On the Functions associated with the Parabolic Cylinder in Harmonic Analysis. By E. T. WHITTAKER.

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1. *Introduction.*

The following paper is a study of the functions defined by the differential equation

$$\frac{d^2y}{dz^2} + Zy = 0,$$

where Z is a quadratic function of z. To these functions the name *functions of the parabolic cylinder* may be given, as it was shown by Weber (*Math Ann.*, Vol. I.) that the harmonic functions applicable to the parabolic cylinder satisfy this equation; some power series which represent these harmonic functions and satisfy the differential equation have been found by Baer (*Diss.*, *Cüstrin*) and Haentzschel (*Zeitschrift für Math.*, Vol. XXXIII.)

The differential equation, however, has an importance quite apart from its occurrence in harmonic analysis; for, owing to its simplicity, it naturally is presented to us as the first object of study in the general scheme of defining new transcendental functions by means of differential equations; and it will be evident from the results found in the present paper that many definite integrals, which have been suggested at various times by different writers as suitable definitions for new transcendental functions, are really special cases of definite integrals which satisfy the differential equation of the parabolic cylinder.

2. *Description of Paper.*

In the present paper the differential equation associated with the parabolic cylinder is first (§ 3) solved by means of a family of definite integrals. It is then (§ 4) shown that one of the integrals of this family possesses properties which fit it to be taken as a standard solution, and this integral is denoted by $D_n(z)$. It is shown that $D_n(z)$ possesses an asymptotic expansion which is well suited for the purpose of calculating its numerical values, and that in certain cases it degenerates into an elementary function. It is then (§ 5) shown that the general solution of the equation does not introduce any function other than $D_n(z)$, and a second family of definite integrals is found which satisfies the differential equation.

It is then shown (§ 6) that the function $D_n(z)$ possesses two sets of recurrence formulæ, and that in the case in which n is an integer (§ 7) this function can be expressed in a simple form, and (§ 8) that it possesses integral properties and can be used to form expansions of arbitrary functions. A third family of definite integral solutions of the differential equation is found in § 9, and a fourth family in § 10, and the members of these families which represent $D_n(z)$ are determined.

3. *Determination of a Family of Definite Integrals which satisfy the Differential Equation.*

As has already been said, the differential equation found by Weber for the functions of the parabolic cylinder is

$$\frac{d^2y}{dz^2}+Zy=0,$$

where Z is a quadratic function of z. There is clearly no real loss of generality if we write this in the form

$$\frac{d^2y}{dz^2}+(n+\tfrac{1}{2}-\tfrac{1}{4}z^2)\,y=0,$$

where n is a constant. We shall accordingly adopt this as our standard form of the equation.* It is evident that two independent solutions of this equation exist of the forms

$$1-\frac{2n+1}{4}z^2+\frac{4n^2+4n+3}{96}z^4+\dots$$

and

$$z-\frac{2n+1}{12}z^3+\frac{4n^2+4n+7}{480}z^5+\dots$$

respectively. These series are convergent over the entire z-plane, and may be taken as the two fundamental solutions. As, however, series-solutions of this kind have been discussed by the two previous writers, and, as we shall show that it is advantageous to choose the two fundamental solutions otherwise, these series need not further be mentioned.

In order to obtain a pair of fundamental solutions suitable to be taken as standards, we shall first show that a family of definite

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* This differs from the standard forms previously adopted ; the reason for it will appear subsequently.

integrals exists, by means of which the solutions can be represented; for in the differential equation

$$\frac{d^2y}{dz^2} + (n+\tfrac{1}{2}-\tfrac{1}{4}z^2)\,y = 0$$

let a new independent variable be introduced by the relation

$$y = e^{-\frac{1}{4}z^2} z^n u.$$

The equation becomes

$$z^2\frac{d^2u}{dz^2} + (2nz-z^3)\frac{du}{dz} + n(n-1)\,u = 0.$$

Now, if v denote the integral

$$\int_\gamma e^{-t-\frac{1}{2}(t^2/z^2)}\,t^{-n-1}\,dt,$$

where the path of integration γ is for the present unspecified, we find that

$$z^2\frac{d^2v}{dz^2} + (2nz-z^3)\frac{dv}{dz} + n(n-1)\,v$$
$$= -\frac{1}{z^2}\int_\gamma \frac{d}{dt}\left[\left\{t^2-tz^2+(n-1)z^2\right\} t^{-n} e^{-t-\frac{1}{2}(t^2/z^2)}\right] dt.$$

