The Transformation and Division of Elliptic Functions.

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Two important papers, with the title "The Transformation of Elliptic Functions," have been communicated by Professor Felix Klein to the London Mathematical Society, and published in Volumes IX and XI of the *Proceedings*.

Professor Klein has greatly honoured our Society by choosing its *Proceedings* for the publication of such fundamental ideas in his theory—ideas which have subsequently received their fullest development in Klein and Fricke's *Modulfunctionen*.

As the present communication is of the nature of a note or commentary on these two papers of Professor Klein, in Vols. IX and XI, the same title has been adopted; and the object of the present paper is to show how to express the various parameters employed by Klein, Kiepert, Fricke, and others, for a given transformation, explicitly in terms of a *single* parameter.

Thence it is easy to construct numerical cases required in the applications of elliptic functions; two such cases have been worked out in the sequel, and the chief results stated.

Starting with the various modular equations given by Kiepert and Klein, the object of the present paper is to express their parameters in terms of another, in such a manner that it is possible to write down the various division values (*Theil-werthe*) of the second as well as of the first stage (*Stufe*), the parameters of Klein and Kiepert being symmetric functions of these division values; and the method is illustrated at length in its application to the simplest cases.

1. In the second paper, in Vol. XI, a certain quantity is introduced which Klein denotes by z_α; expressed in the Jacobian notation by the Eta-function, this z_α is given by a division value (*Theil-werthe*), in the form

$$(-1)^\alpha z_\alpha = q^{\alpha^2/n} H(2\alpha K'i, q^n), \tag{1}$$

for a transformation of odd order n; and Klein shows (*Math. Ann.*, XIV, XV, and XVII) that the roots of the modular equation, for instance, of order 5, 7, or 11, can be expressed in terms of z_α.

Professor Klein's Note in Vol. XI is a mere statement of the principal results he had then arrived at; the theory is fully developed in Vol. II, Part V, of the *Modulfunctionen* ("*M. F.*"), and it is there shown that these z_α functions satisfy a number of biquadratic relations, which can be derived from the well-known four-part theta-function formula

$$\begin{aligned} &\theta(v+w)\,\theta(v-w)\,\theta(t+u)\,\theta(t-u) \\ +&\theta(w+u)\,\theta(w-u)\,\theta(t+v)\,\theta(t-v) \\ +&\theta(u+v)\,\theta(u-v)\,\theta(t+w)\,\theta(t-w) = 0 \end{aligned} \tag{2}$$

(Brioschi, *Annali di Mathematica*, XII, 1883; XXI, 1893; XXI, 1894; *Rendiconti della R. A. dei Lincei*, 1893).

2. But in working with Halphen's γ_n function, defined in his *Fonctions Elliptiques*, I, p. 102, I have found that the relation connecting Halphen's γ with Klein's z, for a transformation of odd order n, can be written

$$(-1)^\alpha z_\alpha = f^3 x^{-\frac{1}{2}} \lambda^{-\alpha^2/n} \gamma_\alpha, \tag{3}$$

where f is the function employed by Kiepert, defined in his article "Ueber Theilung und Transformation der Elliptischen Functionen," *Math. Ann.*, XXVI, p. 369, this f being connected with the τ employed by Klein by the relation

$$\tau^{n-1} = \Delta^{n-1} f^{24}, \tag{4}$$

where Δ denotes the discriminant, so that

$$\Delta = g_2^3 - 27 g_3^2. \tag{5}$$

We find also that

$$(-1)^\alpha z_\alpha = f^3 e^{-2(\alpha/n)^2 \eta\omega} \sigma \frac{2\alpha\omega}{n} = f^3 \tau \left(\frac{2\alpha\omega}{n}\right) \tag{6}$$

in Kiepert's notation (*Math. Ann.*, XXXIII, p. 7), a different τ function from that employed by Klein, but the reciprocal of Halphen's f function (*F. E.*, III, p. 216).

As for λ, it is a quantity defined by the relations

$$\lambda = \frac{\gamma_{\frac{1}{2}(n+1)}}{\gamma_{\frac{1}{2}(n-1)}}, \quad \lambda^3 = \frac{\gamma_{\frac{1}{2}(n+3)}}{\gamma_{\frac{1}{2}(n-3)}}, \quad \ldots,$$

and generally
$$\lambda^{2p-1} = \frac{\gamma_{\frac{1}{2}(n+2n-1)}}{\gamma_{\frac{1}{2}(n-2p+1)}}, \tag{7}$$

with
$$p = 1, 2, 3, \ldots \tfrac{1}{2}(n-1);$$

and now the relation
$$\gamma_n = 0 \tag{8}$$

is satisfied.

3. Klein's biquadratic relation (9), *M. F.*, II, p. 314, derived from the four-part theta-function formula (2), is now seen to be the equivalent of Halphen's formula (*Fonctions Elliptiques*, I, p. 102)

$$\gamma_{m+n}\gamma_{m-n} = \gamma_{m+1}\gamma_{m-1}\gamma_n^2 - \gamma_{n+1}\gamma_{n-1}\gamma_m^2, \tag{9}$$

or
$$\psi_{m+n}\psi_{m-n} = \psi_{m+1}\psi_{m-1}\psi_n^2 - \psi_{n+1}\psi_{n-1}\psi_m^2, \tag{10}$$

derivable at once from the defining relation

$$\wp nu - \wp mu = \frac{\psi_{m+n}\psi_{m-n}}{\psi_m^2\psi_n^2}$$

$$= x^{\frac{1}{2}}\frac{\gamma_{m+n}\gamma_{m-n}}{\gamma_m^2\gamma_n^2}, \tag{11}$$

and the identity

$$\wp nu - \wp mu = (\wp u - \wp mu) - (\wp u - \wp nu), \tag{12}$$

or of the relation (Halphen, *F. E.*, I, p. 104)

$$\gamma_{m+n}\gamma_{m-n}\gamma_{p+q}\gamma_{p-q} + \gamma_{n+p}\gamma_{n-p}\gamma_{m+q}\gamma_{m-q} + \gamma_{p+m}\gamma_{p-m}\gamma_{n+q}\gamma_{n-q} = 0; \tag{13}$$

of which (9) is a particular case, obtained by putting

$$p = 1 \quad \text{and} \quad q = 0.$$

4. The relation (8) can be treated as the equation of a curve, connecting Halphen's x and y as coordinates; and when x and y can be expressed, rationally or irrationally, as functions of a parameter c, the quantities γ_n and z_n and the roots of the modular equation can be expressed as functions of this parameter.

Moreover, the separate division values, of which Klein and Kiepert's parameters are symmetric functions, can be disentangled from each other and written down.

5. The formulas in the *Modulfunctionen* by which relation (3) can be established are, in the first place, equation (8), *M. F.*, II, p. 282,

$$\frac{(-1)^{a+1}}{\sigma^2_{a,0}} = \wp'_{a,0} \prod_{\lambda=1}^{\frac{1}{2}(n-1)} (\wp_{a,0} - \wp_{\lambda,0}), \tag{14}$$

and equation (1), p. 281,

$$(-1)^a z_a = \sqrt[8]{\frac{\Delta}{\Delta^n}}\, \sigma_{a/n,0}\left(\omega_1, \frac{\omega_2}{n}\right); \tag{15}$$

or, introducing Kiepert's f function, defined in the *Math. Ann.*, XXVI, p. 388, by

$$f^3 = \sqrt[8]{\frac{\Delta}{\Delta^n}} = \frac{(-1)^{\frac{1}{2}(n-1)}}{\prod_{\lambda=1}^{\frac{1}{2}(n-1)} \wp' \frac{2a\varpi}{n}}; \tag{16}$$

then (108) may be written

$$(-1)^a z_a = f^3 \sigma_{a/n,0}\left(\omega_1, \frac{\omega_2}{n}\right). \tag{17}$$

Changing then in formula (14) to the transformed modulus,

$$(-1)^a z_a^n = \frac{f^{3n}}{\sqrt{S_a}\,(s_a - s_1)(s_a - s_2) \dots * \dots (s_a - s_{\frac{1}{2}(n-1)})}, \tag{18}$$

where, with the notation of the article on "Pseudo-Elliptic Integrals," *Proc. Lond. Math. Soc.*, Vol. XXV, p. 200,

$$\wp_{a,0} - \wp_{\lambda,0} = s_a - s_\lambda = -\frac{\psi_{a+\lambda}\psi_{a-\lambda}}{\psi_a^2 \psi_\lambda^2} = -x^{\frac{1}{2}} \frac{\gamma_{a+\lambda}\gamma_{a-\lambda}}{\gamma_a^2 \gamma_\lambda^2}, \tag{19}$$

$$\wp'_{a,0} = \sqrt{S_a} = -\frac{\psi_{2a}}{\psi_a^4} = -x \frac{\gamma_{2a}}{\gamma_a^4}. \tag{20}$$

Thence, from these relations,

$$(-1)^a z_a^n = \frac{\gamma_a^4 \cdot \gamma_a^2 \gamma_1^2 \cdot \gamma_a^2 \gamma_2^2 \dots \gamma_a^2 \gamma_{\frac{1}{2}(n-1)}^2}{x^{\frac{1}{4}n} \gamma_{2a} \cdot \gamma_{a+1}\gamma_{a-1} \cdot \gamma_{a+2}\gamma_{a-2} \dots \gamma_{\frac{1}{2}(n-1)+a}\gamma_{\frac{1}{2}(n-1)-a}}. \tag{21}$$

The numerator N of this expression (21) is given by

$$\begin{aligned} N &= \gamma_a^{n+1} (\gamma_1\gamma_2\gamma_3 \dots \gamma_{a-1})^2 (\gamma_{a+1}\gamma_{a+2} \dots \gamma_{\frac{1}{2}(n-1)})^2 \\ &= \gamma_a^{n-1} (\gamma_1\gamma_2 \dots \gamma_{\frac{1}{2}(n-1)})^2, \end{aligned} \tag{22}$$

while the denominator D is given by

$$\begin{aligned} D = x^{\frac{1}{4}n} \gamma_{2a}\gamma_{a-1}\gamma_{a-2}\gamma_{a-3} \dots \gamma_3\gamma_2\gamma_1 \cdot \gamma_1\gamma_2\gamma_3 \dots \gamma_{a-1}\gamma_a\gamma_{a+1} \\ \dots \gamma_{\frac{1}{2}(n-1)-a} \cdot \gamma_{a+1}\gamma_{a+2} \dots \gamma_{2a-3}\gamma_{2a-2}\gamma_{2a-1}\gamma_{2a+1}\gamma_{2a+2} \dots \gamma_{\frac{1}{2}(n-1)+a} \\ \gamma_{\frac{1}{2}(n-1)-a+1}\gamma_{\frac{1}{2}(n-1)-a+2} \dots \gamma_{\frac{1}{2}(n-1)+a-1}\gamma_{\frac{1}{2}(n-1)+a} \\ = x^{\frac{1}{4}n} (\gamma_1\gamma_2\gamma_3 \dots \gamma_{a-1})^2 \gamma_a (\gamma_{a+1}\gamma_{a+2} \dots \gamma_{\frac{1}{2}(n-1)-a})^2, \end{aligned} \tag{23}$$

and this, from relations (7) and (8), reduces to

$$D = x^{\frac{1}{4}n}(\gamma_1\gamma_2\dots\gamma_{a-1})^2\gamma_a(\gamma_{a+1}\gamma_{a+2}\dots\gamma_{\frac{1}{2}(n-1)-a})^2$$

$$\lambda^{a^2}(\gamma_{\frac{1}{2}(n-1)-a+1}\gamma_{\frac{1}{2}(n-1)-a+2}\dots\gamma_{\frac{1}{2}(n-1)})^2$$

$$= x^{\frac{1}{4}n}\lambda^{a^2}\gamma_a^{-1}(\gamma_1\gamma_2\dots\gamma_{\frac{1}{2}(n-1)})^2; \tag{24}$$

so that, finally,
$$(-1)^a z_a^n = \frac{f^{3n}\gamma_a^n}{x^{\frac{1}{4}n}\lambda^{a^2}}, \tag{25}$$

as in equation (3); and thence Klein's function z_a can be expressed in terms of a single parameter c, when x and y, and therefore γ_a and λ, are given as functions of the parameter c, satisfying Halphen's relation

$$\gamma_n = 0. \tag{26}$$

6. Using the notation σ_a and τ_a to denote $\sigma\left(\frac{2a\omega}{n}\right)$ and $\tau\left(\frac{2a\omega}{n}\right)$, and putting (Halphen, *F. E.*, I, pp. 102 and 198),

$$\gamma_a = \psi_a x^{-\frac{1}{4}(a^2-1)}, \tag{27}$$

$$\psi_a = \frac{\sigma_a}{(\sigma_1)^{a^2}} = \frac{\tau_a}{(\tau_1)^{a^2}}; \tag{28}$$

then (7) may be written

$$\lambda^{2p-1} = \frac{\psi_{\frac{1}{2}(n+2p-1)}}{\psi_{\frac{1}{2}(n-2p+1)}}x^{-\frac{1}{4}n(2p-1)} \tag{29}$$

$$= \frac{\sigma(n+2p-1)\frac{\omega}{n}}{\sigma(n-2p+1)\frac{\omega}{n}}\sigma_1^{-n(2p-1)}x^{-\frac{1}{4}n(2p-1)}$$

$$= e^{(2p-1)(2\eta\omega)/n}\sigma_1^{-(2p-1)n}x^{-\frac{1}{4}(2p-1)n}, \tag{30}$$

in consequence of equation (9), *F. E.*, I, p. 170; so that

$$\lambda = e^{(2\eta\omega)/n}\sigma_1^{-n}x^{-\frac{1}{4}n}, \tag{31}$$

or, in Kiepert's notation,

$$\lambda = \left[\frac{1}{x^{\frac{1}{4}}\tau\left(\frac{2\omega}{n}\right)}\right]^n. \tag{32}$$

Otherwise, writing v for $\frac{2\omega}{n}$,

$$\frac{\sigma(av)}{(\sigma v)^{a^2}} = \psi_a v = x^{\frac{1}{8}(a^2-1)}\gamma_a, \tag{33}$$

$$\frac{\sigma(n-a)v}{(\sigma v)^{(n-a)^2}} = x^{\frac{1}{8}(n-a)^2-\frac{1}{8}}\gamma_{n-a}, \tag{34}$$

and

$$\begin{aligned}\sigma(n-a)v &= \sigma(2\omega-av) \\ &= e^{2\eta(\omega-av)}\sigma(av),\end{aligned} \tag{35}$$

so that

$$\begin{aligned}e^{2\eta(\omega-av)} &= \frac{\sigma(n-a)v}{nav} \\ &= (\sigma v)^{(n-a)^2-a^2} x^{\frac{1}{8}(n-a)^2+\frac{1}{8}a^2}\frac{\gamma_{n-a}}{\gamma_a},\end{aligned} \tag{36}$$

or

$$e^{(2\eta\omega)/n\,(n-2a)} = (\sigma v)^{n(n-2a)} x^{\frac{1}{8}n(n-2a)}\lambda^{n-2a}, \tag{37}$$

$$e^{-(2\eta\omega)/n}(\sigma v)^n = x^{-\frac{1}{8}n}\lambda^{-1}, \tag{38}$$

$$e^{-(2\eta\omega)/n^2}\sigma\frac{2\omega}{n} = x^{-\frac{1}{8}}\lambda^{-1/n}; \tag{39}$$

and Kiepert's function

$$\begin{aligned}\tau\frac{2a\omega}{n} &= e^{-2(a/n)^2\eta\omega}\sigma\frac{2a\omega}{n} = \left(e^{-(2\eta\omega)/n^2}\sigma v\right)^{a^2}\psi_a = x^{-\frac{1}{8}a^2}\lambda^{-a^2/n}x^{\frac{1}{8}(a^2-1)}\gamma_a \\ &= x^{-\frac{1}{8}}\lambda^{-a^2/n}\gamma_a.\end{aligned} \tag{40}$$

7. A number of cases of this relation (8) or (26), for the simplest values of n, have been worked out in my paper on "Pseudo-Elliptic Integrals," *Proc. Lond. Math. Soc.*, Vol. XXV, and by means of these results the roots of the modular equations given by Prof. Klein in *Proc. Lond. Math. Soc.*, Vol. IX, p. 123, can now be expressed in terms of a single parameter.

In his paper "Elliptische Functionen und Gleichungen fünften Grades," *Math. Ann.*, XIV, Professor Klein continues his investigations, and expresses the roots of the modular equations of order $n = 2, 3, 4$, and 5, in terms of a single parameter.

The degree of the modular equation of prime order n being $n+1$, and the roots being denoted by

$$\tau_\infty,\ \tau_0,\ \tau_1,\ \tau_2,\ \dots\ \tau_{n-1},$$

then for $n=2$ (*Math. Ann.*, XIV, p. 153), comparing

$$J=\frac{(4\tau-1)^3}{27\tau}, \tag{41}$$

and

$$J=\frac{4(1-\sigma+\sigma^2)^3}{27\sigma^2(1-\sigma)^2}, \tag{42}$$

the roots, τ_∞, τ_0, τ_1, for a given value of J, are expressed by Klein in terms of σ, the *anharmonic ratio* (*Doppelverhältniss*), or squared Legendrian modulus κ^2, by

$$\left.\begin{aligned}\tau_\infty &= -\frac{(1-\sigma)^2}{4\sigma}\\ \tau_0 &= -\frac{\sigma^2}{4(1-\sigma)}\\ \tau_1 &= \frac{1}{4\sigma(1-\sigma)}\end{aligned}\right\}. \tag{43}$$

For $n=3$, Klein expresses the roots of the modular equation in terms of $x_1 : x_2$, which ratio he calls the *tetrahedron irrationality*, by means of the relation

$$J=\frac{(\tau-1)(9\tau-1)^3}{-64\tau}=-64\frac{(x_1^4-x_1x_2^3)^3}{(8x_1^3x_2+x_2^4)^3}, \tag{44}$$

and he finds

$$\tau_\infty=\frac{x_2^4}{8x_1^3x_2+x_2^4},$$

$$\tau_r=\frac{1}{9}\frac{(2\epsilon^r x_1+x_2)^4}{8x_1^3x_2+x_2^4}, \tag{45}$$

where

$$\epsilon=e^{\frac{2}{3}\pi i},\quad r=0,1,2. \tag{46}$$

For $n=4$, the modular equation

$$J=\frac{(\tau^2+14\tau+1)^3}{108\tau(\tau-1)^4}=\frac{(\eta^8+14\eta^4+1)^3}{108\eta^4(\eta^4-1)^4} \tag{47}$$

gives the six values of τ as

$$\eta^4,\quad \frac{1}{\eta^4},\quad \left(\frac{1\pm\eta}{1\mp\eta}\right)^4,\quad \left(\frac{1\pm\eta i}{1\mp\eta i}\right)^4, \tag{48}$$

in terms of η, called the *octahedron irrationality*; and Legendre's modulus

$$\kappa=\eta^2,\quad \tau=\kappa^2=\eta^4. \tag{49}$$

8. The next important extension is to the case of

$$n = 5,$$

where
$$J = \frac{(r^2-10r+5)^3}{-1728r}, \tag{50}$$

and Klein expresses the six roots of this equation in r in terms of the ratio

$$\eta = \eta_1 : \eta_2, \tag{51}$$

which he calls the *ikosahedron irrationality*, by means of the *ikosahedron form* f, defined by

$$f = \eta_1\eta_2(\eta_1^{10}+11\eta_1^5\eta_2^5-\eta_2^{10}), \tag{52}$$

and the relations
$$r_\infty = \frac{125\eta_1^6\eta_2^6}{f}, \tag{53}$$

$$r_\rho = \frac{(\epsilon^{-\rho}\eta_1^2+\eta_1\eta_2-\epsilon^\rho\eta_2^2)^6}{f}, \tag{54}$$

where
$$\epsilon = e^{\frac{2}{5}\pi i}, \quad \rho = 0, 1, 2, 3, 4. \tag{55}$$

So also for the roots of the quintic resolvent

$$J : J-1 : 1 = (r-3)^3(r^2-11r+64) : r(r^2-10r+45)^2 : -1728, \tag{56}$$

or, putting $r = x^2$, in Brioschi's form,

$$x^5-10x^3+45x+\frac{216g_3}{\surd(-\Delta)} = 0. \tag{57}$$

The values $r = 3, 11, 19$ make $K'/K = \surd 3, \surd(11), \surd(19)$; and lead to interesting numerical results; thus, when $r = 19$,

$$\eta - \frac{1}{\eta} = \tfrac{1}{2}\left[-1+i\surd(19)\right] = -r_\infty^{\frac{1}{2}}.$$

(*Lectures on the Ikosahedron*, p. 60, by Felix Klein, translated by G. G. Morrice, 1888; *Math. Ann.*, XIV, p. 417; *Modulfunctionen*, I, p. 649.)

9. Professor Klein passes on to the solution of the modular equation of order

$$n = 7,$$

in the *Math. Ann.*, XIV, p. 428, and XV, p. 251; also in the *Modulfunctionen*, I, p. 692.

The modular equation (*Proc. Lond. Math. Soc.*, Vol. IX, p. 124) is

$$J = \frac{(\tau^2+13\tau+49)(\tau^2+5\tau+1)^3}{1728\tau}, \tag{58}$$

and Klein shows how the eight roots of this equation may be given in terms of three parameters

$$\lambda,\ \mu,\ \nu, \quad \text{or} \quad z_1,\ z_2,\ z_4,$$

connected by the biquadratic relation

$$\lambda^3\mu+\mu^3\nu+\nu^3\lambda = 0, \tag{59}$$

or

$$z_1^3 z_2 + z_2^3 z_4 + z_4^3 z_1 = 0. \tag{60}$$

In the *Modulfunctionen*, I, p. 701, we must take

$$\lambda,\ \mu,\ \nu = z_4,\ z_2,\ z_1\,; \tag{61}$$

and the biquadratic relation is

$$z_4^3 z_2 + z_2^3 z_1 + z_1^3 z_4 = 0. \tag{62}$$

10. But it was shown in the *Proc. Lond. Math. Soc.*, Vol. XXV, p. 223, that these quantities can be expressed in terms of a single parameter z, there defined, by means of the relations

$$\left.\begin{aligned} \lambda &= z_1 = -z^{\frac{4}{7}}(z-1)^{\frac{2}{7}} \\ \mu &= z_2 = -z^{\frac{2}{7}}(z-1)^{\frac{1}{7}} \\ \nu &= z_4 = z^{\frac{1}{7}}(z-1)^{\frac{4}{7}} \end{aligned}\right\}, \tag{63}$$

satisfying the relations (59) and (60) above.

Halphen's x and y are given in terms of the parameter z by the relations

$$x = z(1-z)^2, \quad y = z(1-z), \tag{64}$$

and thus the relation

$$\gamma_7 = (y-x)\,x - y^3 = 0 \tag{65}$$

is satisfied; also

$$z = \frac{\wp\frac{8}{7}\omega - \wp\frac{4}{7}\omega}{\wp\frac{8}{7}\omega - \wp\frac{2}{7}\omega}. \tag{66}$$

The roots of the modular equation (58) are then shown to be expressible in terms of z by the relations

$$\tau_\infty = -\frac{49z(z-1)}{z^3+5z^2-8z+1}$$

$$= -\frac{49}{z+5+\dfrac{1}{1-z}+\dfrac{z-1}{z}}, \tag{67}$$

$$\tau_r = -\frac{(1+\epsilon^{-r}\alpha_1+\epsilon^{-2r}\alpha_2+\epsilon^{-4r}\alpha_4)^4}{z+5+\dfrac{1}{1-z}+\dfrac{z-1}{z}}, \tag{68}$$

where
$$\epsilon = e^{\frac{2}{7}\pi i}, \quad r = 0, 1, 2, 3, 4, 5, 6, \tag{69}$$

and
$$\alpha_1 = \frac{z_1}{z_4} = -z^{\frac{3}{7}}(z-1)^{-\frac{2}{7}}, \tag{70}$$

$$\alpha_2 = \frac{z_4}{z_2} = -z^{-\frac{1}{7}}(z-1)^{\frac{3}{7}}, \tag{71}$$

$$\alpha_4 = \frac{z_2}{z_1} = \quad z^{-\frac{2}{7}}(z-1)^{-\frac{1}{7}}. \tag{72}$$

Thus the irrationality
$$z^{\frac{3}{7}}(z-1)^{\frac{1}{7}} = \lambda^{\frac{1}{7}}, \tag{73}$$

where
$$\lambda = \frac{\gamma_4}{\gamma_3}, \quad \lambda^3 = \frac{\gamma_5}{\gamma_2} = y-x = z^2(1-z), \tag{74}$$

plays the same part here as the ikosahedron irrationality η in equations (53) and (54).

