

ON THE APPLICATION OF QUATERNIONS TO THE ORTHOGONAL TRANSFORMATION AND INVARIANT THEORY

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THE purpose of the present communication is, by means of known results in quaternions, to present the formulæ for the orthogonal transformation of three and four variables in a more convenient and symmetrical form than has yet been given, to show that the invariant theory is implied by and involved in the quaternion representation, and to establish, in the case of the ternary quantic, the irreducible orthogonal forms which do not exceed the fourth degree in the coefficients in the case of invariants or the third degree in the case of covariants as far as the ternary quantic inclusive.

1. *Orthogonal Ternary Transformation.*

Suppose the ternary quantic $a_x^n = (a_1x_1 + a_2x_2 + a_3x_3)^n$ connected with the form $A_x^n = (A_1x_1 + A_2x_2 + A_3x_3)^n$ by the orthogonal transformation

$$\left. \begin{aligned} x_1 &= p_1X_1 + q_1X_2 + r_1X_3 \\ x_2 &= p_2X_1 + q_2X_2 + r_2X_3 \\ x_3 &= p_3X_1 + q_3X_2 + r_3X_3 \end{aligned} \right\} \quad (1)$$

The quaternion imaginaries i, j, k , I denote by i_1, i_2, i_3 ; so that in Aronhold's notation the quaternion vector

$$x_1i + x_2j + x_3k = x_1i_1 + x_2i_2 + x_3i_3 = x_i.$$

Hamilton, Cayley, and others have shown that the transformation may be represented by the quaternion identity

$$x_i = (\alpha_0 + a_i) X_i (\alpha_0 - a_i), \quad (2)$$

wherein the squared tensor of the quaternion $a_0 + a_i$ is unity, viz.,

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1;$$

so that the identity involves only three independent quantities a . Thus the conjugate quaternions $a_0 + a_i$, $a_0 - a_i$ are such that each is the inverse of the other, $(a_0 + a_i)(a_0 - a_i) = 1$. It is easy to verify the quaternion identity, for

$$x_1 i_1 + x_2 i_2 + x_3 i_3 = (a_0 + a_i) i_1 (a_0 - a_i) X_1 + (a_0 + a_i) i_2 (a_0 - a_i) X_2 + (a_0 + a_i) i_3 (a_0 - a_i) X_3, \quad (3)$$

while, from equations (1),

$$x_1 i_1 + x_2 i_2 + x_3 i_3 = p_i X_1 + q_i X_2 + r_i X_3, \quad (4)$$

giving

$$\left. \begin{aligned} p_i &= (a_0 + a_i) i_1 (a_0 - a_i) \\ q_i &= (a_0 + a_i) i_2 (a_0 - a_i) \\ r_i &= (a_0 + a_i) i_3 (a_0 - a_i) \end{aligned} \right\}, \quad (5)$$

and, squaring each side,

$$p_i^2 = -p_p = -1, \quad q_i^2 = -q_q = -1, \quad r_i^2 = -r_r = -1$$

and

$$p_i q_i + q_i p_i = -p_q = 0, \quad q_i r_i + r_i q_i = -q_r = 0, \quad r_i p_i + p_i r_i = -r_p = 0,$$

the six relations required by orthogonality.

From equations (5) we derive

$$\left. \begin{aligned} (a_0 - a_i) p_i (a_0 + a_i) &= i_1 \\ (a_0 - a_i) q_i (a_0 + a_i) &= i_2 \\ (a_0 - a_i) r_i (a_0 + a_i) &= i_3 \end{aligned} \right\} \quad (6)$$

and also

$$\left. \begin{aligned} p_i (a_0 + a_i) &= (a_0 + a_i) i_1 \\ q_i (a_0 + a_i) &= (a_0 + a_i) i_2 \\ r_i (a_0 + a_i) &= (a_0 + a_i) i_3 \end{aligned} \right\} \quad (7)$$

From equations (5) are derived

$$\left. \begin{aligned} p_1 &= a_0^2 + a_1^2 - a_2^2 - a_3^2, & p_2 &= 2(a_0 a_3 + a_1 a_2), & p_3 &= 2(-a_0 a_2 + a_1 a_3) \\ q_1 &= 2(-a_0 a_3 + a_1 a_2), & q_2 &= a_0^2 - a_1^2 + a_2^2 - a_3^2, & q_3 &= 2(a_0 a_1 + a_2 a_3) \\ r_1 &= 2(a_0 a_2 + a_1 a_3), & r_2 &= 2(-a_0 a_1 + a_2 a_3), & r_3 &= a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{aligned} \right\} \quad (8)$$

and from equations (7)

$$\left. \begin{aligned} \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 &= \alpha_1 \\ \alpha_0 p_1 + \alpha_3 p_2 - \alpha_2 p_3 &= \alpha_0 \\ -\alpha_3 p_1 + \alpha_0 p_2 + \alpha_1 p_3 &= \alpha_3 \\ \alpha_2 p_1 - \alpha_1 p_2 + \alpha_0 p_3 &= -\alpha_2 \\ \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 &= \alpha_2 \\ \alpha_0 q_1 + \alpha_3 q_2 - \alpha_2 q_3 &= -\alpha_3 \\ -\alpha_3 q_1 + \alpha_0 q_2 + \alpha_1 q_3 &= \alpha_0 \\ \alpha_2 q_1 - \alpha_1 q_2 + \alpha_0 q_3 &= \alpha_1 \\ \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 &= \alpha_3 \\ \alpha_0 r_1 + \alpha_3 r_2 - \alpha_2 r_3 &= \alpha_2 \\ -\alpha_3 r_1 + \alpha_0 r_2 + \alpha_1 r_3 &= -\alpha_1 \\ \alpha_2 r_1 - \alpha_1 r_2 + \alpha_0 r_3 &= \alpha_0 \end{aligned} \right\} \quad (9)$$

From equations (1) we obtain $a_x = a_p X_1 + a_q X_2 + a_r X_3$. Hence $A_1 = a_p$, $A_2 = a_q$, $A_3 = a_r$, and, from equation (2), $X_i = (a_0 - a_i) x_i (a_0 + a_i)$, so that, if

$$\begin{aligned} X_1 &= P_1 x_1 + Q_1 x_2 + R_1 x_3, & X_2 &= P_2 x_1 + Q_2 x_2 + R_2 x_3, \\ X_3 &= P_3 x_1 + Q_3 x_2 + R_3 x_3, \end{aligned}$$

$$\begin{aligned} X_1 i_1 + X_2 i_2 + X_3 i_3 \\ &= (a_0 - a_i) i_1 (a_0 + a_i) x_1 + (a_0 - a_i) i_2 (a_0 + a_i) x_2 + (a_0 - a_i) i_3 (a_0 + a_i) x_3 \\ &= P_i x_1 + Q_i x_2 + R_i x_3, \end{aligned}$$

and it is clear that in the formulæ (5), (6), (7), (8), (9) we may change p, q, r into P, Q, R , if we at the same time change a_i into $-a_i$ or a_1, a_2, a_3 into $-a_1, -a_2, -a_3$. Thus

$$\begin{aligned} P_1 &= (\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2), & Q_1 &= 2(\alpha_0 \alpha_3 + \alpha_1 \alpha_2), & R_1 &= 2(-\alpha_0 \alpha_2 + \alpha_1 \alpha_3); \\ P_2 &= 2(-\alpha_0 \alpha_3 + \alpha_1 \alpha_2), & Q_2 &= (\alpha_0^2 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2), & R_2 &= 2(\alpha_0 \alpha_1 + \alpha_2 \alpha_3); \\ P_3 &= 2(\alpha_0 \alpha_2 + \alpha_1 \alpha_3), & Q_3 &= 2(-\alpha_0 \alpha_1 + \alpha_2 \alpha_3), & R_3 &= (\alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2). \end{aligned}$$

Also, from (1) and (10), $x_a = X_a$.

