

XXXII. *Elastic Stability of Long Beams under Transverse Forces.* By A. G. M. MICHELL, M.C.E. Melb.*

I. IT is a matter of common experience that a thin flat bar or blade of elastic material, subjected to bending in its own plane, may be unstable in the plane form and fail by combined lateral displacement and twist. It will appear below that the instability of such a beam is due to its want of torsional, rather than of flexural, rigidity, so that the same kind of instability as occurs in a thin blade may affect beams of other forms. The mode of instability will depend upon the proportions of the beam, but if the length is so great compared with the other dimensions that the ordinary equations for the bending and twisting of a linear rod are applicable, a criterion of instability may be obtained from them.

In engineering construction beams which fail under excessive loads by buckling or lateral displacement are very frequently employed. In many commonly occurring cases the assumptions made below as to the relative dimensions of the beam would be justifiable, and lead to results at least sufficiently accurate for purposes of practical design. In other cases the results might be used as bases for empirical formulæ.

Taking the case of a cylindrical beam subject to forces applied in the axial plane of greatest bending rigidity, denote by β_1 the rigidity in this plane, by β_2 that in the perpendicular axial plane, and by γ the torsional rigidity. Let B_1 , B_2 be the bending moments, at any point of the beam, in the planes of β_1 , β_2 respectively, and G the twisting couple; also let ϕ_1 , ϕ_2 be the angular displacements of the axis in the planes of β_1 and β_2 , and θ the total twist from the origin.

Then

$$\left. \begin{aligned} \beta_1 \frac{d\phi_1}{ds} &= B_1 \\ \beta_2 \frac{d\phi_2}{ds} &= B_2 \\ \gamma \frac{d\theta}{ds} &= G \end{aligned} \right\} \dots \dots \dots (1)$$

are the ordinary approximate equations. It will be supposed throughout that γ/β_1 can be treated as a small fraction. β_2 may be either of the same order as, or much smaller than β_1 , but is supposed not to be small in comparison with γ .

* Communicated by the Author.

II. In rectangular coordinates let the axis of the beam before deformation be the axis of x , and the forces be applied in the plane zx .

Take first a single force N applied at the origin parallel to the axis of z , and suppose that after a small deformation the point of application of N becomes $(0, y_1, z_1)$. The bending and twisting moments at the point (x, y, z) will be given by

$$\left. \begin{aligned} B_1 &= -Nx \\ B_2 &= N \left\{ x\theta - (y - y_1) \frac{dz}{dx} \right\} \\ G &= -N \left\{ x \frac{dy}{dx} - (y - y_1) \right\} \end{aligned} \right\} \dots \quad (2)$$

small quantities obviously of higher order than those retained being neglected.

Equations (1) and (2) may be combined by using the geometrical relations

$$\left. \begin{aligned} \frac{d^2 z}{dx^2} &= -\frac{d\phi_1}{ds} + \frac{d\phi_2}{ds} \theta, \\ \frac{d^2 y}{dx^2} &= \frac{d\phi_2}{ds} + \frac{d\phi_1}{ds} \theta, \\ \frac{d\theta}{dx} &= \frac{d\theta}{ds}. \end{aligned} \right\}$$

It may appear at first sight that in the last terms of the first two of these equations the product of small quantities of the order of the strains are being retained, whereas elsewhere all powers above the first have been rejected. This is, however, not really the case, for it is to be observed that θ is not of the same order as $\frac{d\theta}{ds}$, but of the order of the product of $\frac{d\theta}{ds}$ and the length of the beam, which is supposed large.

Substituting in the geometrical relations from equations (1) there result

$$\left. \begin{aligned} \beta_1 \frac{d^2 z}{dx^2} &= -B_1 + \frac{\beta_1}{\beta_2} B_2 \theta, \\ \beta_2 \frac{d^2 y}{dx^2} &= B_2 + \frac{\beta_2}{\beta_1} B_1 \theta, \\ \gamma \frac{d\theta}{dx} &= G; \end{aligned} \right\}$$

or from the equations (2),

$$\left. \begin{aligned} \frac{\beta_1}{N} \frac{d^2 z}{dx^2} &= x + \frac{\beta_1}{\beta_2} \left\{ x\theta - (y-y_1) \frac{dz}{dx} \right\}, \\ \frac{\beta_2}{N} \frac{d^2 y}{dx^2} &= \frac{\beta_1 - \beta_2}{\beta_1} x\theta - (y-y_1) \frac{dz}{dx}, \\ \frac{\gamma}{N} \frac{d\theta}{dx} &= -x \frac{dy}{dx} + (y-y_1). \end{aligned} \right\}$$

The first term on the right of the first of these equations is finite, the others are infinitesimal and may be rejected. The two terms on the right of the last equation are of course of the same order and must both be retained. Comparing the first two terms of the second with the corresponding terms of the third equation, it is seen that $\left(\frac{dy}{dx} / \theta\right)^2$ is of the

same order as γ/β_2 , so that $\frac{dy}{dx}$ is either of the same or of higher order than θ , and therefore $(y-y_1)$ of the same or of higher order than $x\theta$. Hence the last term of the second equation can be rejected, and the equations become

$$\left. \begin{aligned} \frac{\beta_1}{N} \frac{d^2 z}{dx^2} &= x, \\ \frac{\beta_2}{N} \frac{d^2 y}{dx^2} &= \frac{\beta_1 - \beta_2}{\beta_1} x\theta, \\ \frac{\gamma}{N} \frac{d\theta}{dx} &= -x \frac{dy}{dx} + (y-y_1). \end{aligned} \right\} \dots \dots (3)$$

These equations may be used with two distinct assumptions as to the orders of the quantities.