Also
$$\left.\begin{aligned}
\frac{\alpha_2}{\alpha_1^2} &= -\frac{z-1}{z} = \frac{\wp\frac{4}{7}\omega-\wp\frac{2}{7}\omega}{\wp\frac{8}{7}\omega-\wp\frac{4}{7}\omega} \\
\frac{\alpha_4}{\alpha_2^2} &= \frac{1}{z-1} = -\frac{\wp\frac{8}{7}\omega-\wp\frac{2}{7}\omega}{\wp\frac{4}{7}\omega-\omega\frac{2}{7}\omega} \\
\frac{\alpha_1}{\alpha_4^2} &= -z = -\frac{\wp\frac{8}{7}\omega-\wp\frac{4}{7}\omega}{\wp\frac{8}{7}\omega-\wp\frac{2}{7}\omega}
\end{aligned}\right\}. \tag{75}$$

This parameter z is seen to be intimately bound up with Gierster's *Hauptmodul* M ("Ueber Classenanzahl-relationen," *Math. Ann.*, XVII, p. 81); for
$$M = \frac{\lambda^2\mu}{\nu^3} = -\frac{z}{z-1}, \tag{76}$$

and, as the values of τ are unaltered by the group of substitutions
$$z, \ \frac{1}{1-z}, \ \frac{z-1}{z}, \tag{77}$$

Gierster's M is equivalent to minus the reciprocal of our z.

Put
$$z = -x - \frac{1}{x};$$

then
$$\tau = \frac{1-8z+5z^2+z^3}{z\,(1-z)}$$

$$= \frac{x^6-5x^5-5x^4-11x^3-5x^2-5x+1}{x\,(x^2+1)(x^2+x+1)}, \tag{78}$$

$$\tau+6 = \frac{x^6+x^5+x^4+x^3+x^2+x+1}{x\,(x^2+1)(x^2+x+1)}$$

$$= \frac{x^7-1}{x\,(x^2+1)(x^3-1)}. \tag{79}$$

11. A straightforward algebraical verification of equation (58) by the roots given by (67) and (68) would be very formidable; but meanwhile Mr. T. I. Dewar has performed the verification for the special numerical case corresponding to

$$z = 2,$$

and he finds that this makes

$$\tau_\infty = -\frac{98}{13},$$

and
$$1728J = -8478{\cdot}4438756456985. \tag{80}$$

Also, considering only the real seventh roots,

$$\left.\begin{aligned} \alpha_1 &= -2^{\frac{3}{7}} &&= -1{\cdot}34590019263234 \\ \alpha_2 &= -2^{-\frac{1}{7}} &&= -0{\cdot}90572366426391 \\ \alpha_4 &= 2^{-\frac{2}{7}} &&= +0{\cdot}82033535600764 \end{aligned}\right\}, \tag{81}$$

and these make
$$\tau_0 = -0{\cdot}005323020754, \tag{82}$$

and then
$$1728J = -8478{\cdot}443876. \tag{83}$$

A similar numerical calculation makes

$$\tau_1 = -3{\cdot}555254192736-2{\cdot}712985482813i,$$

and
$$1728J = \frac{30143{\cdot}023137291471+23001{\cdot}895151881873i}{\tau_1}$$

$$= -8478{\cdot}44387579-0{\cdot}000000001688045i, \tag{84}$$

and these three values of J are sufficiently close to serve as an arithmetical verification.

Another numerical case can be constructed by taking

$$z = -2\cos\tfrac{2}{7}\pi, \quad \tau = -6.$$

So also we may verify that the roots of Galois' resolvent of the seventh degree,

$$x^7 + \frac{7+7i\sqrt{7}}{2}\Delta x^4 - \frac{35-7i\sqrt{7}}{2}\Delta^2 x - 12g_3\Delta^2 = 0, \tag{85}$$

are given by

$$x_r = \epsilon^r z_1^2 + \epsilon^{2r} z_2^2 + \epsilon^{4r} z_4^2 - \frac{1+i\sqrt{7}}{2}(\epsilon^{-r} z_1 z_4 + \epsilon^{-2r} z_2 z_1 + \epsilon^{-4r} z_4 z_2),$$

$$r = 1, 2, 3, 4, 5, 6, 7 \tag{86}$$

(Klein, *Math. Ann.*, XIV, pp. 426, 458; *M. F.*, I, p. 754).

12. The highest prime number for which Klein's modular equation is rational in τ is

$$n = 13;$$

and now (*Proc. Lond. Math. Soc.*, Vol. IX, p. 126)

$$J = \frac{(\tau^2+5\tau+13)(\tau^4+7\tau^3+20\tau^2+19\tau+1)^3}{1728\tau}, \tag{87}$$

and J', the transformed absolute invariant, is the same function of τ', where

$$\tau\tau' = 13. \tag{88}$$

It was shown, in the article "Pseudo-Elliptic Integrals," § 50, that the relation

$$\gamma_{13} = 0 \tag{89}$$

can be satisfied when x and y are rational functions of a parameter c and of $\sqrt{C}$, where

$$C = 1+4c+6c^2+2c^3+c^4+2c^5+c^6$$

$$= (1+2c-c^2-c^3)^2+4c^2(1+c)^2, \tag{90}$$

so that the curve of equation (89) has a deficiency 2.

13. Then γ_a can also be expressed as a rational function of c and $\sqrt{C}$, and now we are able, from relation (3) and from the relation

$$\frac{A_a}{A_0} = \frac{z_{2a}}{z_a} \tag{91}$$

(Klein, "Ueber gewisse Theilwerthe der Θ-Function," *Math. Ann.*, XVII, p. 569), to express the fourteen roots of equation (87), the modular equation of the thirteenth order, in terms of c and $\sqrt{C}$, in the form

$$\tau_{\infty} = -\frac{13}{4+c-\frac{1}{1+c}-\frac{1+c}{c}}, \tag{92}$$

$$\tau_r = -\frac{\left(1+\sum_{a=1}^{a=6}\frac{A_a}{A_0}\,\epsilon^{36a^2r}\right)^2}{4+c-\frac{1}{1+c}-\frac{1+c}{c}}, \tag{93}$$

where
$$\epsilon = e^{\frac{2}{13}\pi i}, \quad r = 0,\ 1,\ 2,\ \dots\ 12, \tag{94}$$

the A's being expressions such that A^{13} is a rational function of c and $\sqrt{C}$, and A being taken as the real thirteenth root, the various imaginary roots being obtained by appropriate factors of powers of $e^{\frac{2}{13}\pi i}$ (*Proc. Lond. Math. Soc.*, Vol. xxv, p. 256).

14. In the general case, from (3) and (91),

$$\frac{A_a}{A_0} = (-1)^a\,\lambda^{-(3a^2)/n}\,\frac{\gamma_{2a}}{\gamma_a} = \mu^{a^2}\,\frac{\gamma_{2a}}{\gamma_a}, \tag{95}$$

on putting
$$\lambda^{-3} = -\mu^n, \tag{96}$$

or
$$\left(\frac{A_a}{A_0}\right)^n = (-1)^a\,\lambda^{-3a^2}\left(\frac{\gamma_{2a}}{\gamma_a}\right)^n. \tag{97}$$

15. In the special case of $n = 13$ it was shown (*Proc. Lond. Math. Soc.*, Vol. xxv, pp. 252–255) that the relation (89) is reduced to

$$p^2-(1-c^2-c^3)\,p-c\,(1+c)^2 = 0, \tag{98}$$

by means of the substitutions

$$x = y\,(1-z), \quad y = z-\frac{z^2}{p}, \quad z = c\,(p-1), \tag{99}$$

and then
$$\tau = \frac{1-c-4c^2-c^3}{c\,(1+c)} = -4-c+\frac{1}{1+c}+\frac{1+c}{c}, \tag{100}$$

also
$$\left.\begin{aligned} \lambda &= \frac{\gamma_7}{\gamma_6} = -\frac{yz}{cx^3} \\ \lambda^3 &= \frac{\gamma_8}{\gamma_5} = -y^2z^2\,\frac{1+c}{cp} \end{aligned}\right\}, \tag{101}$$

and
$$p = \frac{\wp 2v-\wp v}{\wp 2v-\wp 5v}, \quad c = \frac{\wp 5v-\wp v}{\wp 3v-\wp v}\,\frac{\wp 3v-\wp 2v}{\wp 2v-\wp 5v}, \quad v = \frac{2\omega}{13}. \tag{102}$$

Denoting $\frac{A_a}{A_0}$ by a_a, then we find

$$a_1^{13} = -\frac{1}{\lambda^8} = \frac{P+Q\sqrt{C}}{2c^7(1+c)^5},$$

$P = 1+4+11+29+70+86+69+84+100+68+35+27+19+7+c^{14}$,

$Q = 1+4+12+10+14+32+29+14+13+13+6+c^{11}$,

$P^2 - Q^2C = -4c(1+c)^{10}$;

$$a_2^{13} = \frac{y^{13}}{\lambda^{12}} = \frac{R+S\sqrt{C}}{2c^3(1+c)^7},$$

$R = 1+7+19+19-9-26+7+26-8-17+8+6-4-1+c^{14}$,

$S = 1+5+8-1-12-1+12-1-7+2+2-c^{11}$,

$R^2 - S^2C = 4c^{17}(1+c)$;

R and S being obtained from P and Q by writing $-\frac{1+c}{c}$ for c;

$$a_3^{13} = -\frac{1}{\lambda^{27}}\left(\frac{yz^2}{p}\right)^{13} = \frac{R-S\sqrt{C}}{2c^3(1+c)^7},$$

$$a_4^{13} = \frac{z^{13}}{\lambda^9} = -\frac{T+U\sqrt{C}}{2c^5(1+c)^2},$$

$T = 1+15+100+388+965+1604+1825+1482+960+581+334$
$+155+50+10+c^{14}$,

$U = 1+13+73+230+443+537+416+216+83+27+7+c^{11}$,

$T^2 - U^2C = 4c^{10}(1+c)^{17}$;

obtained from P and Q by writing $-\frac{1}{1+c}$ for c;

$$a_5^{13} = \frac{\lambda^8}{c^{13}} = \frac{P-Q\sqrt{C}}{2c^7(1+c)^5},$$

$$a_6^{13} = \lambda^9\left(\frac{1+c}{z}\right)^{13} = -\frac{T-U\sqrt{C}}{2c^5(1+c)^2}. \quad (103)$$

Thence $\quad a_1a_5 = -\frac{1}{c}, \quad a_2a_3 = \frac{c}{1+c}, \quad a_4a_6 = 1+c; \quad (104)$

so that a change of $\sqrt{C}$ into $-\sqrt{C}$ interchanges a_1 and a_5, a_2 and a_3, a_4 and a_6, but leaves r unaltered.

So also it will be found that, while r is unaltered,

(i.) the change of c into $-\dfrac{1}{1+c}$ changes

$$\begin{array}{ccc} a_1 & \text{into} & a_4, \\ a_2 & ,, & a_5, \\ a_3 & ,, & a_1, \\ a_4 & ,, & a_2, \\ a_5 & ,, & a_6, \\ a_6 & ,, & a_3; \end{array}$$

(ii.) the change of c into $-\dfrac{1+c}{c}$ changes

$$\begin{array}{ccc} a_1 & \text{into} & a_3, \\ a_2 & ,, & a_6, \\ a_3 & ,, & a_4, \\ a_4 & ,, & a_1, \\ a_5 & ,, & a_2, \\ a_6 & ,, & a_5. \end{array}$$

16. Various other relations connecting the a's can be written down, thus:—

$$a_1^3 a_6 = -\frac{1+c}{z},$$

with similar expressions for

$$a_6^3 a_3, \quad a_3^3 a_5, \quad a_5^3 a_4, \quad a_4^3 a_2, \quad a_2^3 a_1; \tag{105}$$

$$a\ a_3 = -\frac{yz^3}{\lambda^3 p} = \frac{c}{y(1+c)},$$

with similar expressions for $a_3^4 a_4$ and $a_4^4 a_1$; (106)

$$a_2^4 a_6 = \frac{y^4(1+c)}{\lambda^3 z},$$

with similar expressions for $a_6^4 a_5$ and $a_5^4 a_2$; (107)

$$a_1^3 a_3 = \frac{1}{\lambda^9}\left(\frac{yz^2}{p}\right)^4,$$

with similar expressions for $a_3^3 a_4^4$ and $a_4^3 a_1$; (108)

$$a_2^3 a_6^3 = \frac{y^3(1+c)^4}{z^4},$$

with similar expressions for $a_6^3 a_5^4$ and $a_5^3 a_2^4$; (109)

$$\frac{a_1^5 a_2}{a_3} = \frac{p}{z^2}$$

with similar expressions for $\frac{a_2^5 a_4}{a_6}, \frac{a_3^5 a_6}{a_4}, \frac{a_4^5 a_5}{a_1}, \frac{a_5^5 a_3}{a_2}, \frac{a_6^5 a_1}{a_5}$; (110)

and so on.

These relations are important in showing that the irrationality of the thirteenth root may be taken once for all.

17. A direct algebraical verification of equation (87) by the roots given in (92) and (93) would be a task still more formidable than that required for the corresponding case of $n = 7$; but here again Mr. T. I. Dewar has performed the numerical verification for the special case obtained by taking in (90)

$$1+2c-c^2-c^3 = 0, \tag{111}$$

thus making

$$\sqrt{C} = 2c+2c^2. \tag{112}$$

Taking the positive value of $\sqrt{C}$, this makes

$$\begin{aligned} p &= c^2, \\ z &= -c(1-c^2) = \frac{c^3}{1+c}, \\ y &= \frac{c^3}{(1+c)^2}, \\ \lambda^3 &= \frac{c^7}{(1+c)^5}, \end{aligned} \tag{113}$$

and

$$\left.\begin{aligned} a_1^{13} &= \frac{(1+c)^5}{c^7} \\ a_2^{13} &= \frac{c^{11}}{(1+c)^6} \\ a_3^{13} &= \frac{c^2}{(1+c)^7} \\ a_4^{13} &= -c^5(1+c)^2 \\ a_5^{13} &= -\frac{1}{c^6(1+c)^5} \\ a_6^{13} &= -\frac{(1+c)^{11}}{c^5} \end{aligned}\right\}; \tag{114}$$

while the negative value of $\sqrt{C}$ would merely permute

a_1 and a_5, a_2 and a_3, a_4 and a_6.

Writing equation (100) in the form

$$\tau = \frac{1+2c-c^2-c^3}{c(1+c)} - 3, \tag{115}$$

then the roots of (111) make

$$\tau = -3, \quad \tau_\infty = -\frac{13}{3}, \tag{116}$$

and this value of τ_∞, substituted in (87), makes

$$1728 J_\infty = -\frac{2^{12}\times 7\times 17^3 \times 23^3}{3^{13}}$$

$$= -1075008{\cdot}6252986. \tag{117}$$

The three roots of (111) are

$$2\cos\frac{2\pi}{7},\quad 2\cos\frac{4\pi}{7},\quad 2\cos\frac{6\pi}{7}; \tag{118}$$

and we can thus put

$$c = 2\cos\frac{2\pi}{7}$$

$$= 1{\cdot}2469796037217467,$$

$$\left.\begin{aligned} -\frac{1}{1+c} &= 2\cos\frac{4\pi}{7} \\ -\frac{1+c}{c} &= 2\cos\frac{6\pi}{7} \end{aligned}\right\}. \tag{119}$$

Mr. Dewar now calculates

$$\left.\begin{aligned} a_1 &= 1{\cdot}212310485995257 \\ a_2 &= 0{\cdot}829535995876051 \\ a_3 &= 0{\cdot}668998253055627 \\ -a_4 &= 1{\cdot}232994543430555 \\ -a_5 &= 0{\cdot}661495338916045 \\ -a_6 &= 1{\cdot}822375951044485 \end{aligned}\right\}, \tag{120}$$

and thence $\tau_0 = -\frac{1}{3}(1+a_1+a_2+a_3+a_4+a_5+a_6)^2$

$$= -\tfrac{1}{3}(-0\cdot006021232698252)^2$$

$$= -0\cdot000012084542226\,; \qquad (121)$$

$$1728J_0 = \frac{13}{\tau_0} + 746 + 15145\tau_0 + 124852\tau_0^2 + \ldots$$

$$= -1075008\cdot6257122, \qquad (122)$$

a close agreement with the result of (117).

18. Mr. Dewar has gone on further to the calculation of the imaginary root τ_1, and the corresponding value of J; and he finds in a similar manner, from (93),

$$\tau_r = -\tfrac{1}{3}(1+\epsilon^{36r}\alpha_1+\epsilon^{2^2.36r}\alpha_2+\ldots+\epsilon^{6^2.36r}\alpha_6)^2,$$

$$\tau_1 = -\tfrac{1}{3}(1+\epsilon^{10}\alpha_1+\epsilon\alpha_2+\epsilon^{12}\alpha_3+\epsilon^4\alpha_4+\epsilon^8\alpha_5+\epsilon^9\alpha_6)^2$$

$$= -3\cdot521253401250+2\cdot86124793802i, \qquad (123)$$

$$\tau_1^2 = +4\cdot212485752989-20\cdot150358067145i, \qquad (124)$$

$$\tau_1^3 = +42\cdot821940684550+83\cdot007483054998i, \qquad (125)$$

$$\tau_1^4 = -388\cdot291894015211-169\cdot766192550990i, \qquad (126)$$

$$\tau_1^2+5\tau_1+13 = -0\cdot3943781253261-5\cdot844118377045i, \qquad (127)$$

$$\tau_1^4+7\tau_1^3+20\tau_1^2+19\tau_1+1$$

$$= -70\cdot1902408787331+62\cdot642738313476i, \qquad (128)$$

and substituting these values in (87),

$$1728J_1 = \frac{3785377\cdot7728290-3075866\cdot2102691i}{-3\cdot52125340125+2\cdot86124793802i}$$

$$= -1075008\cdot6257-0\cdot00036754i, \qquad (129)$$

a close agreement with (117) and (122).

Mr. Dewar employed in these calculations the new multiplying machine invented by Mr. Macfarlane Gray, which is capable of multiplying together two numbers of sixteen figures.

19. The special case of $n = 9$

may be considered at this stage; this is the case receiving particular attention in Joubert's memoir "Sur les équations qui se rencontrent dans la théorie de la transformation des fonctions elliptiques," Paris, 1876.

The relation $\gamma_0 = 0$,

or
$$y^3(y-x-y^2)-(y-x)^3 = 0, \tag{130}$$

is satisfied by putting

$$x = p^2(1-p)(1-p+p^2), \quad y = p^3(1-p), \tag{131}$$

so that the curve (130) is unicursal; and now

$$\lambda = \frac{\gamma_5}{\gamma_4} = 1-\frac{x}{y} = p-p^2. \tag{132}$$

With these values,
$$\left.\begin{aligned} a_1^9 &= -\frac{1}{p^3(1-p)^3} \\ a_2^9 &= \frac{p^6}{(1-p)^3} \\ a_3^9 &= -1 \\ a_4^9 &= \frac{(1-p)^6}{p^3} \end{aligned}\right\}, \tag{133}$$

and
$$a_1 a_2 a_3 a_4 = 1. \tag{134}$$

Changing p into $\dfrac{1}{1-p}$ changes

$$a_1 \text{ into } a_4, \quad a_4 \text{ into } a_3, \quad a_2 \text{ into } a_1;$$

and changing p into $-\dfrac{1-p}{p}$ changes

$$a_1 \text{ into } a_2, \quad a_2 \text{ into } a_4, \quad a_4 \text{ into } a_1;$$

also
$$a_1+a_2+a_4 = 0. \tag{135}$$

As stated previously, we need only consider the real ninth roots of a^9.

20. It was shown (*Proc. Lond. Math. Soc.*, Vol. XXV, p. 233) that

$$-3\tau = \frac{1-6p+3p^2+p^3}{p(1-p)}; \tag{136}$$

and therefore, to the complementary modulus,

$$\tau_\infty = \frac{3}{\tau} = \frac{-9p(1-p)}{1-6p+3p^2+p^3}, \tag{137}$$

and now all the twelve roots of Gierster's modular equation

$$J = \frac{(\tau-1)^3\{9(\tau-1)^3+8\}^3}{-64\tau\{(\tau-1)^3+1\}} \tag{138}$$

can be expressed in terms of the parameter p.

The τ employed here, distinguished as τ_9, is connected with the τ or τ_3 employed in the modular equation of the third order by the relation

$$(\tau_9-1)^3 = \tau_3-1,$$

or

$$\tau_9 = 1+\omega(\tau_3-1)^{\frac{1}{3}}, \quad \omega^3 = 1, \tag{139}$$

and thence the twelve values of τ_9 can be inferred from the four values of τ_3.

If a denotes the tetrahedron irrationality $x_1 : x_2$ in § 7,

$$\tau_3 = \frac{1}{8a^3+1}, \quad \tau_3' = 8a^3+1,$$

$$\tau_9'-1 = (\tau_3'-1)^{\frac{1}{3}} = 2a, \quad \tau_9' = 2a+1. \tag{140}$$

For instance, in the Transformation of the Nineteenth Order (Fricke, *Math. Ann.*, XL)

$$19\tau = -x, \quad 19\tau' = x',$$

$$x'^2 = 4x^3+(8x+19)^2,$$

$$\tau_{3,\infty} = \frac{1}{513}, \quad a = 4, \quad \tau_{3,\infty}' = 513;$$

$$\tau_{3,0} = \frac{27}{19}, \quad \tau_{3,0}' = \frac{19}{27},$$

$$\tau_{9,0}' = 1+\left(-\frac{8}{27}\right)^{\frac{1}{3}} = \tfrac{1}{3}, \; a = -\tfrac{1}{3}, \; \tau_{9,0} = 9.$$

With
$$\sqrt[3]{J} = -\frac{2^4 . 7}{3 . 19},$$
$$a = 4, \quad \text{or} \quad -\tfrac{1}{3},$$
or
$$19a^2 - 5a + 7 = 0, \quad a = \frac{5 + 13i\sqrt{3}}{38};$$
and
$$r_9 = 9, \quad r_9' = \tfrac{1}{3}$$
give the same value of J, and make
$$M = 1, \quad M' = \tfrac{1}{9},$$
and thus correspond to a multiplication by 3.

So also, for $n = 27$, and $n = 81, 243, \ldots$ (*Math. Ann.*, XXXII, p. 67),

$$\eta - 3 = \xi_3 = \frac{1 - 6p + 3p + p^3}{p(1-p)}, \quad \xi_1 = \frac{p^3(1-p)^3}{1 - p + p^2},$$
$$\xi_5 = \frac{1 - 6p + 3p^2 + p^3}{(-p + p^2)(p - p^2)^{\frac{1}{3}}},$$
$$\xi_2 = \frac{1 + 3p - 6p^2 + p^3 - 3(1 - p + p^2)(p - p^2)^{\frac{1}{3}}}{1 - 6p + 3p^2 + p^3},$$
$$\xi_4 = \frac{(1 - p + p^2)(p - p^2)^{\frac{1}{3}}}{1 + 3p - 6p^2 + p^3 - 3(1 - p + p^2)(p - p^2)^{\frac{1}{3}}},$$
$$w = \frac{(1+p)(2-p)(1-2p)(p-p^2)^{\frac{1}{3}}}{(1 - p + p^2)^2}.$$

21. The expression of x, y, and γ_a as functions of a single parameter has been given in the paper on "Pseudo-Elliptic Integrals," *Proc. Lond. Math. Soc.*, Vol. XXV, for the odd numbers

3, 5, 7, 9, 11, 13, and 15.