It follows from this that *any integral of the type*

$$e^{-\frac{1}{4}z^2} z^n \int_\gamma e^{-t-\frac{1}{2}(t^2/z^2)}\,t^{-n-1}\,dt$$

is a solution of the differential equation of the parabolic cylinder, provided the path of integration γ is such that the quantity

$$\left\{t^2-tz^2-(n-1)z^2\right\} t^{-n} e^{-t-\frac{1}{2}(t^2/z^2)}$$

resumes its initial value after describing the path.

This result furnishes a large number of definite integrals which satisfy the equation; thus, the quantity

$$\left\{t^2-tz^2-(n-1)z^2\right\} t^{-n} e^{-t-\frac{1}{2}(t^2/z^2)}$$

is zero at the points

$$t = \tfrac{1}{2}z^2+\tfrac{1}{2}z\left\{z^2-4(n-1)\right\}^{\frac{1}{2}},$$
$$t = \tfrac{1}{2}z^2-\tfrac{1}{2}z\left\{z^2-4(n-1)\right\}^{\frac{1}{2}},$$
$$t = 0 \quad (\text{when } n \text{ is negative}),$$
$$t = \infty \quad \left(\text{provided the direction at infinity is such that } \frac{t}{z} \text{ is real}\right);$$

and therefore, if γ be a path joining any of these points, the expression

$$e^{-\frac{1}{4}z^2}z^n\int_\gamma e^{-t-\frac{1}{2}(t^2/z^2)}t^{-n-1}\,dt$$

will be a solution of the differential equation.

4. *Asymptotic Expansion of one of the Integrals found in* § 3. *Determination of a Standard Solution.*

Among these solutions there is one which (as will appear later) is especially suited to be taken as a standard solution. This is the integral taken along a path which encircles the origin in the t-plane and begins and ends at infinity in that direction which makes $\frac{t}{z}$ a positive real quantity. This path will be denoted by δ. We shall now show that this solution

$$e^{-\frac{1}{4}z^2}z^n\int_\delta e^{-t-\frac{1}{2}(t^2/z^2)}t^{-n-1}\,dt$$

possesses an asymptotic expansion when z is real and positive.

For let z be any real positive quantity greater than some fixed quantity k^2. Then the path of integration can be divided into two parts, namely, (1) a part δ_1 consisting of the portion of δ which is contained within a circle of radius k and centre the origin, and (2) a part δ_2 consisting of the two portions of δ which are outside this circle. Thus

$$e^{-\frac{1}{4}z^2}z^n\int_\delta e^{-t-\frac{1}{2}(t^2/z^2)}t^{-n-1}\,dt$$

$$= e^{-\frac{1}{4}z^2}z^n\int_\delta e^{-t}t^{-n-1}\left\{1-\frac{t^2}{2z^2}+\frac{1}{2!}\left(\frac{t^2}{2z^2}\right)^2-\ldots+\frac{1}{(r-1)!}\left(\frac{-t^2}{2z^2}\right)^{r-1}\right\}dt$$

$$+e^{-\frac{1}{4}z^2}z^n\int_{\delta_1} e^{-t}\left\{e^{-\frac{1}{2}(t^2/z^2)}-1+\frac{t^2}{2z^2}-\ldots-\frac{1}{(r-1)!}\left(\frac{-t^2}{2z^2}\right)^{r-1}\right\}t^{-n-1}\,dt$$

$$+e^{-\frac{1}{4}z^2}z^n\int_{\delta_2} e^{-t}\left\{e^{-\frac{1}{2}(t^2/z^2)}-1+\frac{t^2}{2z^2}-\ldots-\frac{1}{(r-1)!}\left(\frac{-t^2}{2z^2}\right)^{r-1}\right\}t^{-n-1}\,dt.$$