Now, since $A_1 = a_p$, $A_2 = a_q$, $A_3 = a_r$, $A_i = a_p i_1 + a_q i_2 + a_r i_3$; but $X_i = x_p i_1 + x_q i_2 + x_r i_3$; hence A and a are connected in the same way as X and x , and we have

$$\begin{aligned} a_i &= (a_0 + a_i) A_i (a_0 - a_i), & A_i &= (a_0 - a_i) a_i (a_0 + a_i), \\ a_a &= A_a. \end{aligned}$$

It follows from the above results that the quaternion vectors x_i, a_i may be regarded as pseudo-invariants, and it may be shown that the real invariants may be exhibited as rational integral combinations of pseudo-invariants.

$$\begin{aligned}\text{Since} \quad a_i x_i &= (a_0 + a_i) A_i (a_0 - a_i) (a_0 + a_i) X_i (a_0 - a_i) \\ &= (a_0 + a_i) A_i X_i (a_0 - a_i),\end{aligned}$$

it is seen that every product of powers of vectors $a_i, b_i, c_i, \dots, x_i, y_i, z_i, \dots$, and also every linear function of such products is a pseudo-invariant.

If f be a function of pseudo-invariants which is scalar, and F its transform, $f = (a_0 + a_i) F(a_0 - a_i) = F$, and f is a real invariant.

The problem of the orthogonal ternary invariants is thus reduced to the discovery of those functions of quaternion vectors which are mere scalars: *e.g.*,

$$\begin{aligned}x_i^2 &= -x_1^2 - x_2^2 - x_3^2 = -x_x, \\ a_i^2 &= -a_1^2 - a_2^2 - a_3^2 = -a_a,\end{aligned}$$

showing that x_x and a_a are invariants, for the quaternion formula immediately yields $x_x = X_x$, $a_a = A_a$.*

The form a_a or x_x is the only type of invariant symbolic factor which involves one set of letters or umbræ.

When two sets are involved we have

$$a_i b_i + b_i a_i = -2a_b, \quad a_i x_i + x_i a_i = -2a_x,$$

and we know that no other types exist which involve two sets of letters or umbræ.

The remaining symbolic factors (abc) , (abx) are derived from the formulæ

$$a_i b_i c_i - c_i b_i a_i = -2(abc), \quad a_i b_i x_i - x_i b_i a_i = -2(abx),$$

* The fundamental formula $x_i = (a_0 + a_i) X_i (a_0 - a_i)$ is one of a set of four, for the formulæ of transformation given by (1) and (8) are unchanged

- (i.) by changing the signs of $a_2, a_3, x_2, x_3, X_2, X_3$;
- (ii.) ,, ,, $a_3, a_1, x_3, x_1, X_3, X_1$;
- (iii.) ,, ,, $a_1, a_2, x_1, x_2, X_1, X_2$;

so that, writing in general

$$p_1 i_1 - p_2 i_2 - p_3 i_3 = p'_i, \quad -p_1 i_1 + p_2 i_2 - p_3 i_3 = p''_i, \quad -p_1 i_1 - p_2 i_2 + p_3 i_3 = p'''_i,$$

we obtain the quaternion formulæ

$$x'_i = (a_0 + a'_i) X'_i (a_0 - a'_i), \quad x''_i = (a_0 + a''_i) X''_i (a_0 - a''_i), \quad x'''_i = (a_0 + a'''_i) X'''_i (a_0 - a'''_i).$$

and now we have the complete set of factors $a_a, a_b, (abc), a_x, (abx), x_x$ in terms of which all orthogonal invariants and covariants are expressible.

The result $a_i x_i + x_i a_i = -2a_x$ shows that we may take the fundamental ternary quantic of order n to be $(a_i x_i + x_i a_i)^n$, which on expansion is

$$\begin{aligned} & \{ (a_i x_i)^n + (x_i a_i)^n \} \\ & + \binom{n}{1} \{ (a_i x_i)^{n-2} + (x_i a_i)^{n-2} \} a_a x_x \\ & + \binom{n}{2} \{ (a_i x_i)^{n-4} + (x_i a_i)^{n-4} \} a_a^2 x_x^2 \\ & + \dots, \end{aligned}$$

yielding a fundamental set of covariants of degree 1 in the coefficients, the covariants being of type $a_a^p \{ (a_i x_i)^{n-2p} + (x_i a_i)^{n-2p} \}$ of degree and order 1, $n-2p$, respectively. The number of these is $\frac{1}{2}(n+2)$ or $\frac{1}{2}(n+1)$ according as n is even or uneven.

A formula of reduction for such forms is

$$(a_i x_i)^m + (x_i a_i)^m = -2a_x \{ (a_i x_i)^{m-1} + (x_i a_i)^{m-1} \} - a_a x_x \{ (a_i x_i)^{m-2} + (x_i a_i)^{m-2} \},$$

leading to their rapid calculation.

We may continue the series of scalars

$$\begin{aligned} & a_i b_i c_i d_i + d_i c_i b_i a_i, \\ & a_i b_i c_i d_i e_i - e_i d_i c_i b_i a_i, \\ & a_i b_i c_i d_i e_i f_i + f_i e_i d_i c_i b_i a_i, \\ & \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

indefinitely.

$$\text{I find} \quad a_i b_i c_i d_i + d_i c_i b_i a_i = 2(a_b c_a - a_c b_a + a_d b_a),$$

$$a_i b_i c_i d_i e_i - e_i d_i c_i b_i a_i = 2 \{ a_b(cde) + b_c(ade) - a_c(bde) + d_e(abc) \}.$$

The ten expressions of form $a_b(cde)$ are not independent, for they are connected by four independent syzygies, viz.,

$$a_b(cde) - b_c(ade) + b_d(ace) - b_e(acd) = 0,$$

$$a_c(bde) - b_c(ade) + c_d(abe) - c_e(abd) = 0,$$

$$a_d(bce) - b_d(ace) + c_d(abe) - d_e(abc) = 0,$$

$$a_e(bcd) - b_e(acd) + c_e(abd) - d_e(abc) = 0.$$

Any three of these may be derived from the fourth by interchange of letters. As a consequence, in the theory of the invariants, any one of the equations may be regarded as the fundamental syzygy for the reduction of symbolic expressions.