As a verification, let us examine again the three simplest cases of
$$n = 3, 5, \text{ and } 7.$$

For $n = 3$, the single z function is Dedekind's $\eta(\omega)$ (Brioschi, *Annali di Mathematica*, XII, 1883).

For
$$n = 5, \quad \text{and} \quad \gamma_5 = 0,$$
$$y = x,$$
and
$$\lambda = \frac{\gamma_3}{\gamma_2} = x^{\frac{1}{5}}, \quad f^{-2} = x^2;$$
$$z_1 = -f^3 x^{-\frac{1}{5}} \lambda^{-\frac{2}{5}} = -x^{-\frac{11}{5}},$$
$$z_2 = f^3 x^{-\frac{1}{5}} \lambda^{-\frac{2}{5}} = x^{-\frac{11}{5}};$$
so that
$$\frac{z_1}{z_2} = -\lambda^{\frac{5}{5}} = -x^{\frac{1}{5}}, \tag{141}$$
the ikosahedron irrationality (*Proc. L. M. S.*, XXV, p. 215).

For
$$n = 7 \quad \text{and} \quad \gamma_7 = 0,$$
$$x = z(1-z)^2, \quad y = z(1-z),$$
$$\lambda^3 = \frac{\gamma_5}{\gamma_3} = y - x = z^2(1-z); \tag{142}$$

and
$$\left.\begin{aligned} \left(\frac{A_1}{A_0}\right)^7 &= \left(\frac{z_2}{z_1}\right)^7 = -\lambda^{-3} = -z^{-2}(1-z)^{-1} = a_4^7 \\ \left(\frac{A_2}{A_0}\right)^7 &= \left(\frac{z_4}{z_2}\right)^7 = \lambda^{-12}\gamma_4^7 = z^{-1}(1-z)^{-3} = a_2^7 \\ \left(\frac{A_4}{A_0}\right)^7 &= -\left(\frac{z_1}{z_4}\right)^7 = -\lambda^{15}\gamma_4^{-7} = -z^3(1-z)^{-2} = a_1^7 \end{aligned}\right\}, \tag{143}$$

employing the relations (70), (71), (72).

The case of $n = 13$ has already received a full discussion in §§ 12–18.

22. The case of
$$n = 11$$
is important, as being the earliest number for which the relation
$$f(J, J') = 0, \tag{144}$$
connecting the absolute invariant J and its transformed value J', considered as the equation of a curve, is no longer unicursal, but has a deficiency
$$p = 1.$$

The equation connecting J with the parameter η employed by Kiepert (*Math. Ann.*, XXXIII, p. 97), or with Klein's parameter τ (*Modulfunctionen*, II, p. 440), is, when rationalized, a quadratic in J, and of the twelfth degree in η or τ; and these are connected by the relation
$$\eta + 8 = \frac{1}{\tau}. \tag{145}$$

The relation
$$\gamma_{11} = 0 \tag{146}$$
is reduced to
$$z(1-z) = p^2(1-p),$$
or
$$2z = 1 + \sqrt{\{4p^2(p-1)+1\}}, \tag{147}$$
by the substitutions
$$x = y(1-z), \quad y = z\left(1 - \frac{z}{p}\right);$$

so that
$$\lambda = \frac{\gamma_6}{\gamma_5} = x^{\frac{1}{3}}\frac{y-x-y^2}{y-x} = x^{\frac{1}{3}}\frac{z}{p},$$
$$\lambda^3 = \frac{\gamma_7}{\gamma_4} = \frac{(y-x)\,x-y^3}{y} = -yz^2\frac{p-1}{p}, \tag{148}$$
and to agree with the notation employed in "Pseudo-Elliptic Integrals," p. 241, we must put
$$p = 1+c, \quad z = -q,$$
so that
$$\left.\begin{aligned} q\,(q+1) &= c\,(1+c)^2 \\ 2q+1 &= \sqrt{C} \\ C &= 4c\,(c+1)^2+1 \end{aligned}\right\}. \tag{149}$$

23. Kiepert's f is given by ("Pseudo-Elliptic Integrals," p. 243)
$$f^{-2} = \frac{x^{\frac{10}{3}}\lambda^{10}}{(\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5)^2} = \frac{x^6 z^{10}}{y^4 z^2 p^{10}}, \tag{150}$$
and now, written in the order employed in the *M. F.*, II, p. 403,
$$\left.\begin{aligned} z_1 &= -f^3 x^{-\frac{1}{3}}\lambda^{-\frac{1}{11}}\gamma_1 = -f^3 x^{-\frac{1}{3}}\lambda^{-\frac{1}{11}} \\ z_3 &= -f^3 x^{-\frac{1}{3}}\lambda^{-\frac{9}{11}}\gamma_3 = -f^3 x^{-\frac{1}{3}}\lambda^{-\frac{9}{11}} x^{\frac{1}{3}} \\ z_9 &= -f^3 x^{-\frac{1}{3}}\lambda^{-\frac{81}{11}}\gamma_9 = -f^3 x^{-\frac{1}{3}}\lambda^{-\frac{4}{11}} \\ z_5 &= -f^3 x^{-\frac{1}{3}}\lambda^{-\frac{25}{11}}\gamma_5 = -f^3 x^{-\frac{1}{3}}\lambda^{-\frac{25}{11}} yz \\ z_4 &= f^3 x^{-\frac{1}{3}}\lambda^{-\frac{16}{11}}\gamma_4 = f^3 x^{-\frac{1}{3}}\lambda^{-\frac{16}{11}} y \end{aligned}\right\}. \tag{151}$$

Thence various relations ensue, which are independent of the eleventh root of λ; for instance,
$$\left.\begin{aligned} s_a - s_\lambda &= -f^6\,\frac{z_{a+\lambda}z_{a-\lambda}}{z_a^2 z_\lambda^2}, \\ \sqrt{S_a} &= -f^9\,\frac{z_{2a}}{z_a^4}; \end{aligned}\right\} \tag{152}$$
which is true for all values of n; and
$$\left.\begin{aligned} z_a^2 z_{3a} &= -\frac{f^9}{x\lambda^{a^2}}\gamma_a^2\gamma_{3a}, \\ \frac{z_{5a}}{z_a^3} &= \frac{x^{\frac{1}{3}}}{f^6\lambda^{2a^2}}\,\frac{\gamma_{5a}}{\gamma_a^3}, \quad \text{\&c.}; \end{aligned}\right\} \tag{153}$$
$$\left.\begin{aligned} z_a^3 z_{5a} + z_{9a}^3 z_{4a} + z_{3a}^3 z_a &= 0, \\ z_a^2 z_{5a} z_{9a} + z_a z_{3a} z_{4a}^2 - z_{3a} z_{5a} z_{9a}^2 &= 0; \end{aligned}\right\} \tag{154}$$
(*Math. Ann.*, XVII, p. 567; *M. F.*, II, p. 409; Brioschi, *Annali*, XXI).

Also the invariant of the third order (*Modulfunctionen*, II, p. 410)

$$\Phi(z_a) = z_1^2 z_3 + z_3^2 z_9 + z_9^2 z_5 + z_5^2 z_4 + z_4^2 z_1$$

$$= \frac{f^9}{x}\left(-\frac{x^4}{\lambda} - \frac{x^3}{\lambda^2} - \frac{yz}{\lambda^3} + \frac{y^3z^3}{\lambda^6} - \frac{y^3}{\lambda^3}\right)$$

$$= \Delta f^{12}, \tag{155}$$

after reduction; and the invariant $\Psi(z_a)$ on p. 411 can also be expressed as a rational function of x, y, z, p, f, and λ; and thence as a function of the parameter p or c; but

$$\Psi(z_a) = 0.$$

So also the roots of Galois' resolvent of the Eleventh degree can be written down in terms of c; this resolvent is

$$J : J-1 : 1$$

$$= \left\{\tau^2+3\tau+5-i\sqrt{(11)}\right\}\left\{\tau^3-\tau^2-\frac{3+3i\sqrt{(11)}}{2}\tau-\frac{7-i\sqrt{(11)}}{2}\right\}^3$$

$$: \left\{\tau^3-4\tau^2+\frac{7-5i\sqrt{(11)}}{2}\tau-4+6i\sqrt{(11)}\right\}$$

$$\times\left\{\tau^4+2\tau^3+\frac{3-3i\sqrt{(11)}}{2}\tau^2-\left[5+i\sqrt{(11)}\right]\tau-\frac{15+3i\sqrt{(11)}}{2}\right\}^2$$

$$: 1728; \tag{156}$$

or, putting $x^3 = \Delta\left\{\tau^2+3\tau+5-i\sqrt{(11)}\right\}$,

so that $12g_2 = x\left\{\tau^3-\tau^2-\frac{3+3i\sqrt{(11)}}{2}\tau-\frac{7-i\sqrt{(11)}}{2}\right\}$,

the elimination of τ leads to the resolvent in the form

$$x^{11}-22\Delta x^8+11\left\{9+2i\sqrt{(11)}\right\}\Delta^2x^5-11\,.\,12g_2\Delta^2x^4+88i\sqrt{(11)}\,\Delta^3x^2$$

$$+11\left\{3-i\sqrt{(11)}\right\}6g_2\Delta^3x-144g_2^2\Delta^3=0, \tag{157}$$

the roots of which are given by (*Modulfunctionen*, II, p. 428)

$$4x_r = \sum_{a=1}^{a=5} e^{ra^2}\left(z_a^2-z_{3a}z_{9a}\right)-\tfrac{1}{2}\left\{1-i\sqrt{(11)}\right\}\Sigma e^{-ra^2}z_{4a}z_{9a}. \tag{158}$$

24. Again, as before,

$$\left.\begin{aligned} a_1^{11} &= \left(\frac{A_1}{A_0}\right)^{11} = \left(\frac{z_2}{z_1}\right)^{11} = -\left(\frac{z_0}{z_1}\right)^{11} = -\frac{1}{\lambda^5} \\ a_8^{11} &= \left(\frac{A_8}{A_0}\right)^{11} = \left(\frac{z_6}{z_8}\right)^{11} = -\left(\frac{z_5}{z_8}\right)^{11} = -\frac{y^{11}z^{11}}{\lambda^{16}x^{\frac{11}{2}}} \\ a_9^{11} &= \left(\frac{A_9}{A_0}\right)^{11} = \left(\frac{z_{18}}{z_9}\right)^{11} = -\left(\frac{z_4}{z_9}\right)^{11} = -\frac{y^{11}}{\lambda^{13}} \\ a_5^{11} &= \left(\frac{A_5}{A_0}\right)^{11} = \left(\frac{z_{10}}{z_5}\right)^{11} = -\left(\frac{z_1}{z_5}\right)^{11} = -\frac{\lambda^{24}}{y^{11}z^{11}} \\ a_4^{11} &= \left(\frac{A_4}{A_0}\right)^{11} = \left(\frac{z_8}{z_4}\right)^{11} = -\left(\frac{z_3}{z_4}\right)^{11} = +\frac{\lambda^7 x^{\frac{11}{2}}}{y^{11}} \end{aligned}\right\} \qquad (159)$$

Thence various relations ensue, such as

$$\frac{a_3 a_5}{a_1} = -\frac{z_1^2}{z_8 z_9} = \frac{\lambda}{x^{\frac{1}{2}}} = \frac{z}{p},$$

with similar relations for $\frac{a_9 a_4}{a_3}$, $\frac{a_5 a_1}{a_9}$, $\frac{a_4 a_3}{a_5}$, $\frac{a_1 a_9}{a_4}$; (160)

$$a_1 a_4^2 = -\frac{\lambda x^{\frac{1}{2}}}{y^2} = -\frac{xz}{py^2} = -\frac{1-z}{p-z},$$

with similar relations for $a_3 a_1^2$, $a_9 a_8^2$, $a_5 a_9^2$, $a_4 a_5^2$; (161)

$$\frac{a_5}{a_1^3} = \frac{\lambda^8}{yz} = -z\frac{p-1}{p},$$

with similar relations for $\frac{a_4}{a_8^3}$, $\frac{a_1}{a_9^3}$, $\frac{a_3}{a_5^3}$, $\frac{a_9}{a_4^3}$. (162)

These, and other similar relations, show that one eleventh root will serve for the system; and thereby the appropriate power of $e^{\frac{1}{11}\pi i}$ is settled in the case of the imaginary roots of the modular equation (Klein, *Math. Ann.*, XVII, p. 567).

25. It was shown ("Pseudo-Elliptic Integrals," p. 245) that Klein's τ is connected with the c employed above by the relation

$$\frac{1-10\tau+\tau'}{2\tau^2} = H = \frac{1+4c+2c^2-5c^3-2c^4+c^5}{c^2(1+c)^2}, \tag{163}$$

so that

$$\eta+8 = \frac{1}{\tau} = \frac{10H+11+H'}{2(H-11)}$$

$$= \frac{c(1+c)(10+40c+31c^2-28c^3-9c^4+10c^5) + (2+8c+12c^2+9c^3-c^4-3c^5-c^6)\sqrt{C}}{2c(1+c)(1+4c-9c^2-27c^3-13c^4+c^5)}, \tag{164}$$

$$\tau = \frac{-10H-11+H'}{2H^2}$$

$$= \frac{-c^2(1+c)^2(10+40c+31c^2-28c^3-9c^4+10c^5) + c(1+c)(2+8c+12c^2+9c^3-c^4-3c^5-c^6)\sqrt{C}}{2(1+4c+2c^2-5c^3-2c^4+c^5)^2}, \tag{165}$$

where $$C = 1+4c(1+c)^2, \tag{166}$$

and $$H'^2 = 4H^2(H-11)+(10H+11)^2. \tag{167}$$

Also, we shall find that

$$\frac{d\tau}{\tau'} = \frac{dH}{H'} = \frac{dc}{\sqrt{C}} = \frac{2d\xi}{w}, \tag{168}$$

where $$\frac{1}{\tau} = \xi^2+4\xi+8+\frac{4}{\xi},$$

$$w^2 = (\xi^3+4\xi^2+8\xi+4)(\xi^3+8\xi^2+16\xi+16), \tag{169}$$

so that the above relation (163) is a quintic transformation of the elliptic functions obtained by putting $\tau_5 = 11$, or the ikosahedron irrationality $\eta = 1$, in the Transformation of the Fifth Order.

26. We can put $$c = \wp(u;\, J;\, \omega, \omega')-\tfrac{2}{3}, \tag{170}$$

where $$g_2 = \frac{4}{3},\quad g_3 = -\frac{19}{27},\quad 1728J = -\frac{2^{12}}{11}, \tag{171}$$

and then H and τ will be given by

$$H = \wp\left(u;\, J';\, \frac{\omega}{5},\, \omega'\right)-\frac{14}{3},\quad J' = -\frac{2^6.31^3}{3^3.11^5}, \tag{172}$$

$$11\tau = -\wp\left(u-\frac{2\omega}{25}\right)+\frac{14}{3}. \tag{173}$$

Thus, if we put $$a = \wp\left(u - \frac{2\omega}{25}\right) - \frac{2}{3}, \tag{174}$$

we can write $$r = -\frac{1+4a+2a^2-5a^3-2a^4+a^5}{11a^2(1+a)^2}, \tag{175}$$

and, for any given value of u, r, and H, the five roots of the quintic in c or a will correspond to the group of arguments

$$u, \quad u \pm \tfrac{2}{5}\omega, \quad u \pm \tfrac{4}{5}\omega. \tag{176}$$

27. Putting $$c = -\frac{x}{x^2+x+1} \tag{177}$$

makes $$H = \frac{x^{11}-1}{x^2(x^2+1)^2(x^3-1)}; \tag{178}$$

and therefore the roots of $$H = 0 \tag{179}$$
are given by

$$c_p = -\frac{1}{1+2\cos\dfrac{2p\pi}{11}}, \quad p = 1, 2, 3, 4, 5. \tag{180}$$

The roots of $$H - 11 = 0 \tag{181}$$
correspond to the duplication of the argument in c_p, so that, denoting them by b_p,

$$b_p = \frac{(c_p+1)(c_p^3-c_p^2-c_p-1)}{C_p}; \tag{182}$$

and the substitution $$c = \frac{(b+1)(b^3-b^2-b-1)}{B} \tag{183}$$

reproduces the roots c_p in the order c_5, c_1, c_4, c_2, c_3. (184)

This has been verified numerically by Mr. T. I. Dewar; he finds

$$\left.\begin{array}{ll} c_1 = -0{\cdot}37278, & b_1 = -\ \ 1{\cdot}241098 \\ c_2 = -0{\cdot}54620, & b_2 = -\ \ 0{\cdot}754925 \\ c_3 = -1{\cdot}39788, & b_3 = +\ 14{\cdot}856874 \\ c_4 = +3{\cdot}22871, & b_4 = +\ \ 0{\cdot}346486 \\ c_5 = +1{\cdot}08815, & b_5 = -\ \ 0{\cdot}207337 \end{array}\right\}; \tag{185}$$

we can thus take the correspondence

$$c = b_3,\ b_4,\ b_5,\ b_4,\ 0,\ b_5,\ c_1,\ c_3,\ b_2,\ -1,\ b_1,\ c_3 \tag{186}$$
to the arguments

$$u = (2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20,\ 22,\ 24)\frac{\omega}{25}, \tag{187}$$

where $$c = \wp u - \tfrac{2}{3}. \tag{188}$$

28. The corresponding Transformation of the Twenty-fifth Order

$$P=\frac{1}{M^2}\frac{(c,1)^{25}}{c^2(c+1)^2(c^5-2c^4-5c^3+2c^2+4c+1)^2 \times (c^5-13c^4-27c^3-9c^2+4c+1)^2} \tag{189}$$

will be given by taking Gierster's (*Math. Ann.*, XIV, p. 543)

$$\tau_{25}=1, \quad M=\frac{1}{25}, \tag{190}$$

equivalent to two quintic transformations, with

$$\tau_5=11, \quad \tau_5'=\frac{125}{11},$$

$$\tau_5=\frac{1}{11}, \quad \tau_5'=1375;$$

the first being that given in (163), and the second

$$P=H+\frac{14}{3}+\frac{11^4}{(H-11)^2}+\frac{12.11^2}{H-11}+\frac{11^2}{H^2}+\frac{10.11}{H},$$

with

$$P'^2=4P^3-G_2P-G_3,$$

$$G_2=\frac{4\times 29\times 809}{3}, \quad G_3=\frac{61\times 471281}{27}. \tag{191}$$

It is curious that the special pseudo-elliptic integral considered by Abel (*Œuvres*, I, p. 142),

$$\int\frac{5x-1}{\surd\{(x^2+1)^2-4x\}}dx$$
$$=2\,\mathrm{ch}^{-1}\tfrac{1}{2}x\surd(x^3+x^2+3x-1)$$
$$=2\,\mathrm{sh}^{-1}\tfrac{1}{2}(x^2+x+2)\surd(x-1),$$

introduces elliptic functions of the same modulus and invariants, as is seen when we substitute

$$x-1=1/c.$$

Also, in § 8, Klein's ikosahedron irrationality $\eta=1$, and

$$\tau=11, \quad \tau_\infty=\frac{125}{11}, \quad \tau_0=\frac{1}{11}, \quad \tau=\frac{64}{11}.$$

29. To change from the argument u to the argument

$$v = u - \frac{2r\omega}{25}, \tag{192}$$

we must take

$$c = \frac{2ab\,(a+b)+8ab+2\,(a+b)+1-\surd A\,\surd B}{2\,(a-b)^2}, \tag{193}$$

$$\surd C = \frac{\{(3b^2+4b+1)\,a+b^3+4b^2+3b+1\}\,\surd A - \{(3a^2+4a+1)\,b+a^3+4a^2+3a+1\}\,\surd B}{(a-b)^3}, \tag{194}$$

where $$b_r = \wp\frac{2r\omega}{25} - \frac{2}{3} = -\frac{1}{1+2\cos\dfrac{2r\pi}{11}}, \tag{195}$$

and then $$\surd B = \surd\{4b\,(b+1)^2+1\}$$

$$= \frac{2b_r^4-8b_r^2-6b_r-1}{2b_r+1}$$

$$= -2b_r^2 b_{2r} - 4b_r - 1; \tag{196}$$

also $$b_r = b, \quad b_{4r} = \frac{b^4}{(2b+1)(2b^2-2b-1)},$$

$$b_{5r} = -\frac{1}{b} - \frac{1}{b+1}, \text{ \&c.,} \tag{197}$$

and now τ assumes the form in equation (175); and the division values (*Theilwerthe*)

$$\wp\,(1,\,2,\,3,\,4,\,5)\,\frac{2\omega_3}{11}$$

will presumably assume more symmetrical forms than those given in "Pseudo-Elliptic Integrals," p. 243.

30. The next odd number for which the deficiency of Kiepert's or Klein's modular equation is not zero is

$$n = 15;$$

and the division values in this case have been worked out, in terms of a single parameter c, on p. 260 of "Pseudo-Elliptic Integrals."

Equation (97) can now be written

$$a_a^5 = \left(\frac{A_a}{A_0}\right)^5 = (-1)^a \lambda^{-a^2} \left(\frac{\gamma_{2a}}{\gamma_a}\right)^5, \tag{198}$$

and $\lambda = \dfrac{\gamma_8}{\gamma_7} = (c+1)\, y$

$$= -\frac{c\,(c+1)^2}{2\,(c^2+3c+3)} \left\{(c^2+3c+3)(c^4-2c^2-c+1) + (c^4+2c^3-3c-1)\sqrt{C}\right\}, \tag{199}$$

$$C = (c^2-c-1)(c^2+3c+3). \tag{200}$$

so that

$$\left.\begin{aligned}
a_1^5 &= -\frac{1}{\lambda} = \frac{(c^2+3c+3)(c^4+0-2c^2-c+1)-(c^4+2c^3+0-3c-1)\sqrt{C}}{2c\,(c+1)^3} \\
a_2^5 &= \frac{1}{\lambda^4}\left(\frac{\gamma_4}{\gamma_2}\right)^5 = \frac{y}{(c+1)^4} \\
a_3^5 &= -\frac{1}{\lambda^9}\left(\frac{\gamma_6}{\gamma_3}\right)^5 = -\frac{\gamma_3}{\gamma_{12}}\left(\frac{\gamma_6}{\gamma_5}\right)^5 \\
a_4^5 &= \frac{1}{\lambda^{16}}\left(\frac{\gamma_8}{\gamma_4}\right)^5 = \frac{\gamma_7^5}{\gamma_{13}\gamma_4^5} \\
a_5^5 &= -\frac{1}{\lambda^{25}}\left(\frac{\gamma_{10}}{\gamma_5}\right)^5 = -1 \\
a_6^5 &= \frac{1}{\lambda^{36}}\left(\frac{\gamma_{12}}{\gamma_6}\right)^5 = \lambda^9\left(\frac{\gamma_3}{\gamma_6}\right)^5 \\
a_7^5 &= -\frac{1}{\lambda^{49}}\left(\frac{\gamma_{14}}{\gamma_7}\right)^5 = -\frac{\lambda^{16}}{\gamma_7^5}
\end{aligned}\right\} \tag{201}$$

and $a_a a_{2a} = (-1)^a \lambda^{-a^2} \dfrac{\gamma_{4a}}{\gamma_{2a}}, \qquad a_a a_{3a} = \lambda^{-2a^2} \dfrac{\gamma_{2a}\gamma_{6a}}{\gamma_a \gamma_{3a}},$ &c.