Firstly. If γ is of the same order as β_2 , $\frac{dy}{dx}$ of the same order as θ , and $\frac{\beta_2}{\beta_1}$ small and of the order of $\frac{dz}{dx}$.

Secondly. If γ is small compared to β_2 , $\frac{dy}{dx}$ is to be of higher order than θ , and $\beta_1 \beta_2$ may be of the same order, $\frac{dy}{dx}$ of the order of $\theta \frac{dz}{dx}$, and $\frac{\gamma}{\beta_1}$ and $\frac{\gamma}{\beta_2}$ of the order of $\left(\frac{dz}{dx}\right)^2$.

From the last of equations (3)

$$\frac{d^2 \theta}{dx^2} = -\frac{N}{\gamma} x \frac{d^2 y}{dx^2},$$

and therefore from the second equation

$$\frac{d^2\theta}{dx^2} = - \frac{N^2(\beta_1 - \beta_2)}{\beta_1\beta_2\gamma} x^2\theta.$$

Denoting the expression

$$\frac{\beta_1\beta_2\gamma}{N^2(\beta_1 - \beta_2)}$$

(which is of the same dimensions as the moment of inertia of a plane area) by J , there results as the differential equation of the axis of the slightly displaced beam

$$J \frac{d^2\theta}{dx^2} + x^2\theta = 0. \quad (4)$$

This is a well-known transformation of Riccati's equation, and the solution consists of the sum of two Bessel's Functions of fractional orders. For the present purpose a more convenient form of the solution is obtained directly by determination of the coefficients of the series

$$\theta = A_1 x^{m_1} + A_2 x^{m_2} + \dots + A_n x^{m_n} + \dots$$

The result is

$$\begin{aligned} \theta = A \left\{ 1 - \frac{x^4}{J \cdot 4 \cdot 3} + \frac{x^8}{J^2 \cdot 8 \cdot 7 \cdot 4 \cdot 3} - \dots \right. \\ \left. \pm \frac{x^{4n}}{J^n \cdot 4n(4n-1)(4n-4)(4n-5) \dots 4 \cdot 3} \mp \dots \right\} \\ + Bx \left\{ 1 - \frac{x^4}{J \cdot 5 \cdot 4} + \frac{x^8}{J^2 \cdot 9 \cdot 8 \cdot 5 \cdot 4} - \dots \right. \\ \left. \pm \frac{x^{4n}}{J^n \cdot (4n+1) \cdot 4n(4n-3)(4n-4) \dots 5 \cdot 4} \mp \dots \right\}. \quad (5) \end{aligned}$$

In which A and B are constants to be determined from the end conditions assigned in any particular case. The simplest case to which the above result is applicable is that of a cantilever, rigidly fixed at one end, and supporting a single transverse load at the other end, which is free.

If the origin be taken as the free end, there is no twisting moment and $\frac{d\theta}{dx}$ is zero at that point, hence $B=0$, and

$$\theta = A \left(1 - \frac{x^4}{J \cdot 4 \cdot 3} + \frac{x^8}{J^2 \cdot 8 \cdot 7 \cdot 4 \cdot 3} - \dots \right. \\ \left. \pm \frac{x^{4n}}{J^n \cdot 4n(4n-1) \dots 4 \cdot 3} \mp \dots \right)$$

in which A is the angle through which the free end is turned relatively to its original position.

If now the cantilever be supposed fixed at the point $x=l$, so that θ is zero at that point, there results the equation

$$1 - \frac{R}{4 \cdot 3} + \frac{R^2}{8 \cdot 7 \cdot 4 \cdot 3} - \dots \pm \frac{R^n}{4n(4n-1) \dots 4 \cdot 3} \mp \dots = 0. \quad (6)$$

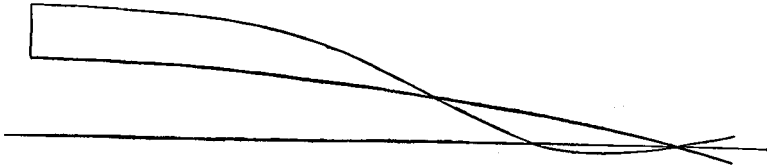
(in which R is put for l^4/J), expressing the condition for the possibility of equilibrium of the cantilever in the slightly displaced form, or, in other words, the condition of critical stability.

The first and second roots of (6) are

$$R=16 \cdot 101 \dots, \text{ and } R=104 \cdot 98 \dots$$

The modes of instability corresponding to both roots are illustrated in the accompanying figure, which represents the

Fig. 1.



projection on the plane xy of the axial plane of greatest rigidity. The extended edge of the plane is represented by