quartic curve of the species (2, 2) of Cayley, may be represented by the equations

$$uy-vx=0, \quad wt-sz=0 \qquad \text{(2)};$$

or, if λ, μ be two indeterminate quantities, the equations (2) may be replaced by the following system:

$$\left.\begin{array}{ll} u=\lambda x, & w=\mu z \\ v=\lambda y, & s=\mu t \end{array}\right\} \qquad \text{(3)}.$$

From these we may eliminate x, y, z, t, and obtain the resultant

$$\left|\begin{array}{llll} \alpha-\lambda & \beta & \gamma & \delta \\ \alpha_1 & \beta_1-\lambda & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2-\mu & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3-\mu \end{array}\right| \qquad \text{(4)},$$

by means of which μ may be determined in terms of λ. This having been done, we may, from any three of the equations (3), obtain values of the ratios $x:y:z:t$ in terms (not, however, generally rational) of one parameter λ. And to every value of λ there will correspond a point, or points, real or imaginary, on the curve.

A solution of the equation (3), in a symbolical form at least, may be thus obtained. If we write,

$$(0)=\alpha,\ (1)=\beta_1,\ (2)=\gamma_2,\ (3)=\delta_3 \qquad \ldots\ldots\ldots\ldots\ldots (5),$$

$$(0,1)=\begin{matrix}\alpha, & \beta,\\ \alpha_1, & \beta_1,\end{matrix}\quad (0,2)=\begin{matrix}\alpha, & \gamma,\\ \alpha_2, & \gamma_2,\end{matrix}\ \ldots\ (2,3)=\begin{matrix}\gamma_2, & \delta_2, & \ldots\\ \gamma_3, & \delta_3,\end{matrix}$$

$$(0,1,2)=\begin{matrix}\alpha, & \beta, & \gamma, & \ldots;\\ \alpha_1, & \beta_1, & \gamma_1\\ \alpha_2, & \beta_2, & \gamma_2\end{matrix}$$

and if it be understood that the symbols 0, 1, 2, 3 are to be combined in the following way,

$$(0).(1)=(0,1),\quad (0).(1).(2)=(0,1,2) \qquad \ldots\ldots\ldots\ldots (6),$$

it will be found that (4) may be written thus

$$(\lambda-0)(\lambda-1)(\mu-2)(\mu-3)=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7);$$

and further, if we put $(\lambda-0)(\lambda-1)=\Lambda$,

the equation (7) may be developed in terms of μ in the following form

$$[\Lambda]\,\mu^2-[\Lambda(2+3)]\,\mu+[\Lambda(23)]=0 \qquad \ldots\ldots\ldots\ldots\ldots (8),$$

in which the terms within brackets may be treated as ordinary algebraical quantities. The solution of (8) will then be

$$2[\Lambda]\,\mu=[\Lambda(2+3)]\pm\sqrt{\{[\Lambda(2+3)]^2-4[\Lambda][\Lambda(2,3)]\}} \ \ldots (9).$$

The terms in this expression, when fully developed, are

$$\begin{aligned}
\Lambda(2+3) = {} & \lambda^2(2+3)\\
& -\lambda\{(0,2)+(0,3)+(1,2)+(1,3)\}\\
& +\ (0,1,2)+(0,1,3)\\
[\Lambda(2+3)]^2 = {} & \lambda^4(2+3)^2\\
& -2\lambda^3(2+3)\{(0,2)+(0,3)+(1,2)+(1,3)\}\\
& +\lambda^2\{[(0,2)+(0,3)+(1,2)+(1,3)]^2\\
& \qquad +2(2+3)[(0,1,2)+(0,1,3)]\}\\
& -2\lambda\{(0,2)+(0,3)+(1,2)+(1,3)\}\{(0,1,2)+(0,1,3)\}\\
& +\ \{(0,1,2)+(0,1,3)\}^2\\
-4[\Lambda][\Lambda(2,3)] = {} & -4\{\lambda^2-(0+1)\lambda+(0,1)\}\\
& \quad\times\{\lambda^2(2,3)-\lambda[(0,2,3)+(1,2,3)]+(0,1,2,3)\}\\
= {} & -4\lambda^4(2,3)\\
& +4\lambda^3\{(0,2,3)+(1,2,3)+(0+1)(2,3)\}\\
& -4\lambda^2\{(0,1,2,3)+(0,1)(2,3)\\
& \qquad +(0+1)[(0,2,3)+(1,2,3)]\}\\
& +4\lambda\{(0+1)(0,1,2,3)+(0,1)[(0,2,3)+(1,2,3)]\}\\
& -4\ (0,1)(0,1,2,3).
\end{aligned}$$

Hence the discriminant under the sign $\surd$ in (9)

$$\begin{aligned}
= \quad & \lambda^4 \{(2+3)^2-4(2,3)\} \\
& +2\lambda^3 \{2(0,2,3)+2(1,2,3)+2(0+1)(2,3) \\
& \qquad -(2+3)[(0,2)+(0,3)+(1,2)+(1,3)]\} \\
& + \lambda^2 \{[(0,2)+(0,3)+(1,2)+(1,3)]^2+2(2+3)[(0,1,2)+(0,1,3)] \\
& \qquad -4(0+1)[(0,2,3)+(1,2,3)]-4(0,1,2,3)\} \\
& +2\lambda \{2(0+1)(0,1,2,3)+2(0,1)[(0,2,3)+(1,2,3)] \\
& \qquad -[(0,2)+(0,3)+(1,2)+(1,3)][(0,1,2)+(0,1,3)]\} \\
& + \{[(0,1,2)+(0,1,3)]^2-4(0,1)(0,1,2,3)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).
\end{aligned}$$

