if we take U_2 so as to coincide with Σ. Thus we have $\Sigma \pm du = 0$; but we have $\Sigma du = 0$; hence we obtain

$$u_1 + u_2 = \sigma, \text{ a constant,}$$
$$u_3 + u_4 = \omega - \sigma \quad \ldots\ldots (17),$$

where $\omega = \Sigma u$. This is evidently true of three such relations corresponding to the three conics of the system which can be described to touch a given line.

Third Paper on Multiple Frullanian Integrals.

By E. B. Elliott.

[*Read November 8th*, 1883.]

The two previous papers, implied in the title of this one, are to be found in the volume of the Society's *Proceedings* for the Session 1876—77. Their main subject was the evaluation, when possible, of the multiple definite integral

$$\int_0^\infty \int_0^\infty \ldots \int_0^\infty \left\{ S(a_1x_1, a_2x_2, \ldots a_nx_n) - S(b_1x_1, b_2x_2, \ldots b_nx_n) \right\} \frac{dx_1 dx_2 \ldots dx_n}{x_1 x_2 \ldots x_n},$$

the function S being a symmetric one in its arguments. The results arrived at in them were rendered at once simpler and more complete in a subsequent paper by Mr. Leudesdorf, which paper, together with a method of arrival at the same conclusions obtained quite independently and given me by Mr. Alfred Lodge, has materially aided me in the following more general investigation.

The object before me now is to replace in the denominator of the expression under the signs of integration the first power of the product by any power whatever which will make the result finite, be that power positive, negative, or vanishing. Thus in any particular subclass of results, such, for instance, as the one below specially considered, in which $S(x_1, x_2, \ldots x_n) \equiv f(x_1) f(x_2) \ldots f(x_n)$, the few isolated forms of function f, for which the result with the first power is finite, may be expected to have corresponding to them a like number of isolated series of forms, each particular form giving a finite result with some other power, integral or fractional, positive or negative, in place of that first power.

The theorem as to single integrals which has to be made fundamental, the first case of a more general one which I gave in the *Messenger of Mathematics* for January last, is readily obtained as follows. Let r be such a real constant and $f(x)$ such a function of x that $\int_0^\infty f(x) \frac{dx}{x^r}$ is finite. This being so, $\frac{f'(x)}{x^{r-1}}$ must vanish both for $x = 0$ and $x = \infty$; and consequently, by the theorem known as

Frullani's, $$\int_0^\infty \left\{ \frac{f'(x)}{x^{r-1}} - \frac{f'(ax)}{(ax)^{r-1}} \right\} \frac{dx}{x} = 0,$$

i.e., $$\int_0^\infty f'(ax) \frac{dx}{x^r} = a^{r-1} \int_0^\infty f'(x) \frac{dx}{x^r}.*$$

Hence, integrating between limits with regard to a,

$$\int_0^\infty \{f(ax) - f(bx)\} \frac{dx}{x^{r+1}} = \frac{a^r - b^r}{r} \int_0^\infty f'(x) \frac{dx}{x^r} \quad \ldots\ldots\ldots\ldots(1),$$

the theorem desired. It must never be forgotten that, in proving it, the integral on its right-hand side has been assumed not infinite. Were it so, the equality could have little meaning, and would be absolutely untrue, unless the difference of limits $\left[\frac{f'(x)}{x^{r-1}}\right]_0^\infty$ vanished.

Before proceeding from this to the general theorem, it seems well first to investigate that sub-class of results which is of greatest simplicity and apparent utility, especially as the application to it of (1) is more entirely free from anything unintelligible or unsatisfactory than may possibly seem to be the case with the use of that equality in the derivation of the more comprehensive result; viz.,

I. *The sub-class*

$$\int_0^\infty \int_0^\infty \ldots \int_0^\infty \left\{ \prod_{s=1}^{s=n} f(a_s x_s) - \prod_{s=1}^{s=n} f(b_s x_s) \right\} \frac{dx_1\, dx_2 \ldots dx_n}{(x_1 x_2 \ldots x_n)^{r+1}}.$$

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* The remark seems worth making, that in general it is or is not lawful to apply the transformation $x = ay$ to the integral $\int_0^\infty F(x)\,dx$ or $\int_{-\infty}^0 F(x)\,dx$ according as, when x becomes in turn positive infinity and zero, or zero and negative infinity, as the case may be, $xF(x)$ does or does not approach limits whose difference is zero. In particular, it is always lawful when the integral is of finite value, since both limiting values in that case vanish.