We can derive such useful syzygies as

$$a_x(bcd) - b_x(acd) + c_x(abd) - d_x(abc) = 0,$$

$$a_a(bcd) - a_b(acd) + a_c(abd) - a_d(abc) = 0,$$

$$x_x(bcd) - b_x(cd x) + c_x(bd x) - d_x(abc) = 0,$$

$$a_a(bc x) - a_b(ac x) + a_c(ab x) - a_x(abc) = 0,$$

which will be of great use in the sequel.

In the case of six sets of umbræ

$$\begin{aligned} & a_i b_i c_i d_i e_i f_i + f_i e_i d_i c_i b_i a_i \\ &= 2(-a_b c_d e_f + a_b c_e d_f - a_b c_f d_e \\ &\quad + a_c b_d e_f - a_c b_e d_f + a_c b_f d_e \\ &\quad - a_d b_c e_f + a_d b_e c_f - a_d b_f c_e \\ &\quad + a_e b_c d_f - a_e b_d c_f + a_e b_f c_d \\ &\quad - a_f b_c d_e + a_f b_d c_e - a_f b_e c_d). \end{aligned}$$

There exist the fundamental identity

$$(abc)(def) = \begin{vmatrix} a_d & a_e & a_f \\ b_d & b_e & b_f \\ c_d & c_e & c_f \end{vmatrix}$$

and the fundamental syzygy

$$\begin{vmatrix} a_e & a_f & a_g & a_h \\ b_e & b_f & b_g & b_h \\ c_e & c_f & c_g & c_h \\ d_e & d_f & d_g & d_h \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & c_3 & 0 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix} \begin{vmatrix} e_1 & f_1 & g_1 & h_1 \\ e_2 & f_2 & g_2 & h_2 \\ e_3 & f_3 & g_3 & h_3 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

2. I propose now to determine the irreducible orthogonal invariants and covariants of the ternary quantics of lower orders when the degree in the coefficients does not exceed 4.

The symbolic expression of any invariant or covariant involves factors of the forms $a_a, a_b, (abc), (abx), a_x, x_x$. We may omit x_x from consideration because it is itself a covariant.

The product of any two determinant factors has been shown to be expressible in terms of symbolic factors a_a, a_b, \dots ; so that we may regard any form as involving at most only one determinant factor. This may be either of the nature (abc) or of the nature (abx) .

If the form involves (abc) and be a covariant, it must contain as well a factor of the form e_x , and, since there exists the syzygy

$$e_x(abc) = a_e(bc_x) - b_e(ac_x) + c_e(ab_x),$$

we can express any such covariant in terms of covariants involving the symbolic factors of nature (ab_x) . In consequence of this fact we are only concerned with the determinant factor (abc) when invariants are in question.

The investigation is thus divisible into four portions—

- (i.) Invariants which involve factors a_a, a_b, \dots only.
- (ii.) Covariants which involve factors a_a, a_b, a_x, \dots only.
- (iii.) Invariants which involve the single determinant factor (abc) .
- (iv.) Covariants which involve the single determinant factor (ab_x) .

Observing that every ternary quantic a_x^n has the special orthogonal covariant x_x , we commence by considering the invariants and covariants of degree 1 in the coefficients.

If the order of the quantic be even and equal to $2m$, we have the irreducible system $a_a^m, a_a^{m-1}a_x^2, a_a^{m-2}a_x^4, \dots, a_x^{2m}$, and, if the order be uneven and equal to $2m+1$, the irreducible system $a_a^m a_x, a_a^{m-1}a_x^3, a_a^{m-2}a_x^5, \dots, a_x^{2m+1}$. The number of invariative forms of the first degree in the coefficients for the quantic of order n is thus the coefficient of x^n in the expansion of the fraction

$$\frac{1}{(1-x)(1-x^2)}.$$

I pass to the forms of the second degree in the coefficients which are without a determinant factor (ab_x) .

Since three letters only appear, viz., a, b, x , there are no syzygies, and the general form to be considered is

$$a_a^{p_{11}} b_b^{p_{22}} a_b^{p_{12}} a_x^{p_1} b_x^{p_2},$$

with the condition $2p_{11} + p_1 = 2p_{22} + p_2$.

For irreducibility p_{12} must be greater than zero; also by interchange of letters we can always take p_{11}, p_{22} to be in descending order of magnitude, so that without loss of generality we may assume

$$p_{11} \geq p_{22};$$

putting then $p_{11} = p_{22} + q, \quad p_2 = p_1 + 2q,$

the general irreducible form is found to be

$$(a_a b_b)^{p_{22}} a_b^{p_{12}+1} (a_x b_x)^{p_1} (a_a b_x^2)^q,$$

and the number of forms for the quantic of order n is the coefficient of x^n in the expansion of

$$\frac{x}{(1-x)^2(1-x^2)^2}.$$

For the quantities of the first few orders, we have

$$\begin{aligned}
 \text{order 1:} & \quad a_b; \\
 \text{order 2:} & \quad a_b^2, \quad a_b a_x b_x; \\
 \text{order 3:} & \quad a_b^3, \quad a_b^2 a_x b_x, \quad a_b a_a b_b, \\
 & \quad a_b a_a b_x^2, \\
 & \quad a_b a_x^2 b_x^2; \\
 \text{order 4:} & \quad a_b^4, \quad a_b^3 a_x b_x, \quad a_b^2 a_a b_b, \quad a_b a_a b_b a_x b_x, \\
 & \quad a_b^2 a_a b_x^2, \quad a_b a_a a_x b_x^3, \\
 & \quad a_b^2 a_x^2 b_x^2, \quad a_b a_x^3 b_x^3.
 \end{aligned}$$

We now come to the invariants of the third degree in the coefficients which have no determinant factor. The form is

$$a_a^{p_{11}} b_b^{p_{22}} c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}},$$

with the conditions

$$2p_{11} + p_{13} = 2p_{22} + p_{23}, \quad 2p_{22} + p_{12} = 2p_{33} + p_{13};$$

we may suppose p_{11}, p_{22}, p_{33} to be in descending order of magnitude, and therefore p_{12}, p_{13}, p_{23} in ascending order; so that, writing

$$p_{11} = p_{33} + p + q, \quad p_{22} = p_{33} + p, \quad p_{13} = p_{12} + 2p, \quad p_{23} = p_{12} + 2p + 2q,$$

we find the form to be

$$(a_a b_b c_c)^{p_{33}} (a_a b_b a_c^2 b_c^2)^p (a_a b^2)^q (a_b a_c b_c)^{p_{12}}.$$