Thence

$$\left.\begin{aligned}
a_3 a_6 &= -1 \\
a_4 a_7 &= -\frac{\lambda}{y} = -c-1 = \frac{1}{a_1 a_2} \\
a_1 a_3 &= \frac{1}{\lambda^2}\frac{\gamma_6}{\gamma_3} = \frac{-c^2-c+1+\sqrt{C}}{2\,(c+1)} \\
a_2 a_4 &= \frac{\gamma_7}{\lambda^8} = \frac{c^2+c+1+\sqrt{C}}{2\,(c+1)}
\end{aligned}\right\}. \tag{202}$$

31. We can also express the ξ parameters employed by Kiepert for $n = 15$ (*Math. Ann.*, XXXII, p. 121) in terms of c.

Starting with Kiepert's relations, for any odd number n,

$$L(n) = Q^{n-1} f(n), \tag{203}$$

$$Q^{24} = \Delta, \tag{204}$$

$$f(n)^{-3} = (-1)^{\frac{1}{2}(n-1)} \prod_{a=1}^{\frac{1}{2}(n-1)} \wp' \frac{2a\omega}{n} \tag{205}$$

(*Math. Ann.*, XXVI, pp. 394, 427), then

$$\xi_1 = \frac{L(15)^3}{L(5)^3 L(3)^3} = \Delta \frac{f(15)^3}{f(5)^3 f(3)^3}$$

$$= \frac{\Delta}{R} = \frac{\Delta}{\wp' \frac{2\omega}{15} \wp' \frac{4\omega}{15} \wp' \frac{8\omega}{15} \wp' \frac{14\omega}{15}}, \tag{206}$$

$$\xi_2 = \frac{L(15)^2 L(3)^2}{L(5)^2} = \Delta \frac{f(15)^2 f(3)^2}{f(5)^2}, \tag{207}$$

so that

$$\xi_2^3 = \Delta^3 \frac{\wp'^2 \frac{2\omega}{5} \wp'^2 \frac{4\omega}{5}}{\wp'^2 \frac{2\omega}{15} \wp'^2 \frac{4\omega}{15} \wp'^2 \frac{6\omega}{15} \wp'^2 \frac{8\omega}{15} \wp'^2 \frac{10\omega}{15} \wp'^2 \frac{12\omega}{15} \wp'^2 \frac{14\omega}{15} \wp'^2 \frac{2\omega}{3}}$$

$$= \frac{\Delta^3}{S_1 S_2 S_4 S_5^2 S_7}$$

$$= \frac{\Delta^3 \gamma_1^8 \gamma_2^8 \gamma_4^8 \gamma_5^{16} \gamma_7^8}{x^{12} \gamma_2^2 \gamma_4^2 \gamma_8^2 \gamma_{10}^4 \gamma_{14}^2}$$

$$= \frac{\Delta^3 \gamma_4^6 \gamma_5^{12} \gamma_7^6}{x^{12} \lambda^{48}}, \tag{208}$$

$$\xi_2 = \frac{\Delta \gamma_4^2 \gamma_5^4 \gamma_7^2}{x^4 \lambda^{16}}, \tag{209}$$

while

$$\xi_1 = \frac{\Delta \gamma_1^4 \gamma_2^4 \gamma_4^4 \gamma_7^4}{x^4 \gamma_2 \gamma_4 \gamma_8 \gamma_{14}}$$

$$= \frac{\Delta \gamma_4^3 \gamma_7^3}{x^4 \lambda^{14}}, \tag{210}$$

so that
$$\xi_4 = \frac{\xi_1}{\xi_2} = \frac{\lambda^2\gamma_4\gamma_7}{\gamma_5^4}$$

$$= -\frac{(c+1)^2(p-z)}{p^3}, \tag{211}$$

reducing to a rational expression ("Pseudo-Elliptic Integrals," p. 258)

$$\xi_4 = \frac{(c+1)^2}{c^3(c^2+3c+3)}. \tag{212}$$

Making use of this value of ξ_4 in Kiepert's equation (448), we find

$$\xi_1 = \frac{-5c(c+1)^2(c^2+3c+3)+(c+1)(c+2)(2c^2+3c+3)\sqrt{C}}{2c(c^2+3c+3)(c^4+3c^3+4c^2+2c+1)}, \tag{213}$$

$$\xi_2 = \frac{\xi_1}{\xi_4} = \frac{-5c^2(c+1)(c^2+3c+3)+c(c+2)(2c^2+3c+3)\sqrt{C}}{2(c+1)(c^4+3c^3+4c^2+2c+1)}, \tag{214}$$

and so forth; thence the values of x, y, z in terms of c in Kiepert's equations (620) *Math. Ann.*, XXXVII, p. 390, can be inferred.

32. In the preceding cases of Transformation the order n has been taken as an odd number, and the resolution of the cubic

$$4\wp^3 - g_2 p - g_3 = 0, \tag{215}$$

or of the associated form, employed in "Pseudo-Elliptic Integrals,"

$$4s(s+x)^2 - \{(y+1)s+xy\}^2 = 0, \tag{216}$$

is not required; and the associated elliptic functions are of the *First Stage* (*Erster Stufe*).

But in most dynamical applications this resolution must be effected, and elliptic functions of the *Second Stage* (*Zweiter Stufe*) must be employed; so that we shall find it useful, for the purpose of mechanical problems, to follow Kiepert, and to determine the modular functions corresponding to an even order.

(Brioschi, *Annali di Matematica*, XXII, 1894.)

33. Referring to Kiepert's paper "Zur Transformation der Elliptischen Functionen," *Math. Ann.*, XXXII, p. 1, for an explanation of the notation, and for the meaning of his parameters denoted by ξ, the following table of results shows the expression of the ξ's in terms of a single parameter, as defined in "Pseudo-Elliptic Integrals":—

$n = 2$ (*Math. Ann.*, XXXII, p. 55).

$$\xi = L(2)^{24} = \Delta f(2)^{24} = -64\tau_2$$
$$= 16\frac{\kappa'^4}{\kappa^2}, \quad 16\frac{\kappa^4}{\kappa'^2}, \quad \text{or} \quad -\frac{16}{\kappa^2\kappa'^2}. \tag{217}$$

$n = 4$

(*Math. Ann.* XXXII, p. 55; and "Pseudo-Elliptic Integrals," p. 211).

$$\left.\begin{aligned} \xi_1 &= \frac{L(4)^{16}}{L(2)^{24}} = 1-16x = \tau_4 = \eta^4 \\ \xi_2 &= L(4)^8 = \frac{1-16x}{x} \\ \xi_3 &= \frac{L(4)^8}{L(2)^{24}} = \frac{f(4)^8}{f(2)^{24}} = x \end{aligned}\right\}. \tag{218}$$

$n = 6$ (p. 83; and p. 216).

$$\left.\begin{aligned} \xi_2 &= \frac{L(6)^4}{L(3)^8 L(2)^4} = \frac{f(6)^4}{f(3)^8 f(2)^4} = y = \frac{-c}{(2-c)(1-2c)} \\ \xi_3 &= \frac{L(6)^3}{L(3)^3 L(2)^9} = \frac{f(6)^3}{f(3)^3 f(2)^9} = \frac{y}{1-y} = \frac{-c}{2(1-c)^2} \\ \xi_1 &= \frac{1-9\xi_2}{1-\xi_2} = \frac{1-9y}{1-y} = \left(\frac{1+c}{1-c}\right)^2 \\ \xi_4 &= \frac{8\xi_1}{1-\xi_1} = \frac{1-9y}{y} = 2\frac{(1+c)^2}{-c} \\ \xi_5 &= \frac{8}{9-\xi_1} = 1-y = \frac{2(1-c)^2}{(2-c)(1-2c)} \\ \xi_6 &= \frac{8\xi_1}{9-\xi_1} = 1-9y = \frac{2(1+c)^2}{(2-c)(1-2c)} \end{aligned}\right\}; \tag{219}$$

and (*Math. Ann.*, XXXVII, p. 385)

$$\left.\begin{aligned} u &= \sqrt{\xi} + \frac{8}{\sqrt{\xi}}, \quad \text{where} \quad \xi = \frac{\xi_6}{\xi_3} \\ v &= \sqrt{\eta} + \frac{1}{\sqrt{\eta}}, \quad \text{where} \quad \eta = \frac{1}{\xi_3\xi_6} \\ w &= \sqrt{\zeta} + \frac{9}{\sqrt{\zeta}}, \quad \text{where} \quad \zeta = \frac{\xi_1}{\xi_3} \end{aligned}\right\}. \tag{220}$$

$n = 8$ (p. 57; and p. 226).

$$\left.\begin{aligned}
\xi_6 &= \frac{f(4)^4}{f(2)^{12}} = \frac{z(1-z)}{(1-2z)^2} = \frac{c(1-c)(1-2c)}{(1-2c+2c^2)^2} \\
\xi_1 &= \frac{1-4\xi_6}{1+4\xi_6} = 1-8z+8z^2 = \left(\frac{1-4c+2c^2}{1-2c^2}\right)^2 \\
\xi_2 &= \frac{1}{1+4\xi_6} = (1-2z)^2 = \left(\frac{1-2c+2c^2}{1-2c^2}\right)^2 \\
\xi_3 &= \frac{\xi_6}{1+4\xi_6} = \frac{1}{2\tau_8} = z(1-z) = \frac{c(1-c)(1-2c)}{(1-2c+2c^2)^2} \\
\xi_4 &= \frac{1}{1-4\xi_6} = \frac{(1-2z)^2}{1-8z+8z^2} = \left(\frac{1-2c+2c^2}{1-4c+2c^2}\right)^2 \\
\xi_5 &= \frac{\xi_6}{1-4\xi_6} = \frac{z(1-z)}{1-8z+8z^2} = \frac{c(1-c)(1-2c)}{(1-4c+2c^2)^2}
\end{aligned}\right\} \tag{221}$$

$n = 10$ (p. 86; and p. 235).

$$\xi_2 = \frac{f(10)^2}{f(5)^4 f(2)^2} = \frac{-a}{1-a-a^2}, \tag{222}$$

because
$$\left.\begin{aligned}
f(10)^2 &= x^{-\frac{9}{3}} \lambda^{-\frac{1}{10}(1^2+2^2+\ldots+9^2)} \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8\gamma_9 \\
f(2)^2 &= x^{-\frac{1}{3}} \lambda^{-\frac{1}{10}5^2} \gamma_5 \\
f(5)^2 &= x^{-\frac{4}{3}} \lambda^{-\frac{1}{10}(2^2+4^2+6^2+8^2)} \gamma_2\gamma_4\gamma_6\gamma_8
\end{aligned}\right\}; \tag{223}$$

and these values lead here, as elsewhere, to the result, employing the values of γ given on p. 204 of "Pseudo-Elliptic Integrals."

Thence
$$\left.\begin{aligned}
\xi_1 &= \frac{1-5\xi_2}{1-\xi_2} = \frac{1+4a-a^2}{1-a^2} \\
\xi_3 &= \frac{1-\xi_1}{4} = \frac{-a}{1-a^2} \\
\xi_4 &= \frac{4\xi_1}{1-\xi_1} = \frac{1+4a-a^2}{-a} \\
\xi_5 &= \frac{4}{5-\xi_1} = \frac{1-a^2}{1-a-a^2} \\
\xi_6 &= \frac{4\xi_1}{5-\xi_1} = \frac{1+4a-a^2}{1-a-a^2}
\end{aligned}\right\}. \tag{224}$$

Thence (*Math. Ann.*, XXXVII, p. 385)

$$\left.\begin{aligned} u, \bar{u} &= \sqrt{\xi} \pm \frac{4}{\sqrt{\xi}}, \quad \text{where} \quad \xi = \xi_4 \xi_5 \\ v, \bar{v} &= \sqrt{\eta} \pm \frac{1}{\sqrt{\eta}}, \quad \text{where} \quad \eta = \frac{1}{\xi_1 \xi_2} \\ w, \bar{w} &= \sqrt{\zeta} \pm \frac{5}{\sqrt{\zeta}}, \quad \text{where} \quad \zeta = \frac{\xi_1}{\xi_2} \end{aligned}\right\}, \tag{225}$$

and putting

$$\begin{aligned} 1-b &= -\frac{1}{a} + 1 + a \\ &= \frac{1}{1-p_{10}} = \frac{\wp \frac{1}{5}\omega - \wp \omega}{\wp \frac{2}{5}\omega - \wp \omega}, \end{aligned} \tag{226}$$

$$\left.\begin{aligned} u^2 = \bar{v}^2 &= \frac{(4+b^2)^2}{b(1-b)(4+b)} \\ v^2 = w^2 &= \frac{(4+2b-b^2)^2}{b(1-b)(4+b)} \\ \bar{u}^2 = \bar{w}^2 &= \frac{(4-8b-b^2)^2}{b(1-b)(4+b)} \end{aligned}\right\}. \tag{227}$$

The eighteen values of Gierster's τ_{10} (*Math. Ann.*, XIV, p. 452) can now be exhibited as a group of substitutions, involving the parameter a.

Passing on here to the case of $n = 20$ (*Math. Ann.*, XXXII, p. 105),

$$\begin{aligned} w &= \sqrt{\{\xi_3(\xi_3+1)(4\xi_3^2+1)\}} \\ &= \frac{\sqrt{(4+b^2)}\sqrt{(1-b)}}{b^2} \\ &= \frac{(1+a^2)\sqrt{A}}{(1-a^2)^2}, \\ A &= -a+a^2+a^3; \\ \eta_1 &= \left\{\frac{\sqrt{(4+b^2)}-2\sqrt{(1-b)}}{b}\right\}^2 \\ &= \left\{\frac{1+a^2-2\sqrt{A}}{1-a^2}\right\}^2, \\ \eta_6 &= \frac{(1-a-a^2)(1-6a-a^2)-5(1+a^2)\sqrt{A}}{(1-a-a^2)(1+4a-a^2)}, \end{aligned}$$

and similarly the values of η_2, η_3, η_4, η_5 can be written down.

$n = 12$ (p. 103; and p. 248).

Beginning with ξ_2, as it does not introduce Δ, we find

$$\left.\begin{aligned}
\xi_2 &= \frac{f(6)^2}{f(3)^4 f(2)^2} = \frac{-a}{1+a+a^2} \\
\text{and thence} \qquad \xi_1 &= \frac{1-3\xi_2}{1+\xi_2} = \frac{1+4a+a^2}{1+a^2} \\
\xi_3 &= \frac{1+\xi_1}{3-\xi_1} = \left(\frac{1+a}{1-a}\right)^2 \\
\xi_4 &= \frac{1-\xi_1}{4} = \frac{-a}{1+a^2} \\
\xi_5 &= 2\,\frac{1+\xi_1}{1-\xi_1} = \frac{(1+a)^2}{-a} \\
\xi_6 &= \frac{1+\xi_1}{2} = \frac{(1+a)^2}{1+a^2} \\
\cdots & \quad \cdots \quad \cdots \quad \cdots
\end{aligned}\right\}, \tag{228}$$

and

$$a = \frac{\wp\frac{1}{3}\omega - \wp\frac{5}{6}\omega}{\wp\frac{1}{3}\omega - \wp\frac{1}{6}\omega}, \tag{229}$$

and thence the twenty-four values of Gierster's r_{12} in terms of a can be exhibited by a group of substitutions.

$n = 14$ (p. 87; and p. 257).

Beginning with ξ_4, we find

$$\begin{aligned}
\xi_4^2 &= \frac{L(14)^2}{L(7)^2 L(2)^{14}} = \frac{f(14)^2}{f(7)^2 f(2)^{14}} \\
&= \frac{x^{-\frac{13}{3}} \lambda^{-\frac{112}{3}} \gamma_1\gamma_2\gamma_3 \cdots \gamma_{12}\gamma_{13}}{x^{-2}\lambda^{-26}\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10}\gamma_{12} \cdot x^{-\frac{7}{3}} \lambda^{-\frac{49}{3}} \gamma_7^7} \\
&= \frac{\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11}\gamma_{13}}{\lambda^8 \gamma_7^7} \\
&= \lambda^{16} \frac{\gamma_1^2\gamma_3^2\gamma_5^2}{\gamma_7^6},
\end{aligned} \tag{230}$$

$$\begin{aligned}
\xi_4 &= \lambda^8 \frac{\gamma_1\gamma_3\gamma_5}{\gamma_7^3} = \frac{\gamma_{11}}{\gamma_3}\,\frac{\gamma_1\gamma_3\gamma_5}{\gamma_7^3} = \frac{\gamma_5\gamma_{11}}{\gamma_7^3} \\
&= \frac{p^3 - c^3p + c + c^2}{(p-1)^2}.
\end{aligned} \tag{231}$$

Putting, as in p. 257 of "Pseudo-Elliptic Integrals,"

$$p = \frac{r+c}{r-1},$$

$$\xi_4 = \frac{r^2 - c^2 r + c + c^2}{1+c} = \frac{c(1+c)^2}{1+2c}, \qquad (232)$$

a rational function of

$$c = \frac{\wp 5v - \wp v}{\wp 3v - \wp v} \frac{\wp 3v - \wp 2v}{\wp 2v - \wp 5v}, \quad v = \tfrac{1}{7}\omega. \qquad (233)$$

With this value of ξ_4, Kiepert's equation (290), p. 90, for ξ_3 becomes

$$\frac{1+c-2c^2-c^3}{1+2c}\xi_3^2 + 7\frac{c(1+c)^2}{1+2c}\xi_3 - \frac{c(1+c)^3(1-6c-16c^2-8c^3)}{(1+2c)^2} = 0,$$

and thence

$$\xi_3 = \frac{-7c(1+c)^2(1+2c) + (1+c)(1+3c+4c^2)\sqrt{C}}{2(1+2c)(1+c-2c^2-c^3)}, \qquad (234)$$

where $C = c(1+2c)(4+5c+2c^2)$, (235)

$$\xi_2 = \frac{\xi_3}{\xi_4} = \frac{-7c(1+c)(1+2c) + (1+3c+4c^2)\sqrt{C}}{2c(1+c)(1+c-2c^2-c^3)}, \qquad (236)$$

and $\xi_1 = \frac{\xi_3^2}{\xi_4} = \frac{1+2c}{c(1+c)^2}\xi_3^2$, (237)

$$w = \frac{8(1+c)(1+3c+4c^2)\sqrt{C}}{(1+2c)^2}. \qquad (238)$$

Also $\frac{1}{\xi_3} = \frac{7c(1+c)^2(1+2c) + (1+c)(1+3c+4c^2)\sqrt{C}}{2c(1+c)(1-6c-16c^2-8c^3)}$; (239)

and thence (*Math. Ann.*, XXXVII, p. 385), for $n = 14$,

$$v^2 - 2 = \eta + \frac{1}{\eta} = \frac{1}{\xi_3} + \xi_3 = y$$

$$= \frac{7(1+2c+4c^2+22c^3+44c^4+32c^5+8c^6)}{2(1+c-2c^2-c^3)(1-6c-16c^2-8c^3)}$$

$$+ \frac{(1+3c+4c^2)(1+4c-4c^2-32c^3-48c^4-32c^5-8c^6)\sqrt{C}}{2c(1+c)(1+2c)(1+c-2c^2-c^3)(1-6c-16c^2-8c^3)}, \qquad (240)$$

and (*Math. Ann.*, XXXVII, p. 386)

$$x = \xi + \frac{8}{\xi} = \xi_2 + \frac{8}{\xi_2}, \qquad (241)$$

where $$\frac{1}{\xi_2} = \frac{\xi_4}{\xi_8} = \frac{7c(1+c)^2(1+2c)+(1+c)(1+3c+4c^2)\sqrt{C}}{2(1+2c)(1-6c-16c^2-8c^3)}, \tag{242}$$

and thence we find that Kiepert's relation

$$x = y-7 = v^2-9 \tag{243}$$

is satisfied.

$n = 16$ (p. 59; and p. 262).

Converting Kiepert's expression for ξ_4 into one involving λ and γ,

$$\xi_4 = \frac{\dfrac{\gamma_8\gamma_4}{\gamma_2^2\gamma_6^2}\,\dfrac{\gamma_6\gamma_2}{\gamma_3^2\gamma_4^2}\,\dfrac{\gamma_{10}\gamma_2}{\gamma_6^2\gamma_4^2}}{\dfrac{\gamma_{12}\gamma_4}{\gamma_4^2\gamma_8^2}\,\dfrac{\gamma_8\gamma_6}{\gamma_1^2\gamma_7^2}\,\dfrac{\gamma_8\gamma_2}{\gamma_3^2\gamma_5^2}} = \frac{\gamma_{10}\gamma_8\gamma_7^2\gamma_5^2\gamma_3^2}{\gamma_{12}\gamma_6^4\gamma_4^2} = \frac{\gamma_8\gamma_7^2\gamma_5^2\gamma_3^2}{\lambda^4\gamma_6^3\gamma_4^3} \tag{244}$$

and this reduces finally to

$$\xi_4 = \frac{a^2-1}{a^2-2a-1}$$

so that

$$\left.\begin{aligned}
\xi_3 &= \frac{1-\xi_4}{2\xi_4} &&= \frac{-a}{a^2-1} \\
\xi_1 &= \frac{1-2\xi_3}{1+2\xi_3} &&= \frac{a^2+2a-1}{a^2-2a-1} \\
\xi_5 &= \frac{\xi_3}{1+2\xi_3} &&= \frac{-a}{a^2-2a-1} \\
\xi_7 &= 1-2\xi_3 &&= \frac{a^2+2a-1}{a^2-1} \\
\xi_8 &= \frac{1-2\xi_3}{\xi_3} &&= \frac{a^2+2a-1}{-a} \\
\xi_2 &= \frac{\sqrt{(1+4\xi_3^2)}}{1+2\xi_3} &&= \frac{a^2+1}{a^2-2a-1} \\
\xi_6 &= \frac{\xi_1}{\xi_2} &&= \frac{a^2+2a-1}{a^2+1} \\
\xi_9 &= \frac{\xi_4}{\xi_2} &&= \frac{a^2-1}{a^2+1} \\
\xi_{10} &= \frac{\xi_5}{\xi_2} &&= \frac{-a}{a^2+1}
\end{aligned}\right\}. \tag{245}$$

Referring to "Pseudo-Elliptic Integrals," p. 263, and *Math. Ann.*, XIV, p. 542, we see that Gierster's

$$\tau_{16} = -\xi_8. \tag{246}$$

$n = 18$ (p. 126; and p. 265).

To agree with the notation of the *Modulfunctionen*, I, p. 685, we must put Gierster's

$$\tau_{18} = -x-2, \tag{247}$$

$$\tau_{12} = \quad y+3, \tag{248}$$

and then (*Math. Ann.*, XIV, pp. 540, 541)

$$2\tau_6+9 = (\tau_{12}-3)^2 = y^2, \tag{249}$$

$$2\tau_6+8 = (\tau_{18}+2)^3 = x^3, \tag{250}$$

$$x^3 = -\frac{\wp\frac{2}{3}\omega-\wp\frac{1}{3}\omega}{\wp\omega-\wp\frac{2}{3}\omega},\quad y^2 = \frac{\wp\omega-\wp\frac{1}{3}\omega}{\wp\omega-\wp\frac{2}{3}\omega} = \frac{(\wp\omega-\wp\frac{1}{3}\omega)^2}{(\wp\omega-\wp\omega')(\wp\omega-\wp\omega'')},$$

so that $$x^3+y^2 = 1. \tag{251}$$

Changing the sign of the x on p. 269 of "Pseudo-Elliptic Integrals," we now find that the relation connecting this x with Kiepert's ξ_3 is

$$\xi_3 = -\frac{1}{x} = \frac{1}{\tau_{18}+2}$$

$$= \frac{-q^3-q^2+2q+1+\sqrt{Q}}{4q\,(q+1)}, \tag{252}$$

so that $$\xi_6 = \frac{1-2\xi_3}{\xi_3} = \tau_{18}$$

$$= \frac{q^3-3q^2-6q-1+\sqrt{Q}}{2q\,(q+1)}, \tag{253}$$

$$\xi_1 = 1-2\xi_3$$

$$= \frac{q^3+3q^2+0-1-\sqrt{Q}}{2q\,(q+1)}, \tag{254}$$

$$\xi_2 = \frac{\xi_3}{1+\xi_3}$$

$$= \frac{q^3+3q^2+0-1-\sqrt{Q}}{2\,(q^3-3q-1)}, \tag{255}$$

$$\xi_4 = \frac{1-2\xi_3}{1+\xi_3}$$

$$= \frac{-q^3-9q^2-6q+1+3\sqrt{Q}}{2\,(q^3-3q-1)}, \tag{256}$$

$$\xi_6 = \frac{1}{1+\xi_3}$$

$$= \frac{q^3-3q^2-6q-1+\sqrt{Q}}{2(q^3-3q-1)}, \qquad (257)$$

so that
$$\frac{\xi_1}{\xi_2} = \frac{\xi_5}{\xi_6} = \frac{q^3-3q-1}{q(q+1)}$$

$$= q-1-\frac{1}{q}-\frac{1}{q+1}. \qquad (258)$$

$n = 22$ (p. 91; and p. 274).