In like manner, $x = y + a$ is a lawful substitution to apply to $\int_{-\infty}^\infty F(x)\,dx$, when, and only when, the difference of limits $F(\infty) - F(-\infty)$ vanishes, as is always the case when the value of the integral is finite, since then each limit is equal to zero.

And, once more, according as the difference of limits

$$[x \log x F(x)]_0^1 \quad \text{or} \quad [x \log x F(x)]_1^\infty$$

vanishes, as will in particular be the case when the value of the integral is finite, or not, so is it allowable or not to transform $\int_0^1 F(x)\,dx$, or $\int_1^\infty F(x)\,dx$, as the case may be, by a substitution like $x = y^a$.

These and other like results are immediate from inspection of Frullani's theorem in the forms

$$\int_0^\infty \{\phi(ax) - \phi(x)\} \frac{dx}{x} = \{\phi(\infty) - \phi(0)\} \log a,$$

$$\int_{-\infty}^\infty \{\phi(x+a) - \phi(x)\}\,dx = \{\phi(\infty) - \phi(-\infty)\}\,a,$$

$$\int_0^1 \{\phi(x^a) - \phi(x)\} \frac{dx}{x \log x} = \{\phi(1) - \phi(0)\} \log a,$$

and others obtained by simple transformations.

The functions, and constants r, dealt with are exactly those for which result (1) has been arrived at.

The double integral of the class,

$$\int_0^\infty \int_0^\infty \{f(a_1x_1)\,f(a_2x_2)-f(b_1x_1)\,f(b_2x_2)\}\,\frac{dx_1\,dx_2}{(x_1x_2)^{r+1}},$$

may be written as the sum of two double integrals, namely,

$$\int_0^\infty f(a_1x_1)\,\frac{dx_1}{x_1^{r+1}}\int_0^\infty \{f(a_2x_2)-f(b_2x_2)\}\,\frac{dx_2}{x_2^{r+1}}$$

$$+\int_0^\infty f(b_2x_2)\,\frac{dx_2}{x_2^{r+1}}\int_0^\infty \{f(a_1x_1)-f(b_1x_1)\}\,\frac{dx_1}{x_1^{r+1}};$$

and this sum is, by (1), equal to

$$\left\{\frac{a_2^r-b_2^r}{r}\int_0^\infty f(a_1x_1)\,\frac{dx_1}{x_1^{r+1}}+\frac{a_1^r-b_1^r}{r}\int_0^\infty f(b_2x_2)\,\frac{dx_2}{x_2^{r+1}}\right\}\int_0^\infty f'(x)\,\frac{dx}{x^r}.$$

If then we have satisfied the one condition

$$a_2^r-b_2^r+a_1^r-b_1^r=0,$$

i.e., $$a_1^r+a^r=b_1^r+b_2^r \qquad\qquad (2),$$

the double integral is equal to

$$-\frac{a_1^r-b_1^r}{r}\int_0^\infty \{f(a_1x)-f(b_2x)\}\,\frac{dx}{x^{r+1}}\cdot\int_0^\infty f'(x)\,\frac{dx}{x^r},$$

which, by (1), $$=-\frac{(a_1^r-b_1^r)(a_1^r-b_2^r)}{r^2}\left\{\int_0^\infty f'(x)\,\frac{dx}{x^r}\right\}^2 \qquad\qquad (3).$$