For irreducibility p and p_{12} must not vanish together; this shows that the number of irreducible forms is the coefficient of x^n in the expansion of

$$\begin{aligned}
 \frac{1}{(1-x^2)^3(1-x^4)} - \frac{1}{(1-x^2)^3} &= \frac{x^2 + x^4 - x^6}{(1-x^2)^3(1-x^4)} \\
 &= x^2 + 4x^4 + 9x^6 + 17x^8 + 28x^{10} + 43x^{12} + \dots
 \end{aligned}$$

Thus for the quadric there is the form

$$a_b a_c b_c;$$

for the quartic the four forms

$$\begin{aligned}
 a_a b_b c_c a_b a_c b_c, & \quad a_a a_b a_c b_c^3, \\
 a_a b_b a_c^2 b_c^2, & \quad a_b^2 a_c^2 b_c^2;
 \end{aligned}$$

and for the sextic the nine forms

$$\begin{aligned}
 a_a^2 b_b^2 c_c^2 a_b a_c b_c, & \quad a_a b_b c_c a_b^2 a_c^2 b_c^2, & \quad a_a^2 a_b a_c b_c^5, \\
 a_a^2 b_b^2 c_c^2 a_c^2 b_c^2, & \quad a_a^2 b_b a_c^2 b_c^4, & \quad a_a a_b^2 a_c^2 b_c^4, \\
 a_a^2 b_b c_c a_b a_c b_c^3, & \quad a_a b_b a_b a_c^3 b_c^3, & \quad a_b^3 a_c^3 b_c^3;
 \end{aligned}$$

and the series can be continued without difficulty.

Taking up now the covariants of the third degree in the coefficients which do not possess any determinant factor, we have to deal with four letters a, b, c, x , and we must take account of the syzygy

$$\begin{vmatrix} a_a & a_b & a_c & a_x \\ b_a & b_b & b_c & b_x \\ c_a & c_b & c_c & c_x \\ a_x & b_x & c_x & x_x \end{vmatrix} = 0.$$

The general form of covariant is $a_a^{p_{11}} b_b^{p_{22}} c_c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}} a_x^{p_1} b_x^{p_2} c_x^{p_3}$, wherein p_{11}, p_{22}, p_{33} may be taken to be in descending order of magnitude.

The conditions

$$2p_{11} + p_{13} + p_1 = 2p_{22} + p_{23} + p_2, \quad 2p_{22} + p_{12} + p_2 = 2p_{33} + p_{13} + p_3$$

$$p_{11} \geq p_{22} \geq p_{33}$$

are easily dealt with, and the result is that we have the twelve ground symbolic products

$$\begin{aligned} & a_b c_x, \quad a_c b_x, \quad b_c a_x, \quad a_x b_x c_x, \\ & a_a b_c^2, \quad a_a b_b c_c, \quad a_a b_x^2 c_x^2, \quad a_a b_b c_x^2, \\ & a_a b_c b_x c_x, \quad a_b a_c b_c, \\ & a_a b_b a_c b_c c_x, \\ & a_a b_b a_c^2 b_c^2. \end{aligned}$$

Four of these appertain to the linear quantic and are visibly reducible.

In the case of the quadric the only form that is not visibly reducible is $a_b a_c b_x c_x$, but, reducing the fundamental syzygy, we find that

$$2a_b a_c b_x c_x - 2c_c a_b a_x b_x + a_a b_b c_x^2 - a_b^2 c_x^2$$

contains x_x as a factor; hence $a_b a_c b_x c_x$ for the quadric is reducible.

For the cubic the irreducible forms may be

$$\begin{aligned} & a_b^2 a_c b_c c_x, & a_a b_b a_c b_c c_x, & a_b a_c b_c a_x b_x c_x, \\ & a_a a_b b_c^2 c_x, & a_b b_c^2 a_x^2 c_x, & a_a a_b b_c b_x c_x^2; \end{aligned}$$

but, multiplying the fundamental syzygy by

$$a_b c_x$$

and reducing for the cubic, we find that

$$2a_a a_b b_c b_x c_x^2 - a_b a_c b_c a_x b_x c_x - a_b b_c^2 a_x^2 c_x$$

is reducible. We therefore discard

$$a_b a_c b_c a_x b_x c_x,$$

and take the remaining five forms as the fundamental covariants of the cubic of the third degree in the coefficients.

Passing to the quartic, the possible irreducible forms are found to be

Order 2 in Variables.

$$\begin{aligned} a_a a_b^2 b_c^2 c_x^2 &= A_{42}, & a_b^3 a_c b_c c_x^2 &= E_{42}, \\ a_a a_b a_c b_c^2 b_x c_x &= B_{42}, & a_b^2 a_c^2 b_c b_x c_x &= F_{42}, \\ a_a a_b b_c^3 a_x c_x &= C_{42}, & a_a b_b a_b a_c b_c c_x^2 &= G_{42}, \\ a_a b_b c_c a_b a_c b_x c_x &= D_{42}, & a_a b_b a_c^2 b_c b_x c_x &= H_{42}. \end{aligned}$$

These reduce to five independent forms, since multiplication of the fundamental syzygy by $a_a b_c^2$, $a_b a_c b_c$, $a_a b_b c_c$ shows that the linear functions

$$A_{42} - B_{42} - 2C_{42} + 2H_{42}, \quad 2B_{42} + E_{42} - 2F_{42} - G_{42}, \quad F_{42}$$

are reducible. We take as fundamental forms A_{42} , B_{42} , C_{42} , D_{42} , E_{42} .

Order 4 in Variables.

$$\begin{aligned} a_b^3 a_c b_x c_x^3 &= A_{44}, & a_a b_b c_c a_b a_x b_x c_x^2 &= F_{44}, \\ a_b^2 a_c^2 b_x^2 c_x^2 &= B_{44}, & a_a b_b a_b a_c b_x c_x^3 &= G_{44}, \\ a_b^2 a_c b_c a_x b_x c_x^2 &= C_{44}, & a_a b_b a_c b_c a_x b_x c_x^2 &= H_{44}, \\ a_a b_c^3 a_x^2 b_x c_x &= D_{44}, & a_a a_b^2 b_c b_x c_x^3 &= I_{44}, \\ a_a a_b b_c^2 a_x b_x c_x^2 &= E_{44}, & a_a a_b a_c b_c b_x^2 c_x^2 &= J_{44}, \end{aligned}$$

and now, multiplying the syzygy by $a_a b_c b_x c_x$, $a_a b_b c_x^2$, $a_b^2 c_x^2$, $a_b a_c b_x c_x$, we find that the linear functions

$$\begin{aligned} 2E_{44} - 2H_{44} - I_{44} + J_{44}, & \quad 2G_{44} + H_{44}, & \quad 2A_{44} - B_{44} + C_{44} - 2I_{44}, \\ 2A_{44} - 2B_{44} - 3C_{44} + 4E_{44} + 2J_{44} \end{aligned}$$

are reducible.