34. Expressed in terms of our λ and γ, we find that Kiepert's three parameters ξ, ξ_1, ξ_3 are given by

$$\xi = \frac{L(22)^2 L(2)^2}{L(11)^2} = \Delta \frac{f(22)^2 f(2)^2}{f(11)^2}$$

$$= \Delta \frac{x^{-7}\lambda^{-\frac{391}{3}}\gamma_1\gamma_2\gamma_3 \cdots \gamma_{20}\gamma_{21} \cdot x^{-\frac{1}{3}}\lambda^{-\frac{11}{3}}\gamma_{11}}{x^{-\frac{10}{3}}\lambda^{-70}\gamma_2\gamma_4\gamma_6 \cdots \gamma_{18}\gamma_{20}}$$

$$= \frac{\Delta}{x^4\lambda^{86}}\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11}^2\gamma_{13}\gamma_{15}\gamma_{17}\gamma_{19}\gamma_{21}$$

$$= \frac{\Delta}{x^4\lambda^{26}}(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^2, \qquad (259)$$

$$\xi_1 = \frac{L(22)^4}{L(11)^8 L(2)^4} = \frac{f(22)^4}{f(11)^8 f(2)^4}$$

$$= \lambda^{10}\left(\frac{\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9}{\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10}}\right)^4, \qquad (260)$$

$$\xi_3 = \frac{L(22)^8}{L(11)^4 L(2)^8} = \Delta^5 \frac{f(22)^8}{f(11)^4 f(2)^8}$$

$$= \frac{\Delta^5}{x^{20}\lambda^{76}} \frac{(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9)^8}{(\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^2}, \qquad (261)$$

and we have to express these quantities as functions of a single parameter c by means of the relations on p. 274 of "Pseudo-Elliptic Integrals."

35. Judging by analogy with preceding cases, the parameter ξ_1 seemed likely to assume the simplest form, because it did not involve the discriminant Δ; and now we shall find that, expressed in terms of the q and c of p. 274, "Pseudo-Elliptic Integrals,"

$$\xi_1 = \frac{c^3q^4(q-1-c-c^2)^3(q+1+c)^5(q-c^2)}{(1+c)(q-1)^5(q+c)^2(q-c-c^2)^9}, \tag{262}$$

where q is given as a function of c by the quartic equation

$$(q+c)(q-c-c^2)^2\{cq^2-(1+2c+c^2+c^3)q+c^2+c^3\}$$
$$-cq(q-1-c-c^2)(q-c^2)(q+1+c)^2 = 0, \tag{263}$$

which can also be written, according to the calculations of Mr. St. Bodfan Griffiths, of University College, Bangor, in a form ready for solution,

$$\{2(1+c)^2q^2-(2c+5c^2+4c^3+2c^4)q-c^4(1+c)\}^2$$
$$= (4c+8c^2+4c^3+c^4)\{(1+2c)q-c^2(1+c)\}^2, \tag{264}$$

or $c^2\{cq^2-(2+4c+c^2+c^3)q+2c^2(1+c)\}^2$

$$= (4c+8c^2+4c^3+c^4)q^2(q-c-c^2)^2. \tag{265}$$

Thus, from (263), we can also write

$$\xi_1 = \frac{c^2q^3(q+1+c)^3(q-1-c-c^2)^2\{cq^2-(1+2c+c^2+c^3)q+c^2+c^3\}}{(1+c)(q+c)(q-1)^5(q-c-c^2)^7}, \tag{266}$$

and the elimination of q between this and (263) will lead to a quartic equation in ξ_1, which is discussed in the sequel.

36. To connect up these values of q and c, which may be distinguished when required by q_{22} and c_{22}, with the q_{11} and c_{11}, employed in the Transformation of the Eleventh Order in § 22, we notice that

$$q_{11} = -z_{11} = -\frac{\wp\frac{6\omega}{11}-\wp\frac{4\omega}{11}}{\wp\frac{6\omega}{11}-\wp\frac{2\omega}{11}}$$

$$= -\frac{\wp\frac{12\omega}{22}-\wp\frac{8\omega}{22}}{\wp\frac{12\omega}{22}-\wp\frac{4\omega}{22}}$$

$$= -\frac{\dfrac{\gamma_{10}\gamma_2}{\gamma_6^2\gamma_4^2}}{\dfrac{\gamma_8\gamma_4}{\gamma_6^2\gamma_2^2}}$$

$$= -\frac{\gamma_{10}}{\gamma_8\gamma_4^3}$$

$$= -\frac{(q+c)(q-c-c^2)}{(q-c^2)^2}, \tag{267}$$

$$1+c_{11} = p_{11}$$

$$= \frac{\wp\dfrac{8\omega}{22}-\wp\dfrac{4\omega}{22}}{\wp\dfrac{20\omega}{22}-\wp\dfrac{4\omega}{22}}$$

$$= \frac{\gamma_{10}^2\gamma_6}{\gamma^{12}\gamma_8\gamma_4^2}$$

$$= \frac{(q+c)(q-c-c^2)^2}{(q+1+c)(q-c^2)^2}, \tag{268}$$

so that
$$\frac{q_{11}}{1+c_{11}} = -\frac{z_{11}}{p_{11}} = -\frac{q+1+c}{q-c-c^2}, \tag{269}$$

$$\frac{1+c_{11}+q_{11}}{1+c_{11}} = \frac{-(1+c)^2}{q-c-c^2}, \tag{270}$$

and these are true in general when $z_{11}, p_{11}, q_{22}, c_{22}$ are replaced by z_n, p_n, q_{2n}, c_{2n}.

37. In Kiepert's notation (*Math. Ann.*, XXXII, p. 96)

$$\frac{1}{\tau} = \eta+8 = \xi^2+4\xi+8+\frac{4}{\xi},$$

$$\frac{\tau'}{\tau^2} = W = (\xi+2-2\xi^{-2})\, w,$$

$$w^2 = (\xi^3+4\xi^2+8\xi+4)(\xi^3+8\xi^2+16\xi+16),$$

so that
$$-\frac{\tau'}{\tau^3}\frac{d\tau}{d\xi} = 2\,(\xi+2-2\xi^{-2}),$$

and
$$\frac{d\tau}{\tau'} = -\frac{2d\xi}{w}. \tag{271}$$

Again (*Math. Ann.*, XXXVII, p. 386), putting

$$u,\quad \bar{u} = \sqrt{\xi} \pm \frac{2}{\sqrt{\xi}}, \tag{272}$$

$$\frac{4}{\sqrt{\xi}} = u - \sqrt{(u^2-8)},$$

$$-\frac{2d\xi}{\xi^{\frac{3}{2}}} = du - \frac{u\,du}{\sqrt{(u^2-8)}},$$

while
$$\frac{w^2}{\xi^3} = u^6 - 4u^2 + 4, \tag{273}$$

so that, from (168),

$$\frac{dc}{\sqrt{C}} = \frac{d\tau}{\tau'} = \frac{du}{\sqrt{(u^6-4u^2+4)}} - \frac{u\,du}{\sqrt{\{(u^2-8)(u^6-4u^2+4)\}}}. \tag{274}$$

Put
$$u^2 = \frac{1}{a+1}; \tag{275}$$

then
$$\frac{du}{\sqrt{(u^6-4u^2+4)}} = \frac{-\frac{1}{2}\,da}{\sqrt{A}}, \tag{276}$$

$$A = 4a\,(a+1)^2+1. \tag{277}$$

Put
$$\bar{u}^2 = u^2 - 8 = \frac{11^2}{b-11}; \tag{278}$$

then
$$\frac{u\,du}{\sqrt{\{(u^2-8)(u^6-4u^2+4)\}}} = \frac{-\frac{1}{2}db}{\sqrt{B}}, \tag{279}$$

$$B = 4b^2\,(b-11) + (10b+11)^2, \tag{280}$$

the same function as H'^2 in (167), § 25.

Thus
$$\frac{dc}{\sqrt{C}} = \frac{-\frac{1}{2}\,da}{\sqrt{A}} + \frac{\frac{1}{2}db}{\sqrt{B}}, \tag{281}$$

and $\dfrac{db}{\sqrt{B}}$ can be reduced to the form of $\dfrac{dc}{\sqrt{C}}$

by means of the quintic transformation (163), so that the preceding relations conceal an elliptic function relation, the interpretation of which is given hereafter, in § 61.

38. In Kiepert's notation, distinguishing this new η by an accent,

$$2\sqrt{\eta'} = v+\bar{v}, \quad 2\sqrt{\zeta} = w+\bar{w},$$

so that
$$\frac{4}{\xi_1} = 4\sqrt{(\eta'\zeta)} = (v+\bar{v})(w+\bar{w}). \tag{282}$$

Also
$$2\sqrt[4]{\eta'} = \sqrt{(v+2)} + \sqrt{(v-2)},$$
$$2\sqrt[4]{\zeta} = \sqrt{(w+22)} + \sqrt{(w-22)},$$

$$\frac{4}{\sqrt{\xi_1}} = \{\sqrt{(v+2)}+\sqrt{(v-2)}\}\{\sqrt{(w+22)}+\sqrt{(w-22)}\}$$
$$= 2(\xi+3)\sqrt{(\xi^3+8\xi^2+16\xi+16)} + 2(\xi+5)\sqrt{(\xi^3+4\xi^2+8\xi+4)}. \tag{283}$$

These relations seem to show that u, v, w should be determined, as the simplest functions of a single parameter c.

$n = 26$ (*Math. Ann.*, XXXII, p. 98).

39. We notice here that Kiepert's ξ_1 is the same as Klein's τ_{13}, so that we can put

$$\xi_1 = \frac{1-c-4c^2-c^3}{c(1+c)}, \tag{284}$$

and thus ξ can be determined as a function of c by the solution of Kiepert's cubic equation (333).

To obtain the ξ's as explicit functions of a parameter, we should have to discuss Halphen's relation

$$\gamma_{26} = 0; \tag{285}$$

but this leads to difficulties not yet surmounted.

40. The connexion between Kiepert's ξ parameters and the function p_n employed by Abel in the expression of the square root of a quartic in the form of a continued fraction is remarkable (Abel, *Œuvres*, II, p. 157).

Expressed by Halphen's γ functions, we find that Abel's

$$q_m = -2x^3 \frac{\gamma_m \gamma_{m+2}}{\gamma_{m+1}^2}, \tag{286}$$

and
$$p_0 = p = -4x; \tag{287}$$

thence
$$p_1 = \frac{2q_1}{p}, \tag{288}$$

and, since
$$p_m p_{m-1} = 2q_m, \tag{289}$$

$$p_{2a} = \frac{q_{2a}}{q_{2a-1}} \frac{q_{2a-2}}{q_{2a-3}} \dots \frac{q_2}{q_1} p = -4x \left(\frac{\gamma_2\gamma_4\gamma_6 \dots \gamma_{2a}}{\gamma_1\gamma_3\gamma_5 \dots \gamma_{2a-1}}\right)^4 \frac{\gamma_{2a+2}}{\gamma_{2a+1}^3}, \tag{290}$$

$$p_{2a+1} = 2\frac{q_{2a+1}}{q_{2a}} \frac{q_{2a-1}}{q_{2a-2}} \dots \frac{q_3}{q_2} \frac{q_1}{p} = \left(\frac{\gamma_1\gamma_3\gamma_5 \dots \gamma_{2a+1}}{\gamma_2\gamma_4\gamma_6 \dots \gamma_{2a+2}}\right)^4 \frac{\gamma_{2a+2}\gamma_{2a+3}}{x^3}. \tag{291}$$

Written in Halphen's notation (*Fonctions Elliptiques*, II, p. 582) Abel's continued fraction expression is

$$\sqrt{\{(x^2+ax+b)^2+px\}}$$
$$= (x^2+ax+b)+1 : \frac{2(x+g)}{p}+1 : \frac{2(x+g_1)}{p_1}+1 : \dots$$
$$\dots : \frac{2(x+g_m)}{p_m}+1 : \dots$$
$$= (x^2+ax+b)+p : 2(x+g)+pp_1 : 2(x+g_1)+p_1p_2 : \dots$$
$$: 2(x+g_{m-1})+p_{m-1}p_m : 2(x+g_m)+p_m p_{m+1} : \dots, \tag{292}$$

and we find
$$g_m = \tfrac{1}{2}\frac{\wp'(m+1)v-\wp' v}{\wp(m+1)v-\wp v}, \tag{293}$$

with
$$p_{m-1}p_m = 2q_m = 4\{\wp(m+1)v-\wp v\}; \tag{294}$$

so that the continued fraction is readily written down when it is periodic, and
$$\gamma_n = 0, \tag{295}$$

leading to a pseudo-elliptic integral.

41. But without having recourse to the transformations of the even orders, we can obtain the resolution of the cubic

$$S = 4s(s+x)^2 - \{(y+1)s+xy\}^2 \tag{296}$$

by means of Halphen's expressions for his x and y in terms of a and γ on p. 377, t. II, *Fonctions Elliptiques*,

$$x = -\frac{\{a^2-2a\gamma(2\gamma^2-3\gamma+2)+\gamma^4\}^3}{2^8a^2(a-1)^2\gamma^4(\gamma-1)^4}, \tag{297}$$

$$y = -\frac{(\gamma^2-a)(\gamma^2-2\gamma+a)(\gamma^2-2a\gamma+a)}{2^3a(a-1)\gamma^3(\gamma-1)^2}. \tag{298}$$

Now, if s_α, s_β, s_γ denote the roots of the cubic (1), we can put

$$s_\alpha = M^2(\gamma^2-2\gamma+\alpha)^2, \tag{299}$$

$$s_\beta = M^2(\gamma^2-2\alpha\gamma+\alpha)^2, \tag{300}$$

$$s_\gamma = M^2(\gamma^2-\alpha)^2, \tag{301}$$

where
$$M = \frac{\alpha^2-2\alpha\gamma(2\gamma^2-3\gamma+2)+\gamma^4}{2^4\alpha(\alpha-1)\gamma^2(\gamma-1)^2}. \tag{302}$$

42. But, if we put

$$\alpha = \frac{q^2}{p^2}\,\frac{1-p^2}{1-q^2}, \quad \gamma = \frac{1-p^2}{1-q^2}, \tag{303}$$

then
$$s_\alpha = N^2(1+p^2-q^2)^2, \tag{304}$$

$$s_\beta = N^2(1-p^2+q^2)^2, \tag{305}$$

$$s_\gamma = N^2(1-p^2-q^2)^2, \tag{306}$$

where
$$N = \frac{1-2(p^2+q^2)+(p^2-q^2)^2}{16p^2q^2}, \tag{307}$$

and then

$$\begin{aligned} x &= -\frac{\{1-2(p^2+q^2)+(p^2-q^2)^2\}^3}{2^8p^4q^4} \\ &= -\frac{\{(1+p+q)(1+p-q)(1-p+q)(1-p-q)\}^3}{2^8p^4q^4}, \end{aligned} \tag{308}$$

$$y = -\frac{(1+p^2-q^2)(1-p^2+q^2)(1-p^2-q^2)}{2^3p^2q^2}. \tag{309}$$

43. In the poristic problem of the polygon of n sides, inscribed in a circle of radius R and circumscribed to a circle of radius r, the centres being a distance c apart, we may put

$$\left.\begin{aligned} p &= \operatorname{cn}\frac{2K}{n} = \frac{r}{R-c} \\ q &= \operatorname{sn}\left(K-\frac{2K}{n}\right) = \frac{r}{R+c} \end{aligned}\right\}, \tag{310}$$

and
$$\left.\begin{aligned} \kappa^2 = 1-\alpha &= \frac{p^2-q^2}{p^2-p^2q^2} = \frac{4Rc}{(R+c)^2-r^2} \\ \gamma = \operatorname{dn}^2\left(K-\frac{2K}{n}\right) &= \left(\frac{R-c}{R+c}\right)^2 \end{aligned}\right\}. \tag{311}$$

44. Putting
$$\left.\begin{aligned}4\mu &= 1-(p+q)^2\\ 4\nu &= 1-(p-q)^2\end{aligned}\right\}, \tag{312}$$

then
$$N = \frac{\mu\nu}{(\mu-\nu)^2}, \tag{313}$$

$$x = -\frac{16\mu^3\nu^3}{(\mu-\nu)^4}, \tag{314}$$

$$y = -\frac{(\mu+\nu)(\mu+\nu-4\mu\nu)}{(\mu-\nu)^2} \tag{315}$$

and then, putting
$$\mu+\nu = 2a, \quad \mu\nu = \beta,$$

$$(\mu-\nu)^2 = 4\,(a^2-\beta), \tag{316}$$

then
$$x = -\frac{\beta^3}{(a^2-\beta)^2}, \tag{317}$$

$$y = -\frac{a\,(a-2\beta)}{a^2-\beta}, \tag{318}$$

$$N = \frac{\beta}{4\,(a^2-\beta)}, \tag{319}$$

and
$$s_\gamma = \frac{a^2\beta^3}{(a^2-\beta)^2}. \tag{320}$$

45. We shall find it convenient to use the symbol s_r to denote the value of s corresponding to the aliquot part $2r\omega_\gamma/n$ of the period; it also simplifies the expressions to put

$$\beta = ma;$$

and now
$$x = -\frac{m^3a}{(a-m)^2}, \tag{321}$$

$$y = -\frac{(1-2m)\,a}{a-m}, \tag{322}$$

$$y+1 = \frac{(2a-1)\,m}{a-m}; \tag{323}$$

and we find, after reduction

$$s_\gamma - s_1 = s_\gamma + x \quad = \frac{m^2 a}{a-m}, \tag{324}$$

$$s_\gamma - s_2 = s_\gamma \quad = \frac{m^2 a^2}{(a-m)^2}, \tag{325}$$

$$s_\gamma - s_3 = s_\gamma + x - y = \frac{(1-m)^2 a}{a-m}, \tag{326}$$

so that

$$\frac{s_\gamma - s_2}{s_\gamma - s_1} = \frac{a}{a-m}, \tag{327}$$

$$\frac{s_\gamma - s_3}{s_\gamma - s_1} = \left(\frac{1-m}{m}\right)^2, \tag{328}$$

$$s_\gamma - s_4 = \left\{\frac{m(1-2m)a - m^2(1-m)}{(1-2m)(a-m)}\right\}^2, \tag{329}$$

$$s_\gamma - s_5 = \frac{m^2 a}{a-m}\left\{\frac{(1-2m)a - m^2(1-m)}{(1-2m)a - m(1-m)^2}\right\}^2, \tag{330}$$

$$s_\gamma - s_6 = \frac{a^2}{(a-m)^2}\left\{\frac{(1-2m+2m^2)(1-2m)a - m(1-m)(1-3m+3m^2)}{2(1-m)(1-2m)a - m(1-m)^2}\right\}^2, \tag{331}$$

$$s_\gamma - s_7 = \frac{m^2 a}{a-m} \times \left\{\frac{(1-2m)^3 a^2 - m(1-m)(1-2m)(2-3m)a + m^2(1-m)^4}{(1-2m)^3 a^2 - m(1-m)(1-2m)(1-3m)a - m^4(1-m)^3}\right\}^2, \tag{332}$$

$$s_\gamma - s_8 = \frac{m^2}{(1-2m)^2(a-m)^2} \times \left\{\frac{(1-2m)^3 a^3 - m(1-m)(1-2m)^2(1+2m-2m^2)a^2 + 4m^3(1-m)^3(1-2m)a - m^4(1-m)^4}{\{(1-2m)a - m(1-m)\} \times \{(1-2m)a - m(1-m)(1-2m+2m^2)\}}\right\} \tag{333}$$

Mr. G. H. Stuart is engaged on the calculation of the succeeding equations, and he has found

$$s_\gamma - s_9 = \frac{m^2 a}{a-m} - x^2\frac{\gamma_8\gamma_{10}}{\gamma_9^2}$$

$$= \frac{(1-m)^2 a}{a-m}\left\{\frac{Aa^3 - Ba^2 + Ca - D}{Pa^3 - Qa^2 + Ra - S}\right\}^2, \tag{334}$$

where

$$\left.\begin{aligned} A &= (1-2m)^3\,(1-2m+4m^2) \\ B &= 2m\,(1-m)(1-2m)^2\,(1-3m+5m^2) \\ C &= m^2\,(1-m)^2\,(1-2m)(1-4m+7m^2) \\ D &= m^6\,(1-m)^3 \\ P &= (1-2m)^3\,(3-6m+4m^2) \\ Q &= 2m\,(1-m)(1-2m)^2\,(3-7m+5m^2) \\ R &= m^2\,(1-m)^2\,(1-2m)(4-10m+7m^2) \\ S &= m^3\,(1-m^6) \end{aligned}\right\}, \tag{335}$$

$$s_\gamma - s_{10} = \frac{m^2\alpha^2}{(\alpha-m)^2}$$

$$\times\left[\frac{A\alpha^4 - B\alpha^3 + C\alpha^2 - D\alpha + E}{\{(1-2m)\,\alpha - m\,(1-m)^2\}\,\{(1-2m)\,\alpha - m^2(1-m)\}\,(P\alpha^2 - Q\alpha + R)}\right]^2,$$

where

$$\left.\begin{aligned} A &= (1-2m)^6 \\ B &= 2m\,(1-m)(1-2m)^3\,(2-7m+7m^2) \\ C &= 2m^2\,(1-m)^2\,(1-2m)^2\,(3-11m+13m^2-4m^3+2m^4) \\ D &= m^3\,(1-m)^3\,(1-2m)(4-17m+27m^2-20m^3+10m^4) \\ E &= m^4\,(1-m)^4\,(1-5m+10m^2-10m^3+5m^4) \\ P &= (1-2m)^4 \\ Q &= m\,(1-m)(1-2m)(1-6m+6m^2) \\ R &= -m^3\,(1-m)^3 \end{aligned}\right\}. \tag{336}$$

So also for $s_\gamma - s_{11},\ s_\gamma - s_{12}$, &c.

The form of the denominator of

$$s_\gamma - s_n$$

can be inferred by putting

$$s_\gamma - s_p = s_\gamma - s_{n-p},$$

and of the numerator by putting

$$(s_\gamma - s_p)(s_\gamma - s_{n-p}) = (s_\alpha - s_\gamma)(s_\beta - s_\gamma)\,; \tag{337}$$

and these relations serve as a check upon the preceding results.

These equations, (324) to (337), ..., provide a simple method of determining the division values of elliptic functions of the second stage in terms of a single parameter; for putting

$$s_\gamma - s_n = 0, \quad \text{or} \quad s_\gamma - s_k = s_\gamma - s_{n-k}, \tag{338}$$

gives a relation by which it is possible to express a and m in terms of a single parameter.

Thus, for instance, from the relation

$$s_\gamma - s_9 = s_\gamma - s_{10}, \tag{339}$$

we obtain the elliptic functions, sn, cn, and dn, of the nineteenth part of a period.