Now the symmetrical forms of the double integral and of the condition (2) tell us that there are three other forms in which the coefficient of the integral factor in this result might with equal accuracy have been obtained, viz.,

$$-\frac{1}{r^2}(a_2^r-b_1^r)(a_2^r-b_2^r),\qquad +\frac{1}{r^2}(a_1^r-b_1^r)(a_2^r-b_1^r),$$

and $$+\frac{1}{r^2}(a_1^r-b_2^r)(a_2^r-b_2^r).$$

It can, in fact, be immediately seen that the condition (2) is merely the one, and the only one, necessitated by the equality of these four expressions. Calling, in fact, either of them $\frac{1}{r^2}k_2$, their equality may be expressed by saying that, if $\phi(z)=0$ be the quadratic of which b_1^r, b_2^r are the roots, then a_1^r, a_2^r are the roots of $\phi(z)=-k_2$, or, in other words, that for all values of z

$$(z-a_1^r)(z-a_2^r)\equiv(z-b_1^r)(z-b_2^r)+k_2.$$

Hence, equating coefficients, we have

$$a_1^r + a_2^r = b_1^r + b_2^r,$$

the sole condition, and $\quad a_1^r a_2^r = b_1^r b_2^r + k_2,$

which determines the most symmetrical value for k_2 when it is satisfied. Thus we have, under the condition (2),

$$\int_0^\infty \int_0^\infty \{f(a_1x_1)\,f(a_2x_2) - f(b_1x_1)\,f(b_2x_2)\} \frac{dx_1\,dx_2}{(x_1x_2)^{r+1}} = \frac{A^2}{r^2}\,(a_1^r a_2^r - b_1^r b_2^r) \quad \text{..................(4)},$$

where A denotes $\int_0^\infty f'(x)\,\frac{dx}{x^r}$.*

A theorem has then been established for the cases $n=1$ and $n=2$, which may now be stated in general, and proved to hold for all cases by the method of mathematical induction; viz.:

If $f(x)$ be such a function and r such a constant that $\int_0^\infty f'(x)\,\frac{dx}{x^r}$ is of finite or vanishing value A, and if the constants $a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n$ be connected, and k_n determined in terms of them by the n conditions which express the identity

$$(z-a_1^r)(z-a_2^r)\ldots(z-a_n^r) \equiv (z-b_1^r)(z-b_2^r)\ldots(z-b_n^r) + (-1)^n k_n \ldots (5);$$

in other words, if $a_1^r, a_2^r, \ldots a_n^r$ and $b_1^r, b_2^r, \ldots b_n^r$ be the roots of two equations of the n^{th} degree which differ only in their constant terms, and if k_n be the difference with proper sign of these constant terms, then

$$\int_0^\infty \int_0^\infty \ldots \int_0^\infty \{f(a_1x_1)\,f(a_2x_2)\ldots f(a_nx_n) - f(b_1x_1)\,f(b_2x_2)\ldots f(b_nx_n)\}$$
$$\times \frac{dx_1\,dx_2\ldots dx_n}{(x_1x_2\ldots x_n)^{r+1}} = \frac{A^n}{r^n}\,k_n \quad \text{....................(6)}.$$

---

* The reasoning above is inapplicable to the limiting case when $r=0$. Indeed, in that case, the equation (2) is nothing more than an obvious identity, and must, it is clear, be replaced by the one obtained by taking the limit when r vanishes of

$$\frac{a_1^r - b_1^r}{r} + \frac{a_2^r - b_2^r}{r} = 0, \quad \textit{i.e.}, \text{ by } \log a_1 + \log a_2 = \log b_1 + \log b_2.$$

Similar independent processes would have to be gone through at all subsequent steps. For this reason results (4), (6), (12), (13), &c., though correct for all non-vanishing values of r, do not stand the test of proceeding to the limit.

The necessary modifications of the work for the limiting case are, however, with ease developed. The results are already known (see Mr. Leudesdorf's paper), and are, that the coefficient of A^n in (6), or of the multiple integral on the right in (12), has to be replaced by

$$\log a_1 \log a_2 \ldots \log a_n - \log b_1 \log b_2 \ldots \log b_n,$$

which call k_n', the connecting conditions being those which express the identity of

$$(z-\log a_1)\,(z-\log a_2)\ldots(z-\log a_n)$$

with $\quad (z-\log b_1)\,(z-\log b_2)\ldots(z-\log b_n) + (-1)^n k_n'.$

The method to be followed is the usual one of assuming this for the value n and deducing it for $n+1$. Having succeeded in this, since it has been shown true for $n=2$, it will have been proved universally.