We have thus six independent forms of order 4 in the variables which we take to be A_{44} , B_{44} , C_{44} , D_{44} , E_{44} , F_{44} .

Order 6 in Variables.

$$\begin{aligned} a_b^2 a_c a_x b_x^2 c_x^3 &= A_{46}, & a_a a_b a_c b_x^3 c_x^3 &= D_{46}, \\ a_b a_c b_c a_x^2 b_x^2 c_x^2 &= B_{46}, & a_a a_b b_c a_x b_x^2 c_x^3 &= E_{46}, \\ a_a b_c^2 a_x^2 b_x^2 c_x^2 &= C_{46}, \end{aligned}$$

Multiplying the syzygy by $a_b a_x b_x c_x^2$, $a_a b_x^2 c_x^2$, we find that the linear functions

$$A_{46} + B_{46} - 2E_{46}, \quad D_{46} + 2E_{46}$$

are reducible. We therefore take the three independent covariants of order 6 in the variables to be

$$A_{46}, B_{46}, C_{46}.$$

Order 8 in Variables.

$$a_b a_c a_x^2 b_x^8 c_x^8.$$

When the syzygy is multiplied by $a_x^2 b_x^2 c_x^2$, this single form is at once seen to be reducible. This completes the discussion of covariants of degree 3 in the coefficients as far as the ternary quartic inclusive, the symbolic forms being free from a determinant factor.

For the invariants of the fourth degree in the coefficients which do not involve a determinant factor the form is

$$a_a^{p_{11}} b_b^{p_{22}} c_c^{p_{33}} d_d^{p_{44}} a_b^{p_{12}} a_c^{p_{13}} a_d^{p_{14}} b_c^{p_{23}} b_d^{p_{24}} c_d^{p_{34}},$$

with the conditions $2p_{11} + p_{13} + p_{14} = 2p_{22} + p_{23} + p_{24}$,

$$2p_{22} + p_{12} + p_{24} = 2p_{33} + p_{13} + p_{34},$$

$$2p_{33} + p_{13} + p_{23} = 2p_{44} + p_{14} + p_{34},$$

and we may take $p_{11} \geq p_{22} \geq p_{33} \geq p_{44}$,

implying $p_{24} \geq p_{13}$, $p_{34} \geq p_{12}$.

Put therefore $p_{33} = p_{44} + q_{33}$, $p_{22} = p_{44} + q_{22} + q_{33}$,

$$p_{11} = p_{44} + q_{11} + q_{22} + q_{33}, \quad p_{24} = p_{13} + q_{24}, \quad p_{34} = p_{12} + q_{34},$$

and the form becomes

$$(a_a b_b c_c d_d)^{p_{44}} (a_a b_b c_c)^{q_{33}} (a_a b_b)^{q_{22}} a_d^{q_{11}} (a_b c_d)^{p_{12}} (a_c b_d)^{p_{13}} a_d^{p_{14}} b_c^{p_{23}} b_d^{q_{24}} c_d^{q_{34}},$$

where $q_{24} = q_{11} + q_{33}$, $q_{34} = q_{11} + 2q_{22} + q_{33}$.

Hence the form is

$$(a_a b_b c_c d_d)^{p_{44}} (a_a b_b c_c b_d c_d)^{q_{33}} (a_a b_b c_d^2)^{q_{22}} (a_a b_d c_d)^{q_{11}} (a_b c_d)^{p_{12}} (a_c b_d)^{p_{13}} a_d^{p_{14}} b_c^{p_{23}},$$

with the single condition $q_{11} + p_{14} = q_{33} + p_{23}$.

Hence it is easy to see that the form finally is composed of powers of the symbolic factors

$$\begin{aligned} & a_b c_d, \quad a_c b_d, \quad a_d b_c, \\ & a_a b_b c_c d_d, \quad a_a b_b c_d^2, \quad a_a b_c b_d c_d, \\ & a_a b_b c_c a_d b_d c_d, \\ & a_a^2 b_b c_c b_d^2 c_d^2. \end{aligned}$$

There are obviously no irreducible forms for the quantic of the first order.

For the quadric the only possible form is $a_b a_c b_d c_d$, as other forms are visibly reducible.

As regards this form observe that the syzygy

$$\begin{vmatrix} a_a & a_b & a_c & a_d \\ a_b & b_b & b_c & b_d \\ a_c & b_c & c_c & c_d \\ a_d & b_d & c_d & d_d \end{vmatrix} = 0$$

reduces for the quadric to

$$6a_b a_c b_d c_d - 8a_a b_c b_d c_d - 3a_b^2 c_d^2 + 6a_a b_b c_d^2 - a_a b_b c_c d_d = 0,$$

showing that $a_b a_c b_d c_d$ is in fact reducible. Therefore the quadric has no invariant of the fourth degree in the coefficients.

For the cubic the possible irreducible forms are

$$a_b^2 a_c b_d c_d^2, \quad a_b a_c a_d b_c b_d c_d, \quad a_a b_b a_c b_d c_d^2, \quad a_a a_b b_c b_d c_d^2, \quad a_a b_b c_c a_d b_d c_d,$$

and these are in one syzygy, for on multiplying the fundamental syzygy by $a_b c_d$ and reducing we find that

$$2a_b^2 a_c b_d c_d^2 + 2a_b a_c a_d b_c b_d c_d + 4a_a b_b a_c b_d c_d^2 - 8a_a a_b b_c b_d c_d^2 - a_a b_b c_c a_d b_d c_d$$

is reducible. There is no other syzygy; so that there are four fundamental invariants for the cubic, and we will take the first four above written as exemplar forms.

For the quartic the possible irreducible forms are

$$\begin{array}{ll} a_b^3 a_c b_d c_d^3 & \text{(I.),} \\ a_b^2 a_c a_d b_c b_d c_d^2 & \text{(II.),} \\ a_b^2 a_c^2 b_d^2 c_d^2 & \text{(III.),} \\ a_a b_b c_c d_d a_b a_c b_d c_d & \text{(IV.),} \\ a_a b_b a_c^2 b_d^2 c_d^2 & \text{(V.),} \end{array} \quad \begin{array}{ll} a_a b_b a_b a_c b_d c_d^3 & \text{(VI.),} \\ a_a b_b a_c a_d b_c b_d c_d^2 & \text{(VII.),} \\ a_a a_b^2 b_c b_d c_d^3 & \text{(VIII.),} \\ a_a a_b a_d b_c^2 b_d c_d^2 & \text{(IX.),} \\ a_a b_b c_c a_b a_d b_d c_d^2 & \text{(X.),} \end{array}$$

ten in number; but these are connected by a number of syzygies with reducible forms. These are five in number, found by multiplying the fundamental syzygy by

$$a_a b_b^2 c_c d_d, \quad a_a b_b c_d^2, \quad a_b^2 c_d^2, \quad a_a b_c b_d c_d, \quad a_b a_d b_c c_d.$$

We then find that the following linear functions are reducible, viz. :—

IV.,

V. — 2VI. — VII. + 2X.,

2I. + II. — III. + 2V. — 4VIII.,

2VII. + VIII. — 2IX. — X.,

2I. — 3II. — 2III. — 4VI. — 2VII. + 8IX.,

and we may consider I., II., III., V., VI. to be the ground forms.