46. Expressed by a linear and quadratic factor,

$$S = 4s(s+x)^2 - \{(y+1)s + xy\}^2$$

$$= \left\{s - \frac{m^2a^2}{(a-m)^2}\right\}\left\{4s^2 + \frac{4(1-2m)a-1}{(a-m)^2}m^2s + \frac{m^4(1-2m)^2a^2}{(a-m)^4}\right\}, \tag{340}$$

so that, denoting the roots of the quadratic factor by s_α and s_β, we find

$$(s_\alpha - s_\gamma)(s_\beta - s_\gamma) = \frac{m^4a^2(a-m+1)}{(a-m)^3}, \tag{341}$$

$$(s_\alpha - s_\beta)^2 = m^4\frac{1-8(1-2m)a}{16(a-m)^4}, \tag{342}$$

and, according to the order of magnitude of s_α, s_γ, s_β, we may put

$$\frac{\kappa^2}{\kappa'^4}, \text{ or } \frac{\kappa'^2}{\kappa^4}, \text{ or } -\kappa^2\kappa'^2$$

$$= \frac{(s_\alpha - s_\gamma)(s_\beta - s_\gamma)}{(s_\alpha - s_\beta)^2}$$

$$= 16\frac{a^2(a-m)(a-m+1)}{1-8(1-2m)a}. \tag{343}$$

We may also determine Kiepert's function T (*Math. Ann.*, XXXII, p. 26) by the relation

$$T^2 = \prod_{r=1}^{r=\frac{1}{2}(n-1)} \frac{s_r - s_\gamma}{s_r - s_{2r}}. \tag{344}$$

47. According to Halphen (*F. E.*, II, p. 407) x and y are given as functions of v by the relations

$$x = \gamma_3^3(v) = \frac{\sigma^3 3v\,\sigma^5 v}{\sigma^8 2v}, \tag{345}$$

$$y = \gamma_4(v) = \frac{\sigma 4v\,\sigma^4 v}{\sigma^5 2v}. \tag{346}$$

Denoting the values when v is changed into pv by x_p and y_p, then (*F. E.*, I, p. 106)

$$x_p = \frac{\gamma_{3p}^3 \gamma_p^5}{\gamma_{2p}^8} = \frac{(s_{2p}-s_p)^3}{S_p}, \tag{347}$$

$$y_p = \frac{\gamma_{4p}\gamma_p^4}{\gamma_{3p}^5} = \sqrt{\frac{S_{2p}}{S_p}}, \tag{348}$$

and now the substitution

$$s - s_{2p} = M^2 t, \tag{349}$$

or

$$s - s_p = M^2(t + x_p), \tag{350}$$

where

$$M^2 x_p = s_{2p} - s_p$$

$$= x^{\frac{1}{3}} \frac{\gamma_{3p}}{\gamma_{2p}^2 \gamma_p}, \tag{351}$$

or

$$M = x^{\frac{1}{6}} \frac{\gamma_{2p}^3}{\gamma_p^3 \gamma_{3p}}, \tag{352}$$

changes $S(s;\, x, y) = 4s(s+x)^2 - \{(y+1)s + xy\}^2$

into $M^6 S(t;\, x_p, y_p)$,

and makes

$$\frac{M\,ds}{\sqrt{\{S(s;\, x, y)\}}} = \frac{dt}{\sqrt{\{S(t;\, x_p, y_p)\}}}; \tag{353}$$

in this way the various permutations of the division-values of argument pv are obtained.

48. Thus, for instance, when $n = 11$, and

$$\gamma_{11} = 0,$$

we find
$$1-z_3 = \frac{x_3}{y_3} = \frac{\gamma_6^3}{\gamma_8\gamma_4^3}$$

$$= \frac{x(y-x-y^2)^3}{y^4\{x(y-x-y^2)-(y-x)^2\}}$$

$$= \frac{z^3(1-z)}{(p-z)^3(1-z-p)}$$

$$= \frac{q^3(1+q)}{(1+c+q)^3(q-c)}$$

$$= \frac{cq(1+c)^2}{(1+c+q)^3(q-c)}$$

$$= \frac{4c(1+c)^3(\sqrt{C}-1)^2}{(1+2c+\sqrt{C})^3(-1-2c+\sqrt{C})}$$

$$= \frac{(1+c)(1+2c+2c^2-\sqrt{C})}{2c^3}.$$

But, if
$$c_3 = \frac{1+2c-\sqrt{C}}{2c^2},$$

$$\sqrt{C_3} = \frac{1+3c+4c^2+c^3-(1+c)\sqrt{C}}{c^3},$$

so that
$$1-z_3 = \frac{\sqrt{C_3}+1}{2},$$

while
$$1-z_1 = \frac{\sqrt{C}+1}{2},$$

and z_1 is thus changed into z_3 by changing c into

$$\frac{1+2c-\sqrt{C}}{2c^2};$$

and the same substitution changes x_1 and y_1 into $M_3^{-2}x_3$ and $M_3^{-2}y_3$, where

$$M_3 = x^3\frac{\gamma_4^3}{\gamma_6} = \frac{(1+2c+\sqrt{C})^3}{4(1+c)}.$$

Similarly, a change of

$$c \text{ into } \frac{-1-4c-2c^2-\sqrt{C}}{2(1+c)^2}$$

changes $x_1, y_1, z_1 \ldots$ into $M_3^{-2}x_3, M_3^{-2}y_3, z_3, \ldots$;

and so on.

Calculating the division-values

$$\wp(1, 2, 3, 4, 5)\frac{2\omega_3}{11},$$

for $x = x_1, \quad y = y_1,$

we find, with $v = \dfrac{2\omega_3}{11}$,

$$24(1+c)^2\wp v = -2+2c+27c^2+42c^3+18c^4-2c^5-2c^6$$
$$+(6c+13c^2+8c^3+2c^4)\sqrt{C},$$

$$24(1+c)^2\wp 2v = -2-10c-33c^2-66c^3-66c^4-26c^5-2c^6$$
$$+(-6c-23c^2-28c^3-10c^4)\sqrt{C},$$

$$24(1+c)^2\wp 3v = -2-10c-33c^2-30c^3-6c^4-2c^5-2c^6$$
$$+(-6c+c^2+8c^3+2c^4)\sqrt{C},$$

$$24(1+c)^2\wp 4v = -2-22c-69c^2-102c^3-78c^4-26c^5-2c^6$$
$$+(6c+13c^2+8c^3+2c^4)\sqrt{C},$$

$$24(1+c)^2\wp 5v = -2-10c-21c^2-18c^3-6c^4-2c^5-2c^6$$
$$+(-6c-11c^2-4c^3+2c^4)\sqrt{C},$$

so that

$$24(1+c)^2 G_1 = 24\sum_{r=1}^{r=5}\wp rv$$
$$= -10-50c-129c^2-174c^3-138c^4-58c^5-10c^6$$
$$+(-6c-7c^2-8c^3-2c^4)\sqrt{C},$$

and the preceding group of substitutions merely permutes these division-values, and changes the homogeneity factor M.

49. The value of $x' = x_{4n}$ may be derived from $x = x_{2n}$ in the following manner:—

Put
$$S = 4s(s+x)^2 - \{(y+1)s+xy\}^2$$
$$= 4(s-s_\alpha)(s-s_\beta)(s-s_\gamma),$$

so that
$$s_\alpha+s_\beta+s_\gamma = \tfrac{1}{4}(y+1)^2-2x,$$
$$s_\beta s_\gamma+s_\gamma s_\alpha+s_\alpha s_\beta = x^2-\tfrac{1}{2}xy(y+1),$$
$$s_\alpha s_\beta s_\gamma = -x^2y^2;$$

and put
$$s\left(\frac{\omega}{n}\right)-s\left(\frac{\omega}{2n}\right) = x_{4n} = x',$$
$$s\left(\frac{\omega}{n}\right) = x_{2n} = x,$$

so that
$$s\left(\frac{\omega}{2n}\right) = x-x'.$$

Then from the formula
$$2s\left(\frac{\omega}{2n}\right)+s\left(\frac{\omega}{n}\right)-s_\alpha-s_\beta-s_\gamma = \tfrac{1}{4}\left\{\frac{s''\dfrac{\omega}{2n}}{s'\dfrac{\omega}{2n}}\right\}^2,$$

where
$$s'^2\frac{\omega}{2n} = 4(x-x'-s_\alpha)(x-x'-s_\beta)(x-x'-s_\gamma),$$
$$\tfrac{1}{2}s''\frac{\omega}{2n} = (x-x'-s_\beta)(x-x'-s_\gamma)$$
$$+(x-x'-s_\gamma)(x-x'-s_\alpha)$$
$$+(x-x'-s_\alpha)(x-x'-s_\beta),$$

we obtain, after reduction, the equation
$$x'^4-x(y+1)x'^2-2x^2x'-x^3 = 0,$$

or, putting
$$x' = \frac{x}{r},$$
$$r = \frac{x_{2n}}{x_{4n}} = \frac{s\left(\dfrac{2\omega}{n}\right)-s\left(\dfrac{\omega}{n}\right)}{s\left(\dfrac{\omega}{n}\right)-s\left(\dfrac{\omega}{2n}\right)},$$
$$r^4+2r^3+(y+1)r^2-x = 0.$$

To solve this quartic equation, write it in the form

$$(r^2+r+t)^2 = (2t-y)\,r^2+2tr+t^2+x\,;$$

when the right-hand side will be a perfect square if

$$(t^2+x)(2t-y)-t^2 = 0,$$

or $$2t^3-t^2\,(y+1)+2tx-xy = 0,$$

or $$4t^2\,(t^2+x)^2-\{(y+1)\,t^2+xy\}^2 = 0,$$

so that we can take, from (321), (322), (325),

$$t^2 = s_\gamma,$$

$$t = \sqrt{s_\gamma} = \frac{ma}{a-m},$$

$$(2t-y)\,r^2+2tr+t^2+x = \frac{a}{a-m}\,(r+m)^2,$$

and thus the quartic for r or x' may be resolved.

As a preliminary verification, take $2n = 6$; then we can put

$$s_\gamma = y^2, \quad t = y$$

(*Proc. Lond. Math. Soc.*, Vol. xxv, p. 216); then

$$(r^2+r+y)^2 = y\,(r+1)^2,$$

$$r^2-r\,(\sqrt{y}-1)+\sqrt{y}\,(\sqrt{y}-1) = 0,$$

$$\left(r-\frac{\sqrt{y}-1}{2}\right)^2 = \frac{(\sqrt{y}-1)(-3\sqrt{y}-1)}{4}, \text{ or } \frac{(1-\sqrt{y})(1+3\sqrt{y})}{4}.$$

This quantity $y = y_6$ is found to be connected with the parameter $a = a_{12}$ by the relation

$$y = \frac{a^2}{(1+a+a^2)^2},$$

and taking $$\sqrt{y} = -\frac{a}{1+a+a^2},$$

$$1-\sqrt{y} = \frac{(1+a)^2}{1+a+a^2},$$

$$1+3\sqrt{y} = \frac{(1-a)^2}{1+a+a^2},$$

so that $$r = -\frac{a+a^2}{1+a+a^2},$$

and this agrees with the value of

$$r = \frac{x_6}{x_{12}} = \frac{s\frac{2}{3}\omega - s\frac{1}{3}\omega}{s\frac{1}{3}\omega - s\frac{1}{6}\omega} = \frac{-x + x\dfrac{y-x}{y^2}}{x} = \frac{y-x-y^2}{y^2}$$

$$= \frac{z}{p-z} = -\frac{a+a^2}{1+a+a^2}.$$

Passing on to the case of $2n = 10$, we have (*Proc. Lond. Math. Soc.*, Vol. xxv, p. 236)

$$x = x_{10} = -\frac{a(1+a)}{(1-a)(1-a-a^2)^2},$$

$$y = y_{10} = \frac{-a(1+a)}{(1-a)(1-a-a^2)},$$

$$t = \sqrt{s_7} = \frac{-a}{(1-a)(1-a-a^2)},$$

$$\left\{r^2 + r - \frac{a}{(1-a)(1-a-a^2)}\right\}^2 = \frac{-a}{1-a-a^2}\left(r + \frac{1}{1-a}\right)^2,$$

so that, putting $A = -a + a^2 + a^3$,

$$r^2 + r - \frac{a}{(1-a)(1-a-a^2)} = \frac{(1-a)r+1}{(1-a)(1-a-a^2)}\sqrt{A},$$

$$\left(r + \frac{1}{2}\,\frac{1-a-a^2+\sqrt{A}}{1-a-a^2}\right)^2 = \frac{(1+a)(1+a^2-2\sqrt{A})}{4(1-a)(1-a-a^2)},$$

and thence $r = \dfrac{s\frac{2}{5}\omega - s\frac{1}{5}\omega}{s\frac{1}{5}\omega - s\frac{1}{10}\omega}$ and $s\left(\dfrac{\omega}{10}\right)$

can be found; so that the case of $\mu = 20$ can now be considered as solved.

With $2n = 12$,

$$r = \frac{s\frac{1}{3}\omega - s\frac{1}{6}\omega}{s\frac{1}{6}\omega - s\frac{1}{12}\omega},$$

and now we take ("Pseudo-Elliptic Integrals," p. 248)

$$t = -\frac{a+a^2+a^3}{1-a} = \frac{A}{1-a},$$

so that $\left(r^2 + r + \dfrac{A}{1-a}\right)^2 = \left(r + \dfrac{1}{1-a}\right)^2 A,$

and thence r; this solves the case of $\mu = 24$.

With $$2n = 14,$$

$$r = \frac{\wp\frac{2}{7}\omega - \wp\frac{1}{7}\omega}{\wp\frac{1}{7}\omega - \wp\frac{1}{14}\omega},$$

and we take

$$t = \sqrt{s_7} = \frac{c(1+c)}{2(1+c-2c^2-c^3)}\{c(1+c)(1-2c)+(1-c)\sqrt{C}\},$$

where $$C = c(1+2c)(4+5c+2c^2),$$

$$y = y_{14} = c\,\frac{0+3c+6c^2-4c^3-10c^4-c^5+(1+2c-2c^2-2c^3)\sqrt{C}}{2(1+c)(1+c-2c^2-c^3)},$$

$$x = x_{14} = c(1+c)\,\frac{\begin{array}{c}0+3c+6c^2-9c^3-21c^4+0+16c^6+8c^7\\ +(1+2c-3c^2-5c^3+2c^4+4c^5)\sqrt{C}\end{array}}{(1+c-2c^2-c^3)^2},$$

so that $\wp\frac{1}{14}\omega$ can be found, which solves the case of $\mu = 28$.

In a similar way the case of $\mu = 32$ can be derived from $\mu = 16$, $\mu = 36$ from $\mu = 18$, $\mu = 44$ from $\mu = 22$, &c.

50. Considering now the transformation of order n and $2n$ together, the $x, y, z, \ldots$ obtained for a transformation of order n will be the $x_2, y_2, z_2, \ldots$ for the order $2n$.

Thus, starting with $$n = 5,$$

the relation $$\gamma_5 = y - x = 0, \tag{354}$$

leads to $$a = \frac{m(1-m)^2}{1-2m}, \tag{355}$$

$$a - m = \frac{m^3}{1-2m}, \tag{356}$$

so that $$y = x = -\left(\frac{1-m}{m}\right)^2(1-2m). \tag{357}$$

But, from the transformation of the Tenth order ("Pseudo-Elliptic Integrals," p. 235),

$$y_5 = \frac{\gamma_8}{\gamma_5^4} = -\frac{y^3z^2(z+p-1)}{y^5p} = \frac{a^2-a^3}{1-a}, \tag{358}$$

so that we must take $$m = \frac{1}{1+a}. \tag{359}$$

Otherwise, $$\frac{s_7-s_3}{s_7-s_1}=\frac{s_5'-s_4'}{s_5'-s_2'}\tag{360}$$

where the accented letters refer to the transformation of order $2n$; so that

$$\left(\frac{1-m}{m}\right)^2=\frac{\gamma_9\gamma_1}{\gamma_5^2\gamma_4^2}\frac{\gamma_5^2\gamma_3^2}{\gamma_7\gamma_3}=\frac{\gamma_9}{\gamma_7\gamma_4^3\gamma_3}=\frac{p+c}{p-z}.\tag{361}$$

Therefore, for the Tenth order,

$$\left(\frac{1-m}{m}\right)^2=\frac{p-1-a}{p-z}=a^2,\tag{362}$$

as before in (359).

With $$n=6,$$

$$x=y-y^2;\tag{363}$$

and therefore either $$m=1,\tag{364}$$

or $$\alpha=\frac{m(1-m)}{2(1-2m)}.\tag{365}$$

With $$m=1,$$

$$\frac{\alpha}{\alpha-1}=\frac{s_7-s_3}{s_7-s_1}=\frac{s_6'-s_4'}{s_6'-s_2'}$$

$$=\frac{\gamma_{10}}{\gamma_8\gamma_4^3}=\frac{\lambda^4}{\gamma_4^4},\tag{366}$$

$$\sqrt{\left(\frac{\alpha}{\alpha-1}\right)}=\frac{\gamma_7}{\gamma_5\gamma_4^2}=-\frac{z^2}{cpy}$$

$$=\frac{z}{(1+a)(p-z)}$$

$$=\frac{-a}{1+a+a^2}\tag{367}$$

("Pseudo-Elliptic Integrals," p. 248),

$$\alpha=\frac{-a^3}{(1+a)^2(1+a^2)}.\tag{368}$$

But, with (365),

$$a = \frac{m(1-m)}{2(1-2m)},$$

$$\frac{a}{a-m} = -\frac{1-m}{1-3m} = \frac{a^2}{(1+a+a^2)^2},$$

$$m = \frac{1+2a+4a^2+2a^3+a^4}{1+2a+6a^2+2a^3+a^4}, \tag{369}$$

$$a = \frac{a^2(1+2a+4a^2+2a^3+a^4)}{(1+2a+6a^2+2a^3+a^4)(1+2a+8a^2+2a^3+a^4)}. \tag{370}$$

51. But, without these details, we notice that the transformation is effected, in terms of a single parameter, either by putting

$$s_\gamma - s_n = \infty, \tag{371}$$

for the order n; or $$s_\gamma - s_n = 0, \tag{372}$$

$$s_\gamma - s_{2n} = \infty, \tag{373}$$

for the order $2n$.

In this way we obtain either

$$m = 1,$$

or $$a = \frac{m(1-m)}{2(1-2m)},$$

for the order $2n = 6$.

For the Eighth order, put

$$s_\gamma - s_4 = 0,$$

or (329) $$a = \frac{m(1-m)}{1-2m}, \quad a-m = \frac{m^3}{1-2m}; \tag{374}$$

so that $$x = -(1-m)(1-2m), \tag{375}$$

$$y = -\frac{(1-m)(1-2m)}{m}, \tag{376}$$

and $$m = 1-z \tag{377}$$

(" Pseudo-Elliptic Integrals," p. 226).

For the Tenth order, put

$$s_\gamma - s_5 = 0,$$

or (330)
$$a = \frac{m^3(1-m)}{1-2m}, \tag{378}$$

$$x = -\frac{m^3(1-m)(1-2m)}{(1-3m+m^2)^2}, \tag{379}$$

$$y = \frac{m(1-m)(1-2m)}{1-3m+m^2}, \tag{380}$$

so that
$$m = \frac{1}{1-a} \tag{381}$$

("Pseudo-Elliptic Integrals," p. 236).

For the Twelfth order, put (331)

$$s_\gamma - s_6 = 0, \quad \text{or} \quad a = \frac{m(1-m)(1-3m+3m^2)}{(1-2m)(1-2m+2m^2)}, \tag{382}$$

$$a - m = \frac{m^4}{(1-2m)(1-2m+2m^2)}, \tag{383}$$

$$x = -\frac{(1-m)(1-2m)(1-2m+2m^2)(1-3m+3m^2)}{m^4}, \tag{384}$$

$$y = -\frac{(1-m)(1-2m)(1-3m+3m^2)}{m^3}, \tag{385}$$

$$1-z = \frac{y}{x} = \frac{1-2m+2m^2}{m} = 1 + \frac{a+a^2}{1-a}, \tag{386}$$

so that
$$m = \frac{1}{1-a} \tag{387}$$

("Pseudo-Elliptic Integrals," p. 248).

52. For the Fourteenth order, put (332)

$$s_\gamma - s_7 = 0, \tag{388}$$

or $(1-2m)^3 a^2 - m(1-m)(1-2m)(2-3m)a + m^2(1-m)^4 = 0,$ (389)

so that, putting $(1-2m)a = m(1-m)\gamma,$

$$(1-2m)\gamma^2 - (2-3m)\gamma + (1-m)^2 = 0. \tag{390}$$

We connect up with the results on p. 257 of "Pseudo-Elliptic Integrals," by calculating

$$\left(\frac{1-m}{m}\right)^2 = \frac{s_7-s_3}{s_7-s_1} = \frac{\gamma_{10}\gamma_4}{\gamma_8\gamma_6\gamma_3^2} = \frac{\lambda^4\gamma_4^2}{\gamma_6^2\gamma_3^2}, \tag{391}$$

$$\frac{1-m}{m} = \frac{\lambda^2\gamma_4}{\gamma_6\gamma_3} = \frac{\gamma_8\gamma_4}{\gamma_6^2\gamma_3} = -\frac{(1+c)(p-z)}{c(1-z)}$$

$$= \frac{-c+\sqrt{C}}{2(1+c)^2}, \tag{392}$$

so that

$$m = \frac{2+3c+2c^2-\sqrt{C}}{2} \tag{393}$$

and this makes

$$z_7 = -\frac{m}{c}, \tag{394}$$

$$\sqrt{s_7} = \frac{c(1+c)}{2(1+c-2c^2-c^3)}\{c(1+c)(1-2c)+(1-c)\sqrt{C}\}. \tag{395}$$

For the Sixteenth order ("Pseudo-Elliptic Integrals," p. 262), with

$$s_7-s_8 = 0, \tag{396}$$

$$\left(\frac{1-m}{m}\right)^2 = \frac{s_8-s_3}{s_8-s_1} = \left(\frac{z}{1-z}\right)^2, \tag{397}$$

so that

$$m = 1-z = \frac{1}{a^2+1}. \tag{398}$$

For the Eighteenth order (p. 265), the various relations

$$\gamma_9 = 0, \quad \text{or} \quad s_5 = s_4, \quad s_6 = s_3, \text{ \&c.}, \tag{399}$$

will be found to lead to a certain equation between a and m; and, putting

$$a = \frac{m(1-m)}{1-2m}\gamma, \tag{400}$$

$$1-2m = \frac{1}{x-1}, \tag{401}$$

$$1-2\gamma = \frac{1}{y-1}, \tag{402}$$

and

$$y = (1+q)x, \tag{403}$$

we are led to the equation

$$(q+1)^2 x^2+(q^3+q^2-2q-1)\,x-2q^2 = 0, \tag{404}$$

having the discriminant

$$(q^3+q^2-2q-1)^2+8q^2\,(q+1)^2$$
$$= q^6+2q^5+5q^4+10q^3+10q^2+4q+1 = Q, \tag{405}$$

so that x has here the same signification as on p. 266 of "Pseudo-Elliptic Integrals"; and now the rest of the identification can be effected.

53. But the Twenty-second order is of importance as affording an independent determination for this order of Kiepert's parameters

$$\xi,\ \xi_1,\ \xi_2.$$

We start by putting

$$s_\gamma-s_5 = s_\gamma-s_6, \tag{406}$$

and obtaining a quintic equation in a, from (330) and (331).

Putting
$$\frac{a}{a-m} = t^2,\quad a = \frac{mt^2}{t^2-1}, \tag{407}$$

and taking the square root, we obtain a quintic equation in t, and in m; and this, on putting

$$m = \frac{1}{1+n},$$

becomes
$$t^5-n\,(2-n)(1+n-n^2)\,t^4-n\,(1+n^2)\,t^3$$
$$+n^2\,(3-n^2)\,t^2+n^3\,(1-n+n^2)\,t-n^3 = 0, \tag{408}$$

a quintic in t, and in n.

But the relation
$$\frac{\gamma_7}{\gamma_4} = \left(\frac{\gamma_6}{\gamma_5}\right)^3,$$

which is the equivalent of $\gamma_{11} = 0$,

leads to the $(a,\,m)$ equation in the form

$$\frac{(1-2m)\,a-m\,(1-m)^2}{(1-2m)(a-m)}+\frac{(1-2m)^2\,a}{m^3}$$
$$= \left\{1+\frac{(1-2m)^2\,a}{(1-2m)\,a-m\,(1-m)^2}\right\}^3. \tag{409}$$

I am indebted to Professor E. B. Elliott for the substitution

$$1+\frac{(1-2m)^2\,a}{(1-2m)\,a-m\,(1-m)^2}=\frac{1-m}{m}\,r,$$

or

$$a=\frac{m\,(1-m)}{1-2m}\;\frac{(1-m)\,r-m}{r-2m}, \tag{410}$$

$$a-m=\frac{m^2}{1-2m}\;\frac{mr+1-3m}{r-2m}, \tag{411}$$

which makes $$s_7-s_5=\frac{1-m}{m}\;\frac{(1-m)\,r-m}{mr+1-3m}\,(r-m)^2, \tag{412}$$

$$s_7-s_6=\frac{\{(1-m)\,r-m\}^2\,(mr+1-2m)^2}{(mr+1-3m)^2\,r_2^2}, \tag{413}$$

and leads to the equation

$$\begin{aligned}4m^4&+2\,(r^4-3r^3+r^2-3r-2)\,m^3\\&-(r^5-r^4-8r^3+r^2-5r-1)\,m^2\\&+r\,(r^4-4r^3-2r^2+0-1)\,m+r^4=0,\end{aligned} \tag{414}$$

a *quartic* in m; but a quintic in r.