Take then the $(n+1)$-tuple integral

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty \{f(a_1x_1)f(a_2x_2)\dots f(a_{n+1}x_{n+1})-f(b_1x_1)f(b_2x_2)\dots f(b_{n+1}x_{n+1})\} \times \frac{dx_1\,dx_2 \dots dx_{n+1}}{(x_1x_2\dots x_{n+1})^{r+1}}.$$

Introducing $n-1$ constants, at present arbitrary, $c_2, \dots c_n$, we write it

$$\int_0^\infty f(a_1x_1)\frac{dx_1}{x_1^{r+1}}\int_0^\infty\int_0^\infty\dots\int_0^\infty \{f(a_2x_2)\dots f(a_nx_n)f(a_{n+1}x_{n+1}) - f(c_2x_2)\dots f(c_nx_n)f(b_{n+1}x_{n+1})\} \times \frac{dx_2\dots dx_n\,dx_{n+1}}{(x_2\dots x_nx_{n+1})^{r+1}}$$

$$+\int_0^\infty f(b_{n+1}x_{n+1})\frac{dx_{n+1}}{x_{n+1}^{r+1}}\int_0^\infty\int_0^\infty\dots\int_0^\infty \{f(a_1x_1)f(c_2x_2)\dots f(c_nx_n) - f(b_1x_1)f(b_2x_2)\dots f(b_nx_n)\} \times \frac{dx_1\,dx_2\dots dx_n}{(x_1x_2\dots x_n)^{r+1}},$$

which, on the supposition that the theorem holds for the value n,

$$=\frac{A^n}{r^n}\left\{k_n\int_0^\infty f(a_1x)\frac{dx}{x^{r+1}}+k_n'\int_0^\infty f(b_{n+1}x)\frac{dx}{x^{r+1}}\right\},$$

provided the $2n$ conditions be satisfied which express the identities

$$(z-a_2^r)\dots(z-a_n^r)(z-a_{n+1}^r) \equiv (z-c_2^r)\dots(z-c_n^r)(z-b_{n+1}^r)+(-1)^n k_n \quad \dots\dots\dots\dots\dots(7),$$

$$(z-a_1)^r(z-c_2^r)\dots(z-c_n^r) \equiv (z-b_1^r)(z-b_2^r)\dots(z-b_n^r)+(-1)^n k_n' \dots(8),$$

and which becomes, if the one additional relation

$$k_n+k_n'=0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(9),$$

holds,
$$\frac{A^n}{r^n}k_n\int_0^\infty \{f(a_1x)-f(b_{n+1}x)\}\frac{dx}{x^{r+1}},$$

i.e., by (1),
$$\frac{A^{n+1}}{r^{n+1}}k_n(a_1^r-b_{n+1}^r) \quad \dots\dots\dots\dots\dots\dots\dots(10).$$

On the whole, the conditions (7, 8, 9) under which this has been arrived at are in number $2n+1$ in the $3n+3$ constants $a_1, \dots a_{n+1}$, $b_1, \dots b_{n+1}$, $c_2, \dots c_n$, k_n, k_n'. The result of removing n of these, k_n' and the c's, as we wish to do, must be to leave $n+1$ conditions in the remaining $2n+3$. Now, these $n+1$ are easily exhibited by multiplying identities (7) and (8) by $z-a_1^r$ and $z-b_{n+1}^r$, respectively, and adding, so as

to get $(z-a_1^r)(z-a_2^r)\ldots(z-a_{n+1}^r) \equiv (z-b_1^r)(z-b_2^r)\ldots(z-b_{n+1}^r)$
$$+(-1)^n\{k_n(z-a_1^r)+k_n'(z-b_{n+1}^r)\}$$
$$\equiv (z-b_1^r)(z-b_2^r)\ldots(z-b_{n+1}^r)+(-1)^{n+1}k_n(a_1^r-b_{n+1}^r),$$
by the use of (9).