3. The Determinant Forms.

For invariants of degree 3 we take the form

$$a_a^{p_{11}} b_b^{p_{22}} c_c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}} (abc).$$

We may take p_{11} , p_{22} , p_{33} and also p_{23} , p_{13} , p_{12} in descending order of magnitude, and that the form may not vanish it is merely necessary that no equalities shall exist between the numbers p_{11} , p_{22} , p_{33} . These conditions are satisfied by writing

$$\begin{aligned} p_{22} &= p_{33} + \kappa + 1, & p_{13} &= p_{12} + 2\kappa + 2, \\ p_{11} &= p_{33} + \kappa + \lambda + 2, & p_{23} &= p_{12} + 2\kappa + 2\lambda + 4, \end{aligned}$$

and then the form becomes

$$(a_a b_b c_c)^{p_{33}} (a_a b_b a_c^2 b_c^2)^\kappa (a_a b_c^2)^\lambda (a_b a_c b_c)^D a_a^2 b_b a_c^2 b_c^4 (abc).$$

Every form contains therefore the factor

$$a_a^2 b_b a_c^2 b_c^4 (abc),$$

and may contain powers of the factors

$$a_a b_b c_c, \quad a_a b_c^2, \quad a_b a_c b_c, \quad a_a b_b a_c^2 b_c^2$$

in addition.

The simplest invariant of the kind is therefore

$$a_a^2 b_b a_c^2 b_c^4 (abc),$$

for the ternary septic and in general for the ternary n -ic, the number of independent invariants of the kind is given by the coefficients of x^n

the expansion of $\frac{x^7}{(1-x^2)^3(1-x^4)}$.

Proceeding to covariants which involve the determinant factor (abx) , observe that syzygies present themselves for the first time when four

letters appear in the forms. Hence the discussion of the form

$$a_a^{p_{11}} b_b^{p_{22}} a_b^{p_{12}} a_x^{p_1} b_x^{p_2} (abx)$$

is very simple.

We may consider $p_{11} > p_{22}$, $p_2 > p_1$ without loss of generality. Put then

$$p_{11} = p_{22} + \kappa + 1, \quad p_2 = p_1 + 2\kappa + 2,$$

so that the form becomes

$$(a_a b_x^2)^\kappa a_b^{p_{12}} (a_x b_x)^{p_1} a_a (abx) b_x^2.$$

Every form must contain $a_a (abx) b_x^2$,

and in addition may contain the factors

$$a_b, \quad a_x b_x, \quad a_a b_x^2$$

any number of times repeated.

The generating function of irreducible forms is

$$\frac{a_a (abx) b_x^2}{1 - a_b \cdot 1 - a_x b_x \cdot 1 - a_a b_x^2},$$

yielding for the cubic $a_a (abx) b_x^2$,

and for the quartic $a_a a_b (abx) b_x^2$, $a_a (abx) a_x b_x^3$.

When we come to the covariants of degree 3 in the coefficients which involve the determinant factor (abx) we reach a complicated system of forms and the indeterminate equations are troublesome to handle. Moreover, syzygies have to be considered. I propose, without exhibiting the whole of the work, to show shortly how the irreducible forms appertaining to the quadric, cubic, and quartic have been obtained.

The general form is

$$a_a^{p_{11}} b_b^{p_{22}} c_c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}} a_x^{p_1} b_x^{p_2} c_x^{p_3} (abx),$$

and we may clearly take p_{11} , p_{22} in descending order of magnitude. The form thus becomes

$$(a_a b_b)^{p_{22}} a_a^\kappa c_c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}} a_x^{p_1} b_x^{p_2} c_x^{p_3} (abx),$$

with the conditions

$$2\kappa + p_{13} + p_1 = p_{23} + p_2, \quad 2p_{22} + p_{12} + p_2 + 1 = 2p_{33} + p_{13} + p_3;$$

from the second of these conditions it is clear that p_{33} , p_{13} , and p_3 cannot be zero simultaneously, and a short examination shows that every form must contain one of the five combinations

$$c_x (abx), \quad c_c a_x b_x (abx), \quad c_c a_b (abx), \quad a_c b_c (abx), \quad a_a c_c b_c b_x (abx),$$

and with each of these we may connect the form

$$(a_a b_b)^{p_{22}} a_a^\kappa c_c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}} a_x^{p_1} b_x^{p_2} c_x^{p_3}$$

with the conditions

$$2\kappa + p_{13} + p_1 = p_{23} + p_2, \quad 2p_{22} + p_{12} + p_2 = 2p_{33} + p_{13} + p_3.$$

The fundamental solutions of these equations are given in the following table :—

p_{22}	κ	p_{33}	p_{12}	p_{13}	p_{23}	p_1	p_2	p_3
			1					1
				1			1	
					1	1		
						1	1	1
	1				2			
1		1						
	1						2	2
1								2
	1				1		1	1
			1	1	1			
1				1	1			1
1				2	2			
		1	1			1	1	
	1	1	1		1		1	
	1	1					2	
		1	2					
		1				2	2	

yielding the seventeen factors :—

$$\begin{aligned} & a_b c_x, \quad a_c b_x, \quad b_c a_x, \quad a_x b_x c_x, \\ & a_a b_c^2, \quad a_a b_b c_c, \quad a_a b_x^2 c_x^2, \quad a_a b_b c_x^2, \quad a_a b_c b_x c_x, \\ & a_b a_c b_c, \quad c_c a_b a_x b_x, \quad a_a c_c b_x^2, \quad c_c a_b^2, \quad c_c a_x^2 b_x^2, \\ & a_a b_b a_c b_c c_x, \quad a_a c_c a_b b_c b_x, \\ & a_a b_b a_c^2 b_c^2. \end{aligned}$$

There is no irreducible form for the quantic of order 1. Taking account of the syzygies, I find, after a laborious process, one irreducible form for the quadric, nine irreducible forms for the cubic, and twenty-five irreducible forms for the quartic, and the whole of the irreducible forms, involving at most four letters, so far as the quartic inclusive are set forth in the sub-joined list.

Irreducible Orthogonal Invariants.

Linear Quantic Invariant : a_b .

Covariant : a_x .

Quadric Invariants : $a_a, \quad a_b^2, \quad a_b a_c b_c$.

Covariants : $a_x^2, \quad a_b a_x b_x, \quad a_c b_x c_x (abx)$.