The first of these integrals can be evaluated by help of the relation

$$\frac{1}{\Gamma(k+1)}=\frac{i}{2\pi}\int_\delta e^{-t}(-t)^{-k-1}\,dt,$$

and thus furnishes the terminating series

$$\frac{2\pi(-1)^{n+1}}{i\Gamma(n+1)}e^{-\frac{1}{4}z^2}z^n\left\{1-\frac{n(n-1)}{2z^2}+\dots\right.$$
$$\left.\dots+\frac{(-1)^{r-1}n(n-1)\dots(n-r+1)}{(r-1)!}\frac{1}{(2z^2)^{r-1}}\right\}.$$

Considering next the second integral, we observe that at all points of its path of integration δ_1 the quantity

$$e^{-\frac{1}{2}(t^2/z^2)}-1+\frac{t^2}{2z^2}-\dots-\frac{1}{(r-1)!}\left(\frac{-t^2}{2z^2}\right)^{r-1}$$

can be expanded as an absolutely convergent series in the form

$$\frac{1}{r!}\left(\frac{-t^2}{2z^2}\right)^r+\frac{1}{(r+1)!}\left(\frac{-t^2}{2z^2}\right)^{r+1}+\dots,$$

from which it follows that, when z is large (r being supposed predetermined independently of z), the second integral can be written

$$e^{-\frac{1}{4}z^2}.z^n.\frac{A}{z^{2r}},$$

where A remains finite when z is indefinitely increased.

Considering, lastly, the third integral, we observe that $|t|$ is greater than k at all points in the range of integration, and therefore (r being supposed predetermined independently of z) the occurrence of the factor e^{-t} in the integrand shows that the integral will at most be comparable with e^{-k} when k is large, *i.e.*, it will for large values of z be small compared with the second integral.

Summing up, we see that, if r be a predetermined integer, then the solution

$$e^{-\frac{1}{4}z^2}\int_\delta e^{-t-\frac{1}{2}(t^2/z^2)}t^{-n-1}\,dt$$

can for large positive values of z be expressed in the form

$$\frac{2\pi(-1)^{n+1}}{i\Gamma(n+1)}e^{-\frac{1}{4}z^2}z^n\left\{1-\frac{n(n-1)}{2z^2}+\dots\right.$$
$$\left.\dots+\frac{(-1)^{r-1}n(n-1)\dots(n-r+1)}{(r-1)!}\frac{1}{(2z^2)^{r-1}}+\frac{A}{z^{2r}}\right\},$$

where A remains finite when z increases indefinitely; that is to say, *the divergent series*

$$\frac{2\pi(-1)^{n+1}}{i\Gamma(n+1)}e^{-\frac{1}{4}z^2}z^n\left\{1-\frac{n(n-1)}{2z^2}+\dots\right.$$
$$\left.+\frac{(-1)^r n(n-1)\dots(n-r)}{2^r r!}\frac{1}{z^{2r}}+\dots\right\}$$

is an asymptotic expansion of this solution for large real positive values of z.

We can now define our standard solution in the following way :—

The integral $$\frac{i\Gamma(n+1)}{2\pi}e^{-\frac{1}{4}z^2}z^n\int_\delta e^{-t-\frac{1}{2}(t^2/z^2)}(-t)^{-n-1}dt$$

will be denoted by $$D_n(z),$$

and will be taken as a standard solution of the differential equation of the parabolic cylinder.

The function $D_n(z)$ is a one-valued analytic function, regular for all finite values of z; and, for large real positive values of z, it possesses the asymptotic expansion

$$D_n(z) = e^{-\frac{1}{4}z^2}z^n\left\{1-\frac{n(n-1)}{2z^2}+\frac{n(n-1)(n-2)(n-3)}{2.4.z^4}-\dots\right\}.$$

It is obvious that when n is a positive integer this last series terminates; in this case the function $D_n(z)$ therefore degenerates into the product of an exponential function and a polynomial. This case will be deferred for the present; but it may be observed that this circumstance is a justification for the choice we have made of a standard solution.