If we then take for the determination of the ratios $x:y:z:t$, the first two and the fourth equations of the system (3), we shall find

$$x:y:z:t = \left| \begin{matrix} \alpha-\lambda, & \beta, & \gamma, & \delta \\ \alpha_1, & \beta_1-\lambda, & \gamma_1, & \delta_1 \\ \alpha_3, & \beta_3, & \gamma_3, & \delta_3-\mu \end{matrix} \right|,$$

and writing

$$\left| \begin{matrix} \alpha-\lambda, & \beta, & \gamma, & \delta \\ \alpha_1, & \beta_1-\lambda, & \gamma_1, & \delta_1 \\ \alpha_3, & \beta_3, & \gamma_3, & \delta_3 \end{matrix} \right| = A,\ B,\ C,\ D,$$

and

$$\left| \begin{matrix} \alpha-\lambda, & \beta, & \gamma \\ \alpha_1, & \beta_1-\lambda, & \gamma_1 \end{matrix} \right| = a,\ b,\ c,$$

we shall find

$$x:y:z:t = (A-a\mu):(B-b\mu):(C-c\mu):D \ldots\ldots\ldots (11).$$

But, according to equation (8), μ is of the form $p\pm\surd q$; hence (11) may be written also in the following form

$$x:y:z:t = (A-ap\pm a\surd q):(B-bp\pm b\surd q):(C-cp\pm c\surd q):D \ldots (12);$$

and consequently, for every value of λ, there are in general two values for each of the coordinates $x:y:z:t$; in other words, each value of λ in general determines two points on the curve. In fact, to each value of λ there correspond two values of μ, say μ_1, μ_2; and then to λ, μ_1 there corresponds one point, and to λ, μ_2 one point; two in all.

If the discriminant has a pair of equal roots, then, as Prof. Cayley suggests, its square root will be of the form

$$(\lambda-a)\surd(P\lambda^2+2Q\lambda+R),$$
$$= (\lambda-a)\surd\{(P\lambda+Q)^2-(Q^2-PR)\},$$

and, assuming

$$P\lambda+Q = \frac{\theta^2+1}{\theta^2-1}\surd(P^2-QR),$$

the foregoing expression becomes

$$(\lambda-a)\frac{(\theta^2+1)}{2\theta\surd P}\surd(P^2-QR);$$

that is, λ and μ may be expressed rationally in terms of θ, and consequently the curve is unicursal.

If the discriminant either has two pairs of equal roots, or vanishes, it is clear that μ can be expressed as the ratio of two rational quadratic functions of λ, and that the curve is unicursal; but I am not at present able to make further statement of the peculiarities of these cases.

On the Explicit Integration of certain Differential Resolvents.

By Sir JAMES COCKLE, M.A., F.R.S.

[*Read Nov. 9th*, 1882.]

1. Let y be any root of

$$y^n - xy + x = 0 \quad\ldots\ldots(1),$$

and let z be any root of $xz^{n-1} - n^3 z + n^3 = 0$ ......(2).

2. Then, if $n = 3$ or 4, we can so determine two numbers λ and μ as that

$$x^{-\lambda} y \text{ and } z^{\mu}$$

shall satisfy the same linear differential equation of the order $n-1$.

3. Put

$$xy^n - y^a + 1 = 0 \quad\ldots\ldots(3),$$

then, y being any root of (3), y^m satisfies the n-ordinal*

$$[D]^a \left[\frac{n-a}{a} D + \frac{m}{a}\right]^{n-a} y^m - \left[\frac{n}{a} D + \frac{m}{a} - 1\right]^n x^a y^m = 0 \quad\ldots(4),$$

for all values of m and for all integral values of n and a, provided that $a < n$.

4. In (3) and (4), substitute x^{-1} for x and 1 for a. Then (3) re-

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* I have elsewhere (*Trans. Royal Society, Victoria*, Vol. vii., pp. 192, 193; 1866) shown this. I do not repeat the proof here because means of verifying the result have already been printed in the *Proceedings*. And if, at pp. 216 and 218 of Mr. Harley's Addendum to Mr. Rawson's paper (*Proc. Lond. Math. Soc.*, Vol. ix., 1878), we make the substitution

$$\begin{pmatrix} y^{-1}, & n-a, & -1, & 1, & n, & 1, & -m \\ y, & r, & b, & a, & m, & c, & n \end{pmatrix},$$

wherein the lower line refers to Mr. Harley's formulæ (i.) and (x.), we get (3) and (4) of the text. The results may moreover be transformed into, and confirmed by, a generalization of a theorem of Boole's, simultaneously and independently arrived at by Mr. Harley in England, and by me in Queensland (see *Report of British Association*, Meeting of 1866, pp. 2, 3 of "Notices and Abstracts").