Cubic Invariants :

$$\begin{aligned} & a_b^3, & a_a b_b a_c b_d c_d^2, & a_a b_b c_c a_d b_d c_d, \\ & a_a b_b a_b, & a_a a_d b_c^2 b_d c_d, & a_b^2 a_c b_d c_d^2. \end{aligned}$$

Covariants :

$$\begin{aligned} & a_a a_x, & a_a b_x^2 (abx), & a_b b_c^2 a_x^2 c_x, & a_c^2 b_x^2 c_x (abx), \\ & a_x^3, & a_a a_c b_c^2 b_x, & a_a b_c^2 c_x (abx), & a_b a_c b_x c_x^2 (abx), \\ & a_b^2 a_x b_x, & a_b^2 a_c b_c c_x, & a_c^2 b_c b_x (abx), & a_a b_c b_x c_x^2 (abx), \\ & a_a a_b b_x^2, & a_a b_b a_c b_c c_x, & c_c a_b a_c b_x (abx), & c_c a_c a_x b_x^2 (abx), \\ & a_b a_x^2 b_x^2, & a_a a_c b_c b_x^2 c_x, & a_a c_a b_c b_x (abx), & a_c a_x b_x^2 c_x^2 (abx). \end{aligned}$$

Quartic Invariants :

$$\begin{aligned} & a_a^2, & a_b^2 a_c^2 b_b c_c, & a_b a_c b_c a_a b_b c_c, & a_b^2 a_c^2 b_d^2 c_d^2, \\ & a_b^4, & a_b^2 a_c^2 b_c^2, & a_b^3 a_c b_d c_d^3, & a_a b_b a_c^2 b_d^2 c_d^2, \\ & a_b^2 a_a b_b, & a_b^3 a_c b_c c_c, & a_b^2 a_c a_d b_c b_d c_d^2, & a_a b_b a_b a_c b_d c_d^3. \end{aligned}$$

Covariants :

$$\begin{array}{lll}
 a_a^2 a_x^2, & a_b^2 a_c b_c a_x b_x c_x^2, & a_a c_c a_c b_c b_x^2 (abx), \\
 a_x^4, & a_a a_b b_c^2 a_x b_x c_x^2, & c_c a_b a_c^2 b_x^2 (abx), \\
 a_b^3 a_x b_x, & a_a b_c^3 a_x^2 b_x c_x, & a_a a_c b_c^2 a_x b_x (abx), \\
 a_a a_b^2 b_x^2, & a_a b_b c_c a_b a_x b_x c_x^2, & b_c^3 a_x^3 c_x (abx), \\
 a_b^2 a_x^2 b_x^2, & a_b^2 a_c a_x b_x^2 c_x^3, & a_c b_c^2 a_x^2 b_x c_x (abx), \\
 a_a b_b a_b a_x b_x, & a_b a_c b_c a_x^2 b_x^2 c_x^2, & a_b b_c^2 a_x^2 c_x^2 (abx), \\
 a_a a_b a_x b_x^3, & a_a b_c^2 a_x^2 b_x^2 c_x^2, & a_a b_c^2 a_x b_x c_x^2 (abx), \\
 a_b a_x^3 b_x^3, & a_a a_c b_c^3 (abx), & c_c a_b a_c a_x b_x^2 c_x (abx), \\
 a_a a_b b_x^2 (abx), & a_a c_c a_b b_c^2 (abx), & a_a c_c a_c b_x^3 c_x (abx), \\
 a_a a_x b_x^3 (abx), & c_c a_b^2 b_c a_x c_x (abx), & a_a b_b a_c b_x c_x^3 (abx), \\
 a_a a_b^2 b_c^2 c_x^2, & a_a b_c^3 a_x c_x (abx), & c_c a_c^2 a_x b_x^3 (abx), \\
 a_a a_b a_c b_c^2 b_x c_x, & a_a a_c b_c^2 b_x c_x (abx), & b_c^2 a_x^2 b_x c_x^2 (abx), \\
 a_a a_b b_c^3 a_x c_x, & a_a b_b c_c a_c b_x c_x (abx), & a_b b_c^2 a_x^2 b_x c_x^3 (abx), \\
 a_a b_b c_c a_b a_c b_x c_x, & a_b a_c^2 b_c b_x c_x (abx), & a_a b_c a_x b_x^3 c_x^3 (abx), \\
 a_b^3 a_c b_c c_x^2, & a_a c_c a_b b_c b_x c_x (abx), & c_c a_c a_x^2 b_x^3 c_x (abx), \\
 a_b^3 a_c b_x c_x^3, & a_c^3 b_c b_x^2 (abx), & b_c a_x^3 b_x^3 c_x^3 (abx). \\
 a_b^2 a_c^2 b_x^2 c_x^2, & &
 \end{array}$$

4. Orthogonal Quaternary Transformation.

Suppose the quantic

$$a_x^n = (a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3)^n$$

to be converted into

$$A_x^n = (A_0 X_0 + A_1 X_1 + A_2 X_2 + A_3 X_3)^n$$

by the orthogonal transformation

$$(x_0, x_1, x_2, x_3) = \begin{pmatrix} p_0 & q_0 & r_0 & s_0 \\ p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \end{pmatrix} (X_0, X_1, X_2, X_3).$$

It is well known that the transformation may be represented by the quaternion identity

$$x_0 + x_i = (\alpha_0 + \alpha_i)(X_0 + X_i)(\beta_0 - \beta_i)$$

or

$$X_0 + X_i = (\alpha_0 - \alpha_i)(x_0 + x_i)(\beta_0 + \beta_i),$$

where

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1;$$

so that only six independent constants are involved.

To verify that the identity does, in fact, give an orthogonal transformation, observe that

$$\begin{aligned} x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 \\ &= (\alpha_0 + \alpha_i)(\beta_0 - \beta_i)X_0 + (\alpha_0 + \alpha_i)i_1(\beta_0 - \beta_i)X_1 \\ &\quad + (\alpha_0 + \alpha_i)i_2(\beta_0 - \beta_i)X_2 + (\alpha_0 + \alpha_i)i_3(\beta_0 - \beta_i)X_3 \\ &= (p_0 + p_i)X_0 + (q_0 + q_i)X_1 + (r_0 + r_i)X_2 + (s_0 + s_i)X_3. \end{aligned}$$

Hence

$$\left. \begin{aligned} p_0 + p_i &= (\alpha_0 + \alpha_i)(\beta_0 - \beta_i) \\ q_0 + q_i &= (\alpha_0 + \alpha_i)i_1(\beta_0 - \beta_i) \\ r_0 + r_i &= (\alpha_0 + \alpha_i)i_2(\beta_0 - \beta_i) \\ s_0 + s_i &= (\alpha_0 + \alpha_i)i_3(\beta_0 - \beta_i) \end{aligned} \right\}, \quad (i.)$$

and, taking the conjugate quaternions of each side of the four identities,

$$\begin{aligned} p_0 - p_i &= (\beta_0 + \beta_i)(\alpha_0 - \alpha_i), \\ q_0 - q_i &= -(\beta_0 + \beta_i)i_1(\alpha_0 - \alpha_i), \\ r_0 - r_i &= -(\beta_0 + \beta_i)i_2(\alpha_0 - \alpha_i), \\ s_0 - s_i &= -(\beta_0 + \beta_i)i_3(\alpha_0 - \alpha_i), \end{aligned}$$

whence, by multiplication,

$$p_0^2 - p_i^2 = p_p = 1, \quad q_0^2 - q_i^2 = q_q = 1, \quad r_0^2 - r_i^2 = r_r = 1, \quad s_0^2 - s_i^2 = s_s = 1.$$

Moreover

$$\begin{aligned} (p_0 + p_i)(q_0 - q_i) + (q_0 + q_i)(p_0 - p_i) \\ &= 2p_q = -(\alpha_0 + \alpha_i)i_1(\alpha_0 - \alpha_i) + (\alpha_0 + \alpha_i)i_1(\alpha_0 - \alpha_i) = 0, \\ (q_0 + q_i)(r_0 - r_i) + (r_0 + r_i)(q_0 - q_i) \\ &= 2q_r = -(\alpha_0 + \alpha_i)i_1i_2(\alpha_0 - \alpha_i) - (\alpha_0 + \alpha_i)i_2i_1(\alpha_0 - \alpha_i) = 0, \end{aligned}$$

and so forth; so that also $p_q = p_r = p_s = q_r = q_s = r_s = 0$, and we have the ten relations between the sixteen coefficients required by orthogonality.

From the relations (i.) we deduce the transformation formulæ

$$\begin{aligned} p_0 &= a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + a_3\beta_3, & r_0 &= a_0\beta_2 + a_1\beta_3 - a_2\beta_0 - a_3\beta_1, \\ p_1 &= -a_0\beta_1 + a_1\beta_0 - a_2\beta_3 + a_3\beta_2, & r_1 &= -a_0\beta_3 + a_1\beta_2 + a_2\beta_1 - a_3\beta_0, \\ p_2 &= -a_0\beta_2 + a_1\beta_3 + a_2\beta_0 - a_3\beta_1, & r_2 &= a_0\beta_0 - a_1\beta_1 + a_2\beta_2 - a_3\beta_3, \\ p_3 &= -a_0\beta_3 - a_1\beta_2 + a_2\beta_1 + a_3\beta_0, & r_3 &= a_0\beta_1 + a_1\beta_0 + a_2\beta_3 + a_3\beta_2, \\ q_0 &= a_0\beta_1 - a_1\beta_0 - a_2\beta_3 + a_3\beta_2, & s_0 &= a_0\beta_3 - a_1\beta_2 + a_2\beta_1 - a_3\beta_0, \\ q_1 &= a_0\beta_0 + a_1\beta_1 - a_2\beta_2 - a_3\beta_3, & s_1 &= a_0\beta_2 + a_1\beta_3 + a_2\beta_0 + a_3\beta_1, \\ q_2 &= a_0\beta_3 + a_1\beta_2 + a_2\beta_1 + a_3\beta_0, & s_2 &= -a_0\beta_1 - a_1\beta_0 + a_2\beta_3 + a_3\beta_2, \\ q_3 &= -a_0\beta_2 + a_1\beta_3 - a_2\beta_0 + a_3\beta_1, & s_3 &= a_0\beta_0 - a_1\beta_1 - a_2\beta_2 + a_3\beta_3, \end{aligned}$$

and the corresponding formulæ for expressing X_0, X_1, X_2, X_3 in terms of x_0, x_1, x_2, x_3 by changing the signs of $a_1, a_2, a_3, \beta_1, \beta_2, \beta_3$.

Note also that, since $(a_0 - a_i)(x_0 + x_i) = (X_0 + X_i)(\beta_0 - \beta_i)$, we obtain by equating scalars $x_a = X_\beta$. Also

$$\begin{aligned} a_0 + a_i &= (a_0 + a_i)(A_0 + A_i)(\beta_0 - \beta_i), \\ a_0 - a_i &= (\beta_0 + \beta_i)(A_0 - A_i)(a_0 - a_i), \end{aligned}$$

and

$$a_a = A_\beta.$$

We can now prove that a_x is an invariant symbol because

$$(a_0 + a_i)(x_0 - x_i) = (a_0 + a_i)(A_0 + A_i)(X_0 - X_i)(a_0 - a_i),$$

and, taking the conjugates of each side,

$$(x_0 + x_i)(a_0 - a_i) = (a_0 + a_i)(X_0 + X_i)(A_0 - A_i)(a_0 - a_i),$$

and, since

$$(a_0 + a_i)(x_0 - x_i) + (x_0 + x_i)(a_0 - a_i) = 2a_0x_0 - a_ix_i - x_ia_i = 2a_x, \text{ a scalar,}$$

we have

$$a_x = A_x,$$

and similarly

$$a_a = A_A, \quad x_x = X_X, \quad a_b = A_B.$$

It is clear that, if f be any quaternion product involving umbræ a, b, c, \dots , or variables x, y, z, \dots , and F its transform, such that

$$f = (a_0 + a_i)F(a_0 - a_i),$$

and \bar{f}, \bar{F} be the quaternions conjugate to f, F respectively,

$$f + \bar{f} = F + \bar{F},$$

and $f + \bar{f}$ or Sf is an invariant combination of symbols.

It can now be shown that the determinant $(abcd)$ is an invariant factor. It is easy to prove that

$$S(a_0 - a_i)(b_0 + b_i)(c_0 - c_i)(d_0 + d_i) = (abcd) + a_b c_d - a_c b_d + a_b c_c,$$

$$S(a_0 + a_i)(b_0 - b_i)(c_0 + c_i)(d_0 - d_i) = -(abcd) + a_b c_d - a_c b_d + a_d b_c.$$

Hence
$$2(abcd) = S(a_0 - a_i)(b_0 + b_i)(c_0 - c_i)(d_0 + d_i)$$

$$- S(a_0 + a_i)(b_0 - b_i)(c_0 + c_i)(d_0 - d_i),$$

and, since

$$(a_0 - a_i)(b_0 + b_i)(c_0 - c_i)(d_0 + d_i)$$

$$= (\beta_0 + \beta_i)(A_0 - A_i)(B_0 + B_i)(C_0 - C_i)(D_0 + D_i)(\beta_0 - \beta_i),$$

$$(a_0 + a_i)(b_0 - b_i)(c_0 + c_i)(d_0 - d_i)$$

$$= (a_0 + a_i)(A_0 + A_i)(B_0 - B_i)(C_0 + C_i)(D_0 - D_i)(a_0 - a_i),$$

it follows that $(abcd) = (ABCD)$. Hence the invariant symbolic factors appertaining to the quaternary quantic are of types $a_a, a_b, a_x, x_x, (abcx), (abcd)$.