The resolution of this quartic was effected by forming its resolving cubic

$$4s^3-g_2s-g_3=0, \tag{415}$$

and noticing that, if it has a rational root in r, this root must be of the form

$$12s=r^5+Ar^4+Br^3+Cr^2+Dr-1. \tag{416}$$

It was then found that the special numerical values of r,

$$r=1,\ -1,\ 2,\ 3,\ \text{made}\ 12s=-1,\ -25,\ -19,\ -97;$$

hence $$A=-7,\quad B=10,\quad C=-5,\quad D=1;$$

and the required root of the general resolving cubic is thus given by

$$12s=r^5-7r^4+10r^3-5r^2+r-1, \tag{417}$$

and this was found to verify; as, putting $12s=e$, the resolving cubic breaks up into the linear factor

$$e-r^5+7r^4-10r^3+5r^2-r+1,$$

and the quadratic factor

$$\begin{aligned}2e^2&+(r^5-7r^4+10r^3-5r^2+r-1)\,e\\&-r^{10}+5r^9-6r^8-3r^7-r^6+17r^5-14r^4+3r^3-2r^2+2r-1.\end{aligned}$$

The discriminant of the quadratic factor is

$$9\,(r^{10}-6r^9+13r^8-14r^7+20r^6-28r^5+19r^4-6r^3+3r^2-2r+1)$$
$$=9\left[\{(r^5-3r^4-3r^3+4r^2+r-1)^2+5r^2(r-1)\}^2+8r^4(r-1)^2\right]$$
$$=9r^4(r-1)^2\{(H+5)^2+8\},$$

on putting $$H=\frac{r^5-3r^4-3r^3+4r^2+r-1}{r^2(r-1)}.$$

The quartic (414) can now be written

$$\{4m^2+(r^4-3r^3+r^2-3r-2)\,m-r\,(r^3-3r^2+r-1)\}^2$$
$$=r^2(r-1)^3(r^3-3r^2-r-1)(m-1)^2, \tag{418}$$

and the resolution of the quartic in m is thus effected.

54. Professor Elliott points out further that, if we put

$$y=-\frac{(2m-1)(m-r)}{r\,(r-1)(m-1)}, \tag{419}$$

then the quartic reduces to a quadratic in y,

$$y^2-(r^2-2r-1)\,y+r=0, \tag{420}$$

and further, putting $$r=\frac{x}{x+1}, \tag{421}$$

this quadratic assumes the symmetrical form

$$x^2y^2+2xy\,(x+y)+x^2+4xy+y^2+x+y=0, \tag{422}$$

or, putting $$x+y=p,\quad xy=q, \tag{423}$$

$$(p+q)^2+p+2q=0. \tag{424}$$

Thence we can deduce

$$x=\frac{2c+1+\sqrt{C}}{2c^2},\quad y=\frac{2c+1-\sqrt{C}}{2c^2}, \tag{425}$$

where $$C=4c\,(c+1)^2+1, \tag{426}$$

so that we may put $$x=\wp\,(u-\tfrac{2}{5}\omega)-\tfrac{2}{3}, \tag{427}$$

$$y=\wp\,(u+\tfrac{2}{5}\omega)-\tfrac{2}{3}, \tag{428}$$

and y is now found to be the reciprocal of c_{22}.

55. Professor Elliott has also made a similar reduction for the equation (263), connecting

$$q = q_{23} \quad \text{and} \quad c = c_{23},$$

by writing it

$$(1+c)^2 q^2 (q-c-c^2)^2 + c^2 q (q-c-c^2)\{(1+2c)\,q-c^2-c^3\}$$
$$-c\{(1+2c)\,q-c^2-c^3\}^2 = 0, \quad (429)$$

and now, if we put

$$z = \frac{q(q-c-c^2)}{(1+2c)\,q-c^2-c^3}, \quad (430)$$

the quartic (429) reduces to the quadratic

$$(1+c)^2 z^2 + c^2 z - c = 0. \quad (431)$$

Here again, by putting

$$c = \frac{1}{y}, \quad z = -x-1, \quad (432)$$

the quadratic (431) becomes the same as (422).

The relations connecting this q and c with m and r may be written

$$r = \frac{m}{1-m}\,\frac{q+1+c}{q-c-c^2}$$

$$= -\frac{1+c}{c}\,\frac{q}{q-c^2}\,\frac{q-c-c^2}{q+1+c}; \quad (433)$$

and the elimination of q between this and (264) leads to the relation

$$(2c+4c^2+c^3)\,r - c^2(1+c) + (cr+1+c)\sqrt{(4c+8c^2+4c^3+c^4)} = 0, \quad (434)$$

or

$$c^2 r^2 - (2c+c^2)\,r - 1 - c = 0, \quad (435)$$

a quadri-quadric relation between c and r, which becomes the same as (422) when we put, as before,

$$c = \frac{1}{y}, \quad r = \frac{x}{x+1}.$$

2 H 2

The elimination of m between (414) and

$$p = \frac{m(1-2m)}{r(r-3)m+r},$$

where $$p = p_{11} = 1 + c_{11},$$

is found to lead to the equation

$$\begin{aligned} p^4 &- (r^4 - 4r^3 + 5r^2 + 0 + 2)\,p^3 \\ &+ (r^6 - 4r^5 + 5r^4 - 4r^3 + 6r^2 + r + 1)\,p^2 \\ &+ r(r+1)(r^3 - 2r^2 + 0 - 1)\,p + r^3 = 0. \qquad (436) \end{aligned}$$

Expressed as the difference of two squares, preparatory to resolution, the equation may be written

$$\begin{aligned} &\left[2p(p - r^2 + r - 1) - r(r^2 - 2r - 1)\{(r-2)p + 1\}\right]^2 \\ &\qquad = r^2(r-1)(r^3 - 3r^2 - r - 1)\{(r-2)p + 1\}^2. \qquad (437) \end{aligned}$$

In Professor Elliott's procedure, this quartic equation is replaced by two quadratic relations, by putting

$$u = \frac{p(p - r^2 + r - 1)}{r(r-2)p + r}, \qquad (438)$$

and then (436) becomes

$$u^2 - (r^2 - 2r - 1)\,u + r = 0, \qquad (439)$$

the same as (420); so that, u and y, if not equal, are the two roots of this quadratic, and

$$\frac{p(p - r^2 + r - 1)}{r(r-2)p + r} + \frac{(2m-1)(m-r)}{r(r-1)(m-1)} = 0,$$

or $$\frac{p(p - r^2 + r - 1)}{r(r-2)p + r} - \frac{(2m-1)(m-r)}{r(r-1)(m-1)} = r^2 - 2r - 1.$$

Here, as before in (419), it is the relation (438) which still requires interpretation, as an elliptic function formula.

56. We connect up these functions m and r with the z_{11} and p_{11}, employed in the transformation of the Eleventh Order, by the relations

$$\frac{z_{11}}{p_{11}}=\frac{s_4-s_3}{s_4-s_1}=\frac{1-m}{m}\,r, \tag{440}$$

after reduction; while

$$z_{11}=\frac{s_3-s_2}{s_3-s_1}=\frac{(1-2m)\,a+m\,(1-m)^2}{(1-2m)(a-m)}$$

$$=\frac{(1-m)(1-2m)}{mr+1-3m}, \tag{441}$$

$$p_{11}=\frac{s_2-s_1}{s_5-s_1}=\frac{m\,(1-2m)}{(mr+1-3m)\,r}. \tag{442}$$

Expressed in terms of m and r,

$$s_\gamma-s_1=m\,(1-m)\,\frac{(1-m)\,r-m}{mr+1-3m}, \tag{443}$$

$$s_\gamma-s_2=(1-m)^2\left\{\frac{(1-m)\,r-m}{mr+1-3m}\right\}^2, \tag{444}$$

$$s_\gamma-s_3=\frac{(1-m)^3}{m}\,\frac{(1-m)\,r-m}{mr+1-3m}, \tag{445}$$

$$s_\gamma-s_4=m^2(1-m)^2\left(\frac{r-1}{mr+1-3m}\right)^2, \tag{446}$$

$$s_\gamma-s_5=\frac{1-m}{m}\,(r-m)^2\,\frac{(1-m)\,r-m}{mr+1-3m}, \tag{447}$$

$$s_\gamma-s_6=\frac{\{(1-m)\,r-m\}^2\,(mr+1-2m)^2}{(mr+1-3m)^2\,r^2},$$

$$s_\gamma-s_7=m\,(1-m)\,\frac{(1-m)\,r-m}{mr+1-3m}\left[\frac{(1-m)\,r^2-mr+m\,(1-2m)}{mr^2+(1-3m)\,r-m\,(1-2m)}\right]^2,$$

&c., &c.,

so that

$$\prod_{r=1}^{r=5}(s_\gamma-s_r)=\frac{m\,(1-m)^9\,(r-1)^2\,(r-m)^2\,\{(1-m)\,r-m\}^5}{(mr+1-3m)^7}, \tag{448}$$

while

$$s_1 - s_2 = \frac{(1-m)(1-2m)(r-2m)\{(1-m)r-m\}}{(mr+1-3m)^2}, \tag{449}$$

$$s_2 - s_4 = \frac{-(1-m)^2(1-2m)r}{(mr+1-3m)^2}, \tag{450}$$

$$s_4 - s_3 = \frac{(1-m)^2\{(1-3m+m^2)r^2+(1-2m)(1-3m)r-m(1-2m)^2\}}{m(mr+1-3m)^2}, \tag{451}$$

$$s_3 - s_5 = \frac{(1-m)(r-1)(r+1-2m)\{(1-m)r-m\}}{m^2(mr+1-3m)}, \tag{452}$$

$$s_5 - s_1 = \frac{(1-m)r(r-2m)\{(1-m)r-m\}}{m(mr+1-3m)}; \tag{453}$$

and therefore Kiepert's f is given by

$$f^{-2} = -(s_1-s_2)(s_2-s_4)(s_4-s_3)(s_3-s_5)(s_5-s_1)$$

$$= \frac{(1-m)^7(1-2m)^2r^2(r-1)(r-2m)^2(r+1-2m)\{(1-m)r-m\}^3}{m^4(mr+1-3m)^8}$$

$$\times\{(1-3m+m^2)r^2+(1-2m)(1-3m)r-m(1-2m)^2\}, \tag{454}$$

whence Kiepert's $$T^2 = f^2\,\Pi(s_\gamma - s_r). \tag{455}$$

57. Also, from (341),

$$(s_\alpha - s_\gamma)(s_\beta - s_\gamma) = \frac{(m-1)^3\{m(r+1)-1\}^2\{m^2(r-3)-m(r+2)+r\}}{\{m(r-3)+1\}^3}, \tag{456}$$

so that, as in Halphen's *F. E.*, III, p. 245,

$$\xi_1 = \frac{L(22)^4}{L(11)^8L(2)^4} = \frac{f(22)^4}{f(11)^8f(2)^4}$$

$$= \frac{\Pi(s_\gamma - s_r)^2}{(s_\alpha - s_\gamma \,.\, s_\beta - s_\gamma)^5}, \tag{457}$$

$$= -\frac{m^2(m-1)^3(r-1)^4(m-r)^4\{m(r-3)+1\}}{\{m^2(r-3)+m(r+2)-r\}^5}. \tag{458}$$

58. Kiepert's parameter ξ can be calculated from the formula

$$\xi = \Delta \frac{f(22)^2 f(2)^2}{f(11)^2} = \Delta f(2)^{24} T^2 = -\frac{16T^2}{\kappa^2\kappa'^2}, \tag{459}$$

or

$$\frac{\xi^2}{\xi_1} = \frac{(\Delta f^2)^2}{s_\alpha - s_\gamma \,.\, s_\beta - s_\gamma}; \tag{460}$$

and then we find, in terms of m and r,

$$\xi = -\frac{(r-1)(m-r)^2(2m-1)^2\{4m^2(r+1)-2m(3r+1)+r\}}{r^2(2m-r-1)(m-1)\{2m^2+m(r^2-3r-1)+r\} \times \{m^2(r-3)+m(r+2)-r\}},$$

and thence Kiepert's η, or Klein and Fricke's τ, by the relation

$$\eta+8 = \frac{1}{\tau} = \frac{\xi^3+4\xi^2+8\xi+4}{\xi}. \tag{461}$$

59. To find the relation between Kiepert's ξ or ξ_1 and our r, we must eliminate m between these equations and the (m, r) equation (414); the work, which is very laborious, has been carried out for me by Mr. G. H. Stuart.

Contrary to anticipation, the equation for ξ_1 in terms of r is the more complicated; it is a quartic in ξ_1, but of the twenty-fifth degree in r; however, it was noticed that the coefficients of ξ_1 could all be expressed rationally in powers of

$$H = \frac{r^5-3r^4-3r^3+4r^2+r-1}{r^2(r-1)}, \tag{462}$$

so that the quartic for ξ_1 could be written

$$\begin{aligned} H^3\xi_1^4 &- (2H^5+80H^4+792H^3+2816H^2+3509H+1331)\,\xi_1^3 \\ &+ (H^5+16H^4+88H^3+184H^2-342H-2651)\,\xi_1^2 \\ &+ (8H^2+75H+143)\,\xi_1 - 1 = 0, \end{aligned} \tag{463}$$

or, resolved as the difference of two squares,

$$\begin{aligned} \{(8H^2+21H+11)\,\xi_1^2+(8H^2+75H+143)\,\xi_1-2\}^2 \\ = \{(H+1)\,\xi_1-(H+9)\}\,H'^2\xi_1^2, \end{aligned} \tag{464}$$

where, as in (167),

$$\begin{aligned} H'^2 &= 4H^3+56H^2+220H+121 \\ &= 4H^2(H-11)+(10H+11)^2. \end{aligned} \tag{465}$$

Similarly, Mr. G. H. Stuart found that the quartic for ξ in terms of r could be written

$$H(\xi^2+4)^2+(4H-11)(\xi^2+4)\,\xi-(H^2+10H+11)\,\xi^2=0, \quad (466)$$

or

$$\xi+\frac{4}{\xi}=\frac{-4H+11+H'}{2H}. \quad (467)$$

The elimination of H between these two equations (463) and (466) will be found to lead to a reciprocal quartic in ξ_1, which breaks up into Kiepert's two quadratic equations (303) and (306) (*Math. Ann.*, XXXII, p. 92); for, from (467), in Kiepert's notation

$$H'=2(u^2-2)\,H-11,$$

so that (464) can be written

$$(8H^2+21H+11)\,\xi_1^2+(8H^2+75H+143)\,\xi_1-2$$
$$+\{(H+1)\,\xi_1^2-(H+9)\,\xi_1\}\{2(u^2-2)\,H-11\}=0, \quad (468)$$

and eliminating H between (466) and (468), two quadratics in H, will lead to the result.

60. Writing equation (467)

$$\left(\sqrt{\xi}-\frac{2}{\sqrt{\xi}}\right)^2=u^2-8=\frac{-12H+11+H'}{2H}, \quad (469)$$

then, from (278),

$$b-11=11^2\,\frac{2H}{-12H+11+H'}$$
$$=11^2\,\frac{12H-11+H'}{2(H-11)^3},$$
$$b=11\,\frac{22H^2+88H+121+11H'}{2(H-11)^2}. \quad (470)$$

We may distinguish this H by writing it $H(\theta)$, where θ denotes the elliptic argument; as

$$H(\theta)=\wp(\theta;\,g_2,g_3)-\tfrac{14}{3},$$
$$H'(\theta)=\wp'(\theta;\,g_2,g_3),$$

where (*M.F.*, II, p. 444) $g_2=\dfrac{124}{3}$, $g_3=\dfrac{41\,.\,61}{27}$,

$$g_2^3-27g_3^2=-11^5, \quad J=-\frac{2^6\,.\,31^3}{3^3\,.\,11^5}; \quad (471)$$

and now we find we can put

$$b = H\left(\theta+\tfrac{8}{5}\omega\right),$$

while

$$\xi+\frac{4}{\xi}=\frac{4b+77}{b-11};$$

$$u^2=\left(\sqrt{\xi}+\frac{2}{\sqrt{\xi}}\right)^2=\frac{8b+33}{b-11},$$

$$\bar{u}^2=u^2-8=\left(\sqrt{\xi}-\frac{2}{\sqrt{\xi}}\right)^2=\frac{121}{b-11}.$$

61. Klein and Fricke's τ is also an elliptic function, which may be distinguished by its elliptic argument ϕ as $\tau(\phi)$; in fact,

$$H(\phi)=-11\tau(\phi),\quad H'(\phi)=11\tau'(\phi).$$

Let ϕ' denote the argument of τ when ξ is changed into $\frac{4}{\xi}$, the effect of which is to change $2\kappa\kappa'$ into its reciprocal, or to change from a positive to a negative discriminant Δ, or from Klein's J to Kiepert's J; then

$$\tau(\phi)=\frac{\xi}{\xi^3+4\xi^2+8\xi+4},$$

$$\tau(\phi')=\frac{\xi^2}{\xi^3+8\xi^2+16\xi+16},$$

$$\tau'(\phi)=\frac{(\xi^3+2\xi^2+0-2)\,w}{(\xi^3+4\xi+8\xi+4)^2},$$

$$\tau'(\phi')=\frac{(\xi^3+0-16\xi-32)w}{(\xi^3+8\xi^2+16\xi+16)^2},$$

where

$$w^2=(\xi^3+4\xi^2+8\xi+4)(\xi^3+8\xi^2+16\xi+16).$$

Thence, by means of the addition formula

$$H(\phi\pm\phi')+H(\phi)+H(\phi')+14=\tfrac{1}{4}\left\{\frac{H'(\phi)-H'(\phi')}{H(\phi)-H(\phi')}\right\}^2,$$

or

$$-11\left\{\tau(\phi\pm\phi')+\tau(\phi)+\tau(\phi')\right\}+14=\tfrac{1}{4}\left\{\frac{\tau'(\phi)\mp\tau'(\phi')}{\tau(\phi)-\tau(\phi')}\right\}^2, \qquad (472)$$

we find, after reduction, that

$$H(\phi+\phi') = \frac{11\xi^2+77\xi+44}{(\xi-2)^2}$$

$$= 11 + \frac{11^2}{\left(\sqrt{\xi}-\frac{2}{\sqrt{\xi}}\right)^2}$$

$$= H(\theta+\tfrac{8}{5}\omega), \tag{473}$$

so that we can take $\phi+\phi' = \theta+\frac{8}{5}\omega.$ (474)

Similarly, with $u^2 = \left(\sqrt{\xi}+\frac{2}{\sqrt{\xi}}\right)^2 = \frac{1}{a+1},$ (475)

so that $\frac{1}{a+1} - \frac{121}{b-11} = 8,$ (476)

as in (275), we find

$$H(\phi-\phi') = \frac{a^5-2a^4-5a^3+2a^2+4a+1}{a^2(a+1)^2}, \tag{477}$$

so that $\phi-\phi' = \theta',$ (478)

if θ' is the argument of the elliptic function a, which is such that

$$a(\theta') = \wp(\theta';\ g_2, g_3) - \tfrac{2}{3},$$

with $g_2 = \frac{4}{3},\quad g_3 = -\frac{19}{27},\quad J = -\frac{2^{12}}{11};$

an elliptic function already employed.

These relations (474) and (478) may serve as interpretations of the elliptic function properties implied in equations (419) and (438).

62. Thence $\phi = \frac{1}{2}(\theta+\theta')+\frac{4}{5}\omega,$

so that, starting with c in the transformation of the Eleventh Order, we may put

$$c_{11} = c = \wp\tfrac{1}{2}(\theta+\theta')-\tfrac{2}{3},\quad \text{with}\quad g_2 = \tfrac{4}{3},\quad g_3 = -\tfrac{19}{27};$$

$$\sqrt{C} = \sqrt{(4c^3+8c^2+4c+1)} = \wp'\tfrac{1}{2}(\theta+\theta'),$$

and then

$$-11\tau = \wp\left\{\tfrac{1}{2}(\theta+\theta')+\tfrac{4}{5}\omega\right\}-\frac{14}{3}, \quad \text{with} \quad g_2=\frac{124}{3}, \quad g_3=\frac{41\,.\,61}{27}.$$

$$11r' = \wp'\left\{\tfrac{1}{2}(\theta+\theta')+\tfrac{4}{5}\omega\right\}.$$

Putting in (462) $\qquad r=\dfrac{c}{c+1}$

gives
$$H(\theta)=\frac{c^5-2c^4-5c^3+2c^2+4c+1}{c^2(c+1)^2}, \tag{479}$$

so that $\quad c=c(\theta)=\wp(\theta)-\frac{2}{3}, \quad$ with $\quad g_2=\frac{4}{3}, \quad g_3=-\frac{19}{27}$;

and then
$$c_{22}=\frac{1}{c\,(p+\frac{2}{5}\omega)}. \tag{480}$$

The duplication formula, for

$$H=H(\theta) \quad \text{and} \quad \tau=\tau(\theta),$$

$$H(2\theta)=-\frac{(H-11)(H^3+11H^2+11H-121)}{4H^3+56H^2+220H+121}, \tag{481}$$

or
$$\tau(2\theta)=-\frac{(1+\tau)(1+\tau-11\tau^2+11\tau^3)}{1-20\tau+56\tau^2-44\tau^3}, \tag{482}$$

will often be required in the numerical applications.

63. So also the relation $\quad s_\gamma-s_{11}=0 \qquad (483)$

may be replaced by (337), in the form

$$(s_\gamma-s_5)(s_\gamma-s_6)=(s_\alpha-s_\gamma)(s_\beta-s_\gamma);$$

and, from (447) and (456),

$$\frac{(1-m)(r-m)^2\{(1-m)\,r-m\}^3(mr+1-2m)^2}{m\,(mr+1-3m)^3\,r^2}$$
$$=\frac{(1-m)^3\{(1-m)\,r-m\}^2\{(1-m-m^2)r-2m+3m^2\}}{(mr+1-3m)^3}, \tag{484}$$

or $(r-m)^2\{(1-m)\,r-m\}(mr+1-2m)^2$
$$=m\,(1-m)^2\{(1-m-m^2)\,r-2m+3m^2\}, \tag{485}$$

a quintic in m, and in r; and Mr. James Hammond has found that this (m, r) relation becomes the same as (414) if we write

$$1-m \text{ for } m, \quad \text{and} \quad \frac{(1-m)\,r}{r-m} \text{ for } r.$$

64. The special numerical values in the cases of Complex Multiplication implied in the Modular Equation of the Eleventh Order provide interesting applications of the preceding theory.

Taking Kiepert's form of the modular equation (325*b*) (*Math. Ann.*, XXXII, p. 98), it will be found that the coefficient of W can be resolved into the factors

$$(\eta+6)(\eta+7)(\eta+1)(\eta+4)(\eta-8)(\eta^2+2\eta-44)(\eta^2+4\eta-16),$$

and these are found to correspond to the cases of complex multiplication where the ratio of the periods

$$\frac{K'}{K}=\sqrt{2},\ \sqrt{7},\ \sqrt{7},\ \sqrt{(19)},\ \sqrt{(43)},\ \sqrt{(10)},\ \sqrt{(35)};$$

and then $L(11)^2$ is found to be the corresponding complex multiplier, so that

$$L(11)^2=3i-\sqrt{2},\quad -2i+\sqrt{7},\quad 2i+\sqrt{7},\quad \tfrac{1}{2}\{-5i+\sqrt{(19)}\},$$
$$\tfrac{1}{2}\{-i+\sqrt{(43)}\},\quad i+\sqrt{(10)},\quad \tfrac{1}{2}\{-3i+\sqrt{(35)}\}.$$

Kiepert's notation can be connected up with that employed by Brioschi (*Annali di Matematica*, XXI, 1893, p. 309) by putting

$$L(11)=z\left\{\text{or } \frac{i\sqrt{(11)}}{z}\right\},\quad l=\frac{z^4}{11^2},\ \&c.$$

65. In a similar manner, when the ratio of the periods

$$\frac{K'}{K}=\sqrt{(22-\rho^2)}=\sqrt{(21)},\ \sqrt{(18)},\ \sqrt{(13)},\ \sqrt{6},$$

we may take $L(22)^2$ as the corresponding complex multiplier, and

$$L(22)^2=1+i\sqrt{(21)},\quad 2+i\sqrt{(18)},\quad 3+i\sqrt{(13)},\quad 4+i\sqrt{6};$$

also,

$$L(2)^{24}=\xi^6\eta'=-64\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^4,\quad 64(\sqrt{3}+\sqrt{2})^8,$$
$$-\{\sqrt{(13)}+3\}^6,\quad 64(\sqrt{2}+1)^4;$$

derived from the corresponding special values of the modulus given in the *Proc. Lond. Math. Soc.*, Vol. XIX, p. 301.