When the real part of n is negative, by deforming the path of integration, we find that *the function $D_n(z)$ can be expressed by the real integral*

$$D_n(z) = \frac{1}{\Gamma(-n)}e^{-\frac{1}{4}z^2}z^n\int_0^\infty e^{-t-\frac{1}{2}(t^2/z^2)}t^{-n-1}dt.$$

5. *Expression of the Second Solution of the Equation in terms of $D_n(z)$. A Second Family of Definite Integrals which satisfy the Equation.*

Since the differential equation

$$\frac{d^2y}{dz^2}+(n+\tfrac{1}{2}-\tfrac{1}{4}z^2)y = 0$$

is unaffected if we simultaneously replace n by $-n-1$ and z by $-iz$, and since from the asymptotic expansion it is evident that $D_n(z)$ and $D_{-n-1}(iz)$ cannot be mere multiples of each other, we see that *the general solution of the differential equation of the parabolic cylinder is expressible in terms of the D-function in the form*

$$aD_n(z)+bD_{-n-1}(iz),$$

where a and b are arbitrary constants.

This transformation shows that *integrals of the type*

$$e^{\frac{1}{4}z^2} z^{-n-1} \int_\gamma e^{-t+\frac{1}{2}(t^2/z^2)} t^n dt,$$

where γ *is any path such that the quantity*

$$\{t^2+tz^2-(n+2)\,z^2\}\, t^{n+1} e^{-t+\frac{1}{2}(t^2/z^2)}$$

resumes its initial value after t *has described* γ, *constitute a second family of definite integrals satisfying the equation.*

6. *Recurrence Formulæ for the Function* $D_n(z)$.

For all real or complex values of n and of z the function $D_n(z)$ satisfies certain recurrence formulæ. For, on integrating by parts the right-hand member of the equation

$$-\frac{2\pi i}{\Gamma(n+1)}\, e^{\frac{1}{4}z^2} z^n D_n(z) = \int_\delta e^{-t-\frac{1}{2}(t^2/z^2)} (-t)^{-n-1} dt,$$

we obtain the result

$$D_n(z) - zD_{n-1}(z) + (n-1)\, D_{n-2}(z) = 0,$$

and in the same way we can obtain a second recurrence formula, namely,

$$\frac{dD_n(z)}{dz} + \tfrac{1}{2} z D_n(z) - nD_{n-1}(z) = 0.$$

7. *The Functions* $D_n(z)$ *in the case in which* n *is an Integer.*

We have already seen (§ 4) that when n is an integer the function $D_n(z)$ ceases to be a new transcendental function and becomes merely the product of an exponential function and a polynomial. Its explicit expression may be found as follows :—We have

$$D_n(z) = \frac{i\,\Gamma(n+1)}{2\pi}\, e^{-\frac{1}{4}z^2} z^n \int_\delta e^{-t-\frac{1}{2}(t^2/z^2)} (-t)^{-n-1} dt.$$

If we take a new variable v defined by the relation $t = z(v-z)$, the resulting integral can, when n is a positive integer, be at once evaluated by Cauchy's formula for the n-th derivate of a function. *We thus obtain the formula*

$$D_n(z) = (-1)^n e^{\frac{1}{4}z^2} \frac{d^n}{dz^n} (e^{-\frac{1}{2}z^2})$$

which represents $D_n(z)$ *when* n *is a positive integer.*

Applying the transformation which changes z to iz and n to $-n-1$, we see that, when n is a negative integer, the expression

$$e^{-\frac{1}{4}z^2}\frac{d^{-n-1}}{dz^{-n-1}}(e^{\frac{1}{2}z^2})$$

is a solution of the differential equation. This gives a second set of solutions expressible as the product of an exponential and a polynomial.

8. *Integral Properties of the Functions $D_n(z)$ when n is an Integer. Expansion of an Arbitrary Function.*

We shall now show that the functions $D_n(z)$ possess properties analogous to those expressed in the theory of Legendre functions by the well known equations

$$\int_{-1}^{1} P_m(z)\,P_n(z) = 0, \quad \int_{-1}^{1} P_n^2(z)\,dz = \frac{2}{2n+1}.$$

For, from the differential equation, we have

$$D_n(z)\frac{d^2D_m(z)}{dz^2} - D_m(z)\frac{d^2D_n(z)}{dz^2} + (m-n)\,D_m(z)\,D_n(z) = 0,$$

and, integrating this relation between $z=-\infty$ and $z=\infty$, *we have the integral relation*

$$\int_{-\infty}^{\infty} D_m(z)\,D_n(z)\,dz = 0,$$

for all unequal integer values of m and n.

Moreover, we have by the recurrence-formulæ

$$\int_{-\infty}^{\infty} D_n^2(z)\,dz = \int_{-\infty}^{\infty} D_n(z)\left\{-2\frac{dD_n(z)}{dz} + (n-1)\,D_{n-1}(z)\right\}dz.$$

Transforming the right-hand member by use of the theorem just established, and by integration by parts, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} D_n^2(z)\,dz &= n\int_{-\infty}^{\infty} D_{n-1}^2(z)\,dz \\ &= n!\int_{-\infty}^{\infty} \{D_0(z)\}^2\,dz \\ &= n!\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz,\end{aligned}$$

whence, *for all integer values of* n, *we have the integral relation*

$$\int_{-\infty}^{\infty} \{D_n(z)\}^2 \, dz = (2\pi)^{\frac{1}{2}} n!\,.$$

From these integral relations it is evident that, *if any function* $f(z)$ *be expanded in a series of the form*

$$f(z) = a_0 D_0(z) + a_1 D_1(z) + a_2 D_2(z) + \ldots,$$

then the coefficients are given by the relations

$$a_n = \frac{1}{(2\pi)^{\frac{1}{2}} n!} \int_{-\infty}^{\infty} D_n(t) f(t) \, dt.$$

9. *A Third Family of Definite Integrals which satisfy the Differential Equation: Expression of* $D_n(z)$ *as an Integral of this Type.*

We shall now find another family of definite integrals which satisfy the differential equation.

For let
$$y = \int_\gamma e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}n-1} (1-t)^{\frac{1}{2}(n-1)} \, dt,$$

where the path of integration is for the present left undetermined. Then

$$\begin{aligned} &\frac{d^2 y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)\, y \\ &= \int_\gamma e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}n-1} (1-t)^{\frac{1}{2}(n-1)} \{\tfrac{1}{2}t + \tfrac{1}{4}z^2 t^2 + n + \tfrac{1}{2} - \tfrac{1}{4}z^2\} \, dt \\ &= -\int_\gamma \frac{d}{dt} \{e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}n} (1-t)^{\frac{1}{2}(n+1)}\} \, dt. \end{aligned}$$

It follows that *the differential equation of the parabolic cylinder is satisfied by any integral of the type*

$$\int_\gamma e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}n-1} (1-t)^{\frac{1}{2}(n-1)} \, dt,$$

provided the path of integration γ *is such that the quantity*

$$e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}n} (1-t)^{\frac{1}{2}(n+1)}$$

resumes its initial value after t *has described* γ.

A large number of paths γ can be found to satisfy this condition; among these we shall select the one which gives our standard solution $D_n(z)$.

Let ϵ denote a path of integration which encircles the point $t = -1$, and begins and ends at infinity in that direction which makes z^2t negative. Then, if we form the integral corresponding to this path, we easily find that it possesses an asymptotic expansion of the form $z^n e^{-\frac{1}{4}z^2} \times$ a power series in $\frac{1}{z}$; and, on evaluating the first coefficient in this series and comparing with the asymptotic expansion of $D_n(z)$, *we have the formula*

$$D_n(z) = -\frac{i\Gamma\left(\frac{n}{2}+1\right)}{2^{-\frac{1}{2}(n-1)}\pi}\int_\epsilon e^{\frac{1}{4}z^2t}(1+t)^{-\frac{1}{2}n-1}(1-t)^{\frac{1}{2}(n-1)}dt,$$

which expresses the function $D_n(z)$ as a member of this family of integrals.