The form of result (10) for the $n+1$-tuple integral, and the $n+1$ conditions expressing this identity found as necessary for its accuracy on the supposition of the soundness of the theorem for n-tuple integrals, are then exactly the form and the $n+1$ conditions which suffice as above to express the theorem for the case of $n+1$. n of the conditions are relations proper in the a's and b's, the $(n+1)^{\text{th}}$ determining the $k_n(a_1^r-b_{n+1}^r)$, or, to use our earlier notation, the k_{n+1}, in terms of them, viz., $k_{n+1}=a_1^ra_2^r\ldots a_{n+1}^r-b_1^rb_2^r\ldots b_{n+1}^r$.

The theorem is then, by the usual reasoning of mathematical induction, completely established.

The following is the transformation of the theorem first proved to which we are led by the substitution $x=e^y$, $a=e^a$, &c. The independent development of it is possibly even less cumbrous than that gone through above. *If the integral* $\int_{-\infty}^{\infty} e^{-ry}\phi'(y)\,dy$, *where r is a real constant, have a finite value A, and, if the conditions be satisfied which express the identity*
$$(z-e^{ra_1})(z-e^{ra_2})\ldots(z-e^{ra_n}) \equiv (z-e^{r\beta_1})(z-e^{r\beta_2})\ldots(z-e^{r\beta_n})+(-1)^n k_n',$$
then $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} e^{-r(y_1+y_2+\ldots+y_n)}\left\{\prod_{s=1}^{s=n}\phi(y_s+a_s)-\prod_{s=1}^{s=n}\phi(y_s+\beta_s)\right\}$$
$$\times\, dy_1dy_2\ldots dy_n$$
$$=\frac{A^n}{r^n}k_n'=\frac{A^n}{r^n}\{e^{r(a_1+a_2+\ldots+a_n)}-e^{r(\beta_1+\beta_2+\ldots+\beta_n)}\}\ldots(11).$$

From the sub-class we now proceed to

II. *The General Integral*
$$\int_0^{\infty}\int_0^{\infty}\ldots\int_0^{\infty}\{S(a_1x_1, a_2x_2, \ldots a_nx_n)-S(b_1x_1, b_2x_2, \ldots b_nx_n)\}\frac{dx_1dx_2\ldots dx_n}{(x_1x_2\ldots x_n)^{r+1}},$$
where S is any suitable symmetric function.

The double integral
$$\int_0^{\infty}\int_0^{\infty}\{S(a_1x_1, a_2x_2)-S(b_1x_1, b_2x_2)\}\frac{dx_1dx_2}{(x_1x_2)^{r+1}}$$
$$=\int_0^{\infty}\int_0^{\infty}\{S(a_1x_1, a_2x_2)-S(a_1x_1, b_2x_2)+S(a_1x_1, b_2x_2)-S(b_1x_1, b_2x_2)\}$$
$$\times\frac{dx_1dx_2}{(x_1x_2)^{r+1}};$$

and this, by two uses of (1), the applicability of which would seem to require that $S(x, c)$ be, for all values of c, a function of x such as the $f(x)$ of that equality,

$$= \frac{a_2^r - b_2^r}{r} \int_0^\infty \int_0^\infty \frac{d}{dx_2} S(a_1 x_1, x_2) \frac{dx_1 dx_2}{x_1^{r+1} x_2^r} + \frac{a_1^r - b_1^r}{r} \int_0^\infty \int_0^\infty \frac{d}{dx_1} S(x_1, b_2 x_2) \frac{dx_1 dx_2}{x_1^r x_2^{r+1}}$$

$$= -\frac{a_1^r - b_1^r}{r} \int_0^\infty \int_0^\infty \frac{d}{dx_2} \{S(a_1 x_1, x_2) - S(b_2 x_1, x_2)\} \frac{dx_1 dx_2}{x_1^{r+1} x_2^r},$$

provided the condition $a_1^r + a_2^r = b_1^r + b_2^r$ be satisfied,

$$= -\frac{(a_1^r - b_2^r)(a_1^r - b_1^r)}{r^2} \int_0^\infty \int_0^\infty \frac{d^2}{dx_1 dx_2} S(x_1, x_2) \frac{dx_1 dx_2}{(x_1 x_2)^r},$$

by a further application of (1) which, since the operation $\frac{d}{dx_2}$ may be considered subsequent to the integration with respect to x_1, appears to necessitate no further limitation of the forms of function S. The work of modifying the factor multiplying the integral in this result proceeds exactly as the corresponding reduction which produced (4), and need not be repeated.