When the ratio of the periods is $\sqrt{(13)}$, we shall find that

$$\xi=2i,\quad \xi^3=-8i,\quad u^2=4,\quad v=10\sqrt{(13)},\quad w=-3\sqrt{(13)},$$
$$\sqrt{\eta'}=5\sqrt{(13)}+18,$$
$$L(2)^{24}=-\frac{16}{\kappa^2\kappa'^2}=\xi^6\eta'=-64\{5\sqrt{(13)}+18\},$$

or

$$2\kappa\kappa'=5\sqrt{(13)}-18=\left\{\frac{\sqrt{(13)}-3}{2}\right\}^3,$$

a verification of the well-known result.

Also $$\tau(\phi) = \frac{2-3i}{26}, \quad \tau'(\phi) = \frac{57+103i}{26\sqrt{(13)}},$$
$$H(\phi+\phi') = -\frac{77}{4}, \quad a(\theta') = -\frac{3}{4}, \text{ \&c.}$$

So also we find that, for

$$K'/K = \sqrt{6}, \quad \xi = -1+i\sqrt{3}, \quad \xi^3 = 8, \quad u^2 = 2, \quad v = 6,$$
$$\sqrt{\eta'} = (\sqrt{2}+1)^2;$$
$$K'/K = \sqrt{(18)}, \quad \xi = 1+i\sqrt{3}, \quad \xi^3 = -8, \quad u^2 = 6, \quad v = 98,$$
$$\sqrt{\eta'} = (\sqrt{3}+\sqrt{2})^4;$$
$$K'/K = \sqrt{(21)}, \quad \xi = i+\sqrt{3}, \quad \xi^3 = 8i,$$
$$u^2 = 4+2\sqrt{3} = (\sqrt{3}+1)^2, \quad v = 32\sqrt{7}+18\sqrt{(21)}, \quad \bar{v} = 84+48\sqrt{3},$$
$$\sqrt{\eta'} = \tfrac{1}{2}(v+\bar{v}) = (2\sqrt{7}+3\sqrt{3})(8+3\sqrt{7}) = \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2.$$

66. With $$\frac{K'}{K} = \sqrt{2},$$
$$L(2)^{24} = \xi^6\eta' = 16\left(\frac{1}{\kappa}-\kappa\right)^2 = 64,$$
$$L(2)^2 = \sqrt{2},$$
and $$\xi^2+4\xi+\frac{4}{\xi} = \eta = -6,$$
$$\xi^3+4\xi^2+6\xi+4 = 0,$$
$$(\xi+2)(\xi^2+2\xi+2) = 0,$$
$$\xi = -2, \quad \text{or} \quad -1+i.$$
Then $$u = 0, \quad v = 2, \quad w = -14, \quad \eta' = 1,$$
$$\tau = \tfrac{1}{2}; \quad \tau' = \tfrac{1}{2}i\sqrt{2}.$$
Also $a = \infty$, so that $\theta' = 0$, and
$$\phi = \phi' = \tfrac{1}{2}\theta + \tfrac{4}{3}\omega;$$
$$H(\phi) = -\frac{11}{2}, \quad H(2\phi) = -\frac{33}{8};$$
$$H(\tfrac{1}{2}\theta) = -8+i\sqrt{2}, \quad H(\theta) = -5+2i\sqrt{2},$$
so that $$H(\theta)^2+10H(\theta)+33 = 0,$$
and the discriminant of the quartics (263) and (414) vanishes, so that the quartics have a pair of equal roots.

When $$H(\phi) = -\frac{33}{8}, \quad \tau(\phi) = \frac{3}{8}, \quad \tau'(\phi) = \frac{11i\sqrt{2}}{16},$$
then $$\xi = -\frac{4+\sqrt[3]{(44)}}{3}.$$

With $$\frac{K'}{K} = \sqrt{7},$$

$$2\kappa\kappa' = \tfrac{1}{8},$$

and $$L(2)^{24} = \xi^6\eta' = -\frac{16}{\kappa^2\kappa'^2} = -2^{12}.$$

This is satisfied by taking

$$\xi = -4, \quad u^2 = -1, \quad \eta' = -1;$$

and then $$\tau(\phi) = \tfrac{1}{7}, \quad \tau(\phi') = 1;$$

$$\tau'(\phi) = \frac{17i\sqrt{7}}{49}, \quad \tau'(\phi') = i\sqrt{7};$$

$$H(\theta + \tfrac{8}{5}\omega_2) = -\frac{22}{9}, \quad \tau(\theta + \tfrac{8}{5}\omega_2) = \frac{2}{9};$$

$$a(\theta') = 0,$$

so that $$\theta' = \tfrac{4}{5}\omega.$$

With $$\frac{K'}{K} = \sqrt{(19)},$$

$$\eta = -4, \quad \tau(\phi) = \tfrac{1}{4}, \quad \tau'(\phi) = \frac{i\sqrt{(19)}}{4},$$

and $$\xi^3 + 4\xi^2 + 4\xi + 4 = 0.$$

With $$\frac{K'}{K} = \sqrt{(43)},$$

$$\eta = 8, \quad \tau(\phi) = \frac{1}{16}, \quad \tau'(\phi) = \frac{i\sqrt{(43)}}{32},$$

and $$\xi^3 + 4\xi^2 - 8\xi + 4 = 0.$$

With $$\frac{K'}{K} = \sqrt{(10)},$$

$$\xi^6\eta' = 16\left(\frac{1}{\kappa} - \kappa\right)^2 = 64\left(\frac{\sqrt{5}+1}{2}\right)^{12},$$

which we find is satisfied by

$$\xi = -3 - \sqrt{5}, \quad u^2 = -2, \quad v = 2, \quad \eta' = 1.$$

Then $$\tau(\phi) = \frac{7-3\sqrt{5}}{4}, \quad \tau'(\varphi) = \frac{11\sqrt{5}-24}{\sqrt{2}}i;$$

$$\tau(\phi') = \frac{7+3\sqrt{5}}{4}, \quad \tau'(\phi') = \frac{11\sqrt{5}+24}{\sqrt{2}}i;$$

$$H(\phi+\phi') = -\frac{11}{10}, \quad \tau(\phi+\phi') = \frac{1}{10}, \quad \tau'(\phi+\phi') = \frac{11i\sqrt{(10)}}{50};$$

$$H(\phi-\phi') = -\frac{43}{18} = H(5a), \quad \text{if} \quad H(a) = -\tfrac{1}{2}.$$

With $$\frac{K'}{K} = \sqrt{(35)},$$

$$\eta = -2+2\sqrt{5}, \quad \tau(\phi) = \frac{(\sqrt{5}-1)^2}{16}, \quad \tau'(\phi) = \frac{i\sqrt{7}}{4}.$$

So also we find that
$$\frac{K'}{K} = \sqrt{(22)}$$
corresponds to
$$L(22)^2 = \sqrt{(22)}, \quad \xi = 2, \quad \tau(\phi) = \tau(\phi') = \frac{1}{22}, \quad \tau'(\phi) = \frac{7\sqrt{2}}{22};$$

$$H(\theta+\tfrac{8}{5}\omega) = \infty, \quad \theta = \tfrac{2}{5}\omega;$$

$$a(\theta') = -\tfrac{7}{8}, \quad a(\tfrac{1}{2}\theta') = -\tfrac{1}{2}.$$

But $$H(\phi) = -\tfrac{1}{2},$$
so that $$\tfrac{1}{2}\theta' = \phi,$$
and $$\phi-\phi' = 2\phi, \quad \text{or} \quad \phi = -\phi'.$$

Now $$u^2 = 8, \quad v = 198, \quad \sqrt{\eta'} = 2(\sqrt{2}+1)^6;$$
so that $$16\left(\frac{1}{\kappa}-\kappa\right)^2 = \xi^6\eta' = 64\eta',$$
$$\frac{1}{\kappa}-\kappa = 2(\sqrt{2}+1)^6;$$
agreeing with the corresponding value of the modulus.

Other numerical cases can be worked out, as further applications, corresponding to

$$\tau(\phi) = -1, \quad \phi = \tfrac{8}{5}\omega, \text{ \&c.};$$

$$H(\theta+\tfrac{8}{5}\omega) = 0, \quad \theta = 0, \quad u^2 = -3, \quad \xi = -\frac{7+\sqrt{(33)}}{2},$$

$$\tau(\phi) = \frac{2\sqrt{3}-\sqrt{(11)}}{\sqrt{(11)}}, \quad \tau'(\phi) = \frac{13\sqrt{(11)}-24\sqrt{3}}{\sqrt{(11)}}, \text{ \&c.};$$

$$H(\theta+\tfrac{8}{5}\omega_2) = 11, \quad H(\theta) = \infty, \text{ \&c.}$$

67. The case of $\quad \dfrac{K'}{K} = \sqrt{(11)}$

corresponds to the vanishing of Kiepert's W; and

$$W = \frac{\tau'}{\tau^3} = \sqrt{(\eta+8)}\sqrt{(\eta^3+4\eta^2-72\eta-364)}$$

$$= (\xi+2-2\xi^{-2})\sqrt{(\xi^3+4\xi^2+8\xi+4)}\sqrt{(\xi^3+8\xi^2+16\xi+16)}.$$

The value $\xi = 0$, or ∞, makes $\tau = \infty$, $\eta = -8$,

and $$J = -\frac{2^9}{3^3},$$

as required in this case (*Proc. Lond. Math. Soc.*, Vol. XIX, p. 306).

When $$\xi+2-2\xi^{-2} = 0,$$

or $$\xi^3+2\xi^2-2 = 0,$$

and we put $$\xi^6 = 16\kappa\kappa',$$

then $$2\sqrt{(\kappa\kappa')}+2\sqrt[3]{(2\kappa\kappa')}-1 = 0,$$

the equation obtained by putting

$$\kappa = \lambda', \quad \kappa' = \lambda,$$

in Schröter's Modular Equation of the Eleventh Order.

When $$\xi^3+4\xi^2+\ 8\xi+\ 4 = 0,$$

or $$\xi^3+8\xi^2+16\xi+16 = 0,$$

then $$w = 0, \quad \tau' = 0,$$

and $$\frac{K'}{K} = 2\sqrt{(11)}, \text{ or } \frac{\sqrt{(11)}}{2}$$

(Klein-Fricke, *Modulfunctionen*, II, p. 437).

[*Note added November*, 1896.

It was considered important, as a check upon the accuracy of the formulas, to have some numerical verifications of the results in special cases of the Transformation of the Eleventh and Twenty-second Order; and this has been carried out by Mr. T. I. Dewar.

The object is to calculate the twelve values of y, namely,

$$y_\infty,\ y_0,\ y_1,\ y_2,\ \dots\ y_{10},$$

the roots of Klein's "Multiplier Equation of the Eleventh Order" (*Math. Ann.*, XV, p. 88; *M. F.*, II, p. 442), in the form

$$y^{12}+11\Delta\,(-90y^6+40.12g_2y^4-15.216g_3y^3+2.144g_2^2y^2)$$
$$-12g_2.216g_3\Delta y-11\Delta y-11\Delta^2=0,$$

equivalent to Kiepert's L equation (147), *Math. Ann.*, XXXI, p. 428, when we take

$$\Delta=1,\quad y=-L^2.$$

If one root y_∞ or L_∞^2 is known, the remaining eleven roots are given by the a's of § 24, from the formula

$$\frac{y_r}{y_\infty}=\frac{L_r^2}{L_\infty^2}=-\frac{1}{11}\,(1+\epsilon^{36r}a_1+\epsilon^{36.4r}a_2+\epsilon^{36.9r}a_3+\epsilon^{36.16r}a_4+\epsilon^{36.25r}a_5)^2,$$
$$\epsilon=e^{\frac{1}{11}\pi i},\quad r=0,1,2,\dots 10$$

(Klein, *Math. Ann.*, XVII, p. 567),

noticing that $a_3=a_9$ in equations (159).

First, with

$$\frac{K'}{K}=\surd(11),$$

and

$$J=-\frac{2^9}{3^3},$$

we take

$$12g_2=32,\quad 216g_3=56\surd(11),$$
$$\Delta=-1,$$
$$y_\infty=-L_\infty^2=\surd(11);$$

and then

$$\tau=\infty,\quad H=0;$$

and therefore, as in (180), we may take

$$c = -\frac{1}{1+2\cos\dfrac{2\pi}{11}}$$

$$= -0{\cdot}372,\ 785,\ 597,\ 771,\ 791,\ 7,$$

and this makes, in (149),

$$\sqrt{C} = \frac{2+5c+0-2c^3}{c}$$

$$= -0{\cdot}642,\ 952,\ 335,\ 136,\ 877,$$

and, in § 22,

$$z = 0{\cdot}821,\ 476,\ 167,\ 568,\ 438,$$

$$y = -0{\cdot}254,\ 428,\ 804,\ 456,\ 417,$$

$$x = -0{\cdot}045,\ 421,\ 605,\ 252,\ 541,$$

$$x^{\frac{1}{3}} = -0{\cdot}356,\ 796,\ 697,\ 749,\ 900,$$

$$\lambda = -0{\cdot}467,\ 304,\ 294,\ 715,\ 509,$$

$$\lambda^{\frac{1}{11}} = -0{\cdot}933,\ 176,\ 117,\ 881,\ 270.$$

Now, in § 24, starting with

$$a_1 = -\lambda^{-\frac{1}{11}},$$

and thence determining a_3, a_9, a_5, a_4 from the relations

$$a_3 a_1^2 = -\frac{yz}{\lambda x^{\frac{1}{3}}},$$

$$a_9 a_3^2 = \frac{y^3 z^2}{\lambda^4 x^{\frac{1}{3}}},$$

$$a_5 a_9^2 = -\frac{y}{z},$$

$$a_4 a_5^2 = \frac{\lambda^5 x^{\frac{1}{3}}}{y^3 z^2},$$

Mr. Dewar finds that

$$a_1 = 1{\cdot}230,\ 578,\ 018,\ 091,\ 480,$$

$$a_3 = -1{\cdot}771,\ 424,\ 180,\ 284,\ 673,$$

$$a_9 = -0{\cdot}583,\ 448,\ 985,\ 672,\ 033,$$

$$a_5 = 0{\cdot}909,\ 841,\ 056,\ 781,\ 324,$$

$$a_4 = -0{\cdot}864,\ 171,\ 339,\ 453,\ 279;$$

so that

$$1+\Sigma a = -0{\cdot}078,\ 625,\ 430,\ 537,\ 181,$$

and $$L_0^2 = \frac{(1+\Sigma a)^2}{\sqrt{(11)}}$$

$$= \quad 0·001,\ 863,\ 927,\ 552,\ 231,$$

agreeing closely with the approximate value given by

$$L_0^2 \simeq \frac{11}{12\gamma_2 . 216\gamma_3},$$

$$12\gamma_2 = 32,$$

$$216\gamma_3 = 56\sqrt{(11)}.$$

So also the imaginary roots are given by

$$L_1^2,\ L_{10}^2 = \quad 2·323362 \pm 1·934112i,$$

$$L_2^2,\ L_9^2 = \quad 1·698161 \pm 0·356919i,$$

$$L_3^2,\ L_8^2 = -2·776133 \pm 2·259814i,$$

$$L_4^2,\ L_7^2 = -0·164155 \pm 3·579350i,$$

$$L_5^2,\ L_6^2 = \quad 0·576146 \pm 0·247194i.$$

Next, with $$\frac{K'}{K} = \sqrt{(22)} \quad \text{or} \quad \sqrt{\left(\frac{2}{11}\right)},$$

we take $$\tau = \frac{1}{22}$$

and $$a = -\frac{1}{2},$$

in (175); and therefore, with our c out of phase with this a by one twenty-fifth of a period, we can take the b and $\sqrt{B}$ the same as the c and $\sqrt{C}$ just employed with

$$\frac{K'}{K} = \sqrt{(11)};$$

and now, from (193),

$$c = \quad 27·028,\ 919,\ 189,\ 803,\ 4,$$

or $$= -\ 1·063,\ 634,\ 337,\ 710,\ 340,$$

according to the sign attributed to

$$\sqrt{A} = \tfrac{1}{2}\sqrt{2}.$$

Taking $c = 27 \cdot 028,\ 919,\ 189,\ 803,\ 4,$

$$\sqrt{C} = 291 \cdot 442,\ 741,\ 535,\ 938,\ 85,$$

$$z = -145 \cdot 221,\ 370,\ 767,\ 969,\ 4,$$

$$y = -897 \cdot 631,\ 636,\ 212,\ 663,\ 5,$$

$$x = -131252 \cdot 928,\ 291,\ 711,\ 92;$$

and a rapid calculation, from the formulas

$$12g_2 = \{(y+1)^2+4x\}^2-24x(y+1),$$

$$216g_3 = \{(y+1)^2+4x\}^3-36x(y+1)\{(y+1)^2+4x\}+216x^2,$$

showed that $\gamma_2 = \dfrac{g_2}{\Delta^{\frac{1}{3}}} = 11 \cdot 325,$

and therefore corresponds to

$$\gamma_2 = 775-540\sqrt{2},$$

$$\frac{K'}{K} = \sqrt{\left(\frac{2}{11}\right)}.$$

Now $\gamma_3 = -7\sqrt{(11)}(\sqrt{2}-1)^4(9\sqrt{2}+2),$

$$L_\infty^2 = \sqrt{(11)}(\sqrt{2}-1),$$

and $x^{\frac{1}{3}} = -50 \cdot 820,\ 195,\ 781,\ 427,\ 47,$

$$\lambda = 263 \cdot 305,\ 853,\ 647,\ 054,$$

$$\lambda^{\frac{1}{11}} = 1 \cdot 659,\ 746,\ 895,\ 262,\ 522,$$

$$a_1 = -0 \cdot 218,\ 712,\ 903,\ 083,\ 518,$$

$$a_3 = 0 \cdot 773,\ 432,\ 664,\ 433,\ 080,$$

$$a_9 = -2 \cdot 053,\ 977,\ 923,\ 433,\ 609,$$

$$a_5 = -1 \cdot 465,\ 129,\ 712,\ 633,\ 559,$$

$$a_4 = 1 \cdot 964,\ 405,\ 935,\ 908,\ 983,$$

so that $1+\Sigma a = 0 \cdot 000,\ 018,\ 061,\ 191,\ 377.$

Taking the second value,

$$c = -1{\cdot}063,\ 634,\ 337,\ 710,\ 340,\ 0,$$
$$\sqrt{C} = 0{\cdot}991,\ 348,\ 563,\ 851,\ 883,\ 5,$$
$$z = 0{\cdot}004,\ 325,\ 718,\ 074,\ 058,\ 25,$$
$$y = 0{\cdot}004,\ 619,\ 769,\ 581,\ 744,\ 308,$$
$$x = 0{\cdot}004,\ 599,\ 785,\ 760,\ 966,\ 573;$$

and these give values to $12g_2$ and $216g_3$, which make

$$\gamma_2 = \frac{g_2}{\Delta^{\frac{1}{3}}} = 1538{\cdot}675,$$

which corresponds to

$$\gamma_2 = 775 + 540\sqrt{2}, \quad K'/K = \sqrt{(22)}.$$

Now $$\gamma_3 = 7\sqrt{(11)}(\sqrt{2}+1)^4(9\sqrt{2}+2),$$
$$L_\infty^2 = -\sqrt{(11)}(\sqrt{2}+1),$$

and $$x^{\frac{1}{3}} = 0{\cdot}166,\ 307,\ 767,\ 947,\ 716,\ 8,$$
$$\lambda = -0{\cdot}011,\ 305,\ 225,\ 190,\ 562,\ 9,$$
$$\lambda^{\frac{1}{11}} = -0{\cdot}665,\ 312,\ 040,\ 870,\ 516,$$
$$a_1 = 3{\cdot}395,\ 657,\ 270,\ 978,\ 186,$$
$$a_3 = -0{\cdot}081,\ 537,\ 962,\ 826,\ 480,$$
$$a_9 = 0{\cdot}614,\ 208,\ 346,\ 818,\ 869,$$
$$a_5 = -2{\cdot}830,\ 940,\ 071,\ 212,\ 028,$$
$$a_4 = -2{\cdot}077,\ 159,\ 775,\ 134,\ 698,$$

so that $$1+\Sigma a = 0{\cdot}020,\ 227,\ 808,\ 623,\ 849.$$

But now, from a consideration of the approximate value of L_0^2 in Kiepert's equation (147),

$$L_0^2 \simeq \frac{11}{12\gamma_2 \,.\, 216\gamma_3},$$

we see that, in accordance with the general principle stated in § 5, these second values of a must be employed with the transformed modulus, corresponding to

$$K'/K = \sqrt{(2 \div 11)}.$$

Conversely, the first series of values of a must be employed when

$$K'/K = \sqrt{(22)};$$

and now $L_\infty^2 = -\sqrt{(11)}(\sqrt{2}+1)$

$= -8{\cdot}007,\ 040,\ 550,\ 218,\ 830,$

$$\frac{L_0^2}{L_\infty^2} = \frac{(1+\Sigma a)^2}{-11}$$

$= -0{\cdot}000,\ 000,\ 000,\ 029,\ 655,\ 148,\ 541,\ 5,$

$L_0^2 = \ \ 0{\cdot}000,\ 000,\ 000,\ 237,\ 449,\ 976,\ 894,\ 553.$

Also $L_1^2,\ L_{10}^2 = \ \ 5{\cdot}96687 \pm 8{\cdot}29103i,$

$L_2^2,\ L_9^2 = -4{\cdot}75298 \pm 7{\cdot}52094i,$

$L_3^2,\ L_8^2 = \ \ 9{\cdot}99196 \pm 3{\cdot}28036i,$

$L_4^2,\ L_7^2 = -7{\cdot}31464 \pm 3{\cdot}83824i,$

$L_5^2,\ L_6^2 = \ \ 0{\cdot}11247 \pm 9{\cdot}63335i.$

With $K'/K = \sqrt{(2 \div 11)},$

and the second series of values of the a's,

$L_\infty^2 = \sqrt{(11)}(\sqrt{2}-1),$

$= \ \ 1{\cdot}373,\ 790,\ 969,\ 468,\ 031,$

$$\frac{L_0^2}{L_\infty^2} = \frac{(1+\Sigma a)^2}{-11}$$

$$= \frac{0{\cdot}000,\ 409,\ 164,\ 241,\ 723,\ 06}{-11},$$

$L_0^2 = -0{\cdot}000,\ 051,\ 100,\ 558,\ 209,\ 852,$

$L_1^2,\ L_{10}^2 = \ \ 2{\cdot}5203793 \pm 1{\cdot}5134366i,$

$L_2^2,\ L_9^2 = \ \ 1{\cdot}8423729 \pm 0{\cdot}0747787i,$

$L_3^2,\ L_8^2 = \ \ 1{\cdot}0898557 \pm 3{\cdot}7294405i,$

$L_4^2,\ L_7^2 = -4{\cdot}0969314 \pm 1{\cdot}4719174i,$

$L_5^2,\ L_6^2 = -2{\cdot}0425644 \pm 3{\cdot}8031382i.$]