When the real part of n is negative, this gives, on deformation of the path, the result

$$D_n(z) = \frac{1}{\pi}\sin\frac{n\pi}{2}\Gamma\left(\frac{n}{2}+1\right)e^{-\frac{1}{4}z^2}z^n\int_0^\infty e^{-s}s^{-\frac{1}{2}n-1}\left(1+\frac{2s}{z^2}\right)^{\frac{1}{2}(n-1)}ds.$$

10. *A Fourth Family of Definite Integrals which satisfy the Differential Equation. Expression of $D_n(z)$ as an Integral of this type.*

By a similar process to that employed in § 3 and § 9, it can be shown that *the differential equation of the parabolic cylinder is satisfied by any integral of the type*

$$z\int_\gamma e^{-\frac{1}{4}z^2t}(t-1)^{-\frac{1}{2}n-\frac{1}{2}}(t+1)^{\frac{1}{2}n}dt,$$

provided the path of integration γ is such that the quantity

$$e^{-\frac{1}{4}z^2t}(t-1)^{-\frac{1}{2}n+\frac{1}{2}}(t+1)^{\frac{1}{2}n+1}$$

resumes its initial value after t has described γ.

A large number of paths γ can be found to satisfy this condition, and these furnish integrals of the equation. Among these we shall now find one which represents the standard function $D_n(z)$.

Let η denote a path of integration which begins and ends at infinity in the direction which makes z^2t positive and which encircles the point $t=1$.

Let y_1 denote the integral taken along this path, so that

$$y_1 = z\int_\eta e^{-\frac{1}{4}z^2t}(t-1)^{-\frac{1}{2}n-\frac{1}{2}}(t+1)^{\frac{1}{2}n}dt.$$

Then, if s be a new variable defined by the relation

$$t = \frac{4s}{z^2} + 1,$$

we have
$$y_1 = 2^{-\frac{1}{2}n+1} e^{-\frac{1}{4}z^2} z^n \int e^{-s} s^{-\frac{1}{2}n-\frac{1}{2}} \left(1 + \frac{2s}{z^2}\right)^{\frac{1}{2}n} ds,$$

where the integration is now taken round a contour which begins and ends at infinity, and encloses the point $s = 0$.

This quantity clearly possesses an asymptotic expansion of the form

$$e^{-\frac{1}{4}z^2} z^n \times \text{a power series in } \frac{1}{z},$$

and is therefore a multiple of $D_n(z)$. To find the constant multiplier, we observe that the first term in the expansion is

$$2^{-\frac{1}{2}n+1} e^{-\frac{1}{4}z^2} z^n \int e^{-s} s^{-\frac{1}{2}n-\frac{1}{2}} ds$$

or
$$\frac{2^{-\frac{1}{2}n+2} \pi i^n}{\Gamma\left(\frac{n+1}{2}\right)} e^{-\frac{1}{4}z^2} z^n.$$

It follows that
$$y_1 = \frac{2^{-\frac{1}{2}n+2} \pi i^n}{\Gamma\left(\frac{n+1}{2}\right)} D_n(z),$$

and therefore that $D_n(z)$ *is expressed as a member of this family of integrals by the formula*

$$D_n(z) = \frac{2^{\frac{1}{2}n-2}\Gamma\left(\frac{n+1}{2}\right)}{\pi i^n} z \int_{\eta} e^{-\frac{1}{8}z^2 t} (t-1)^{-\frac{1}{2}n-\frac{1}{2}} (t+1)^{\frac{1}{2}n} dt.$$

When the real part of n is less than unity, it is easily seen by deformation of the path that *this can be replaced by the real integral*

$$D_n(z) = \frac{1}{\Gamma\left(\frac{1-n}{2}\right)} e^{-\frac{1}{4}z^2} z^n \int_0^\infty e^{-s} s^{-\frac{1}{2}n-\frac{1}{2}} \left(1 + \frac{2s}{z^2}\right)^{\frac{1}{2}n} ds.$$