The general theorem to which we are led may now be stated: viz., *If $S(x_1, x_2, \dots x_n)$ be a symmetric function of its n arguments, restricted in form by a condition, which would seem to be that for all values of $n-1$ of these arguments, it is such a function S of the n^{th} argument x that for a certain value of r $\int_0^\infty \frac{dS}{dx} \cdot \frac{dx}{x^r}$ is finite, then, r having that value,*

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty \{S(a_1 x_1, a_2 x_2, \dots a_n x_n) - S(b_1 x_1, b_2 x_2, \dots b_n x_n)\} \frac{dx_1 dx_2 \dots dx_n}{(x_1 x_2 \dots x_n)^{r+1}}$$

$$= \frac{k_n}{r^n} \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{d^n}{dx_1 dx_2 \dots dx_n} S(x_1, x_2, \dots x_n) \frac{dx_1 dx_2 \dots dx_n}{(x_1 x_2 \dots x_n)^r} \dots (12),$$

if $a_1, a_2, \dots a_n$, $b_1, b_2, \dots b_n$ be constants connected, and k_n one determined in terms of them, by the n conditions which express the identity

$$(z - a_1^r)(z - a_2^r) \dots (z - a_n^r) \equiv (z - b_1^r)(z - b_2^r) \dots (z - b_n^r) + (-1)^n k_n \dots (5).$$

The derivation of this from the case of $n = 2$ by mathematical induction proceeds exactly as in the case of the sub-theorem already demonstrated, as far as conditions are concerned. It will be well, however, to exhibit in outline the deduction of the integral form for case $n+1$ from that for case n. The method is, as before, to introduce $n-1$ new constants $c_2, c_3, \dots c_n$. We thus have

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty \{S(a_1 x_1, \dots a_n x_n, a_{n+1} x_{n+1}) - S(b_1 x_1, \dots b_n x_n, b_{n+1} x_{n+1})\} \times \frac{dx_1 dx_2 \dots dx_n dx_{n+1}}{(x_1 \dots x_n x_{n+1})^{r+1}}$$

$$= \int_0^\infty \int_0^\infty \dots \int_0^\infty \{S(a_1x_1, a_2x_2, \dots a_nx_n, a_{n+1}x_{n+1}) - S(a_1x_1, c_2x_2, \dots c_nx_n, b_{n+1}x_{n+1})\} \times \frac{dx_1 dx_2 \dots dx_{n+1}}{(x_1x_2 \dots x_{n+1})^{r+1}}$$

$$+ \int_0^\infty \int_0^\infty \dots \int_0^\infty \{S(a_1x_1, c_2x_2, \dots c_nx_n, b_{n+1}x_{n+1}) - S(b_1x_1, b_2x_2, \dots b_nx_n, b_{n+1}x_{n+1})\} \times \frac{dx_1 dx_2 \dots dx_{n+1}}{(x_1x_2 \dots x_{n+1})^{r+1}}$$

$$= \frac{k_n}{r^n} \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{d^n}{dx_2 \dots dx_{n+1}} S(a_1x_1, x_2, \dots x_{n+1}) \frac{dx_1 dx_2 \dots dx_{n+1}}{x_1^{r+1}(x_2 \dots x_{n+1})^r}$$

$$+ \frac{k'_n}{r^n} \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{d^n}{dx_1 \dots dx_n} S(x_1, \dots x_n, b_{n+1}x_{n+1}) \frac{dx_1 dx_2 \dots dx_{n+1}}{(x_1 \dots x_n)^r x_{n+1}^{r+1}},$$

by two applications of the theorem for the case n under the proper conditions; and, further, by cyclical interchange of the suffixes of the variables, and an application of (1) under the last condition $k_n + k'_n = 0$,

$$= k_n (a_1^r - b_{n+1}^r) \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{d^n}{dx_1 dx_2 \dots dx_{n+1}} S(x_1, x_2, \dots x_{n+1}) \frac{dx_1 dx_2 \dots dx_{n+1}}{(x_1x_2 \dots x_{n+1})^r},$$

the limitation of $S(x_1, x_2, \dots x_{n+1})$ implied being, it would seem, that stated above, since the operation $\dfrac{d^n}{dx_2 \dots dx_{n+1}}$ may be considered subsequent to the integration with regard to x_1. The remainder of the reasoning is exactly that by which (6) was established.

The companion theorem, derived from the form of (1) which gave us (11), may be stated :—*If* $S(y_1, y_2, \dots y_n)$ *be such a symmetric function of its n arguments as to be, whatever be any* $n-1$ *of them, a function of the* n^{th} *of the nature of the* $\phi(y)$ *introduced in* (11), *then, under conditions which express the identity*

$$(z-e^{r\alpha_1})(z-e^{r\alpha_2}) \dots (z-e^{r\alpha_n}) \equiv (z-e^{r\beta_1})(z-e^{r\beta_2}) \dots (z-e^{r\beta_n}) + (-1)^n k'_n,$$

the equality holds—

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \dots \int_{-\infty}^\infty e^{-r(y_1+y_2+\dots+y_n)} \{S(y_1+\alpha_1, y_2+\alpha_2, \dots y_n+\alpha_n) - S(y_1+\beta_1, y_2+\beta_2, \dots y_n+\beta_n)\} \times dy_1 dy_2 \dots dy_n$$

$$= \frac{k'_n}{r^n} \int_{-\infty}^\infty \int_{-\infty}^\infty \dots \int_{-\infty}^\infty e^{-r(y_1+y_2+\dots+y_n)} \frac{d^n}{dy_1 dy_2 \dots dy_n} S(y_1, y_2, \dots y_n) \times dy_1 dy_2 \dots dy_n \quad \dots\dots\dots (13).$$

The apparent stringency of the limitations to the forms of symmetric function in these two companion forms of result is great, and may probably be modified. It will indeed be strange if in each case what is necessary for the result arrived at to be true is not in reality exactly what is required for the integral on the right-hand side to be intelligible and finite. It may be remarked that the general results give correctly the first obtained ones (5) and (11), and also, with the proper modification of the factor k_n or k'_n according to the note on page 15, the special ones for the case $r=0$ treated of in my own and Mr. Leudesdorf's former papers.

A new point of view of the whole subject of Frullanian integrals, which may possibly indicate its real extensiveness, is suggested by the forms of the general results (12) and (13). Taking the former, and applying to it the successive differentiation $\dfrac{d^n}{da_1 da_2 \dots da_n}$, we arrive at a result which expresses that the integral on the right is one to which it is lawful to apply the transformation which replaces $x_1, x_2, \dots x_n$ by a_1x_1, a_2x_2, a_nx_n. From this starting-point the results obtained might possibly have been arrived at with greater elegance than has here been done. The all-including result would be obtained by starting from the most general equality of transformation which could be proved of the form

$$\int_0^\infty\int_0^\infty \dots \int_0^\infty \frac{d^n}{da_1 da_2 \dots da_n} F(a_1x_1, a_2x_2, \dots a_nx_n) \frac{dx_1 dx_2 \dots dx_n}{\phi(x_1, x_2, \dots x_n)}$$
$$= \psi(a_1, a_2, \dots a_n) \int_0^\infty\int_0^\infty \dots \int_0^\infty \frac{d^n}{dx_1 dx_2 \dots dx_n} F(x_1, x_2, \dots x_n) \frac{dx_1 dx_2 \dots dx_n}{\chi(x_1, x_2, \dots x_n)},$$

and proceeding by integration with regard to the constants.

On Symmetric Functions, and in particular on certain Inverse Operators in connection therewith. By Captain P. A. MacMahon, R.A.

[*Read Nov. 8th*, 1883.]

1. The present paper is more especially concerned with the non-unitary symmetric functions (that is, with those whose partitions contain no unit), their calculation and the development of their properties; for the reason that it has been recently shown (*vide* the author's paper in Vol. vi., No. 2, *American Journal of Mathematics*) that they are, in fact, seminvariants of an allied quantic, and the whole series contains potentially the complete solution of the syzygies which exist between the sources of covariants of binary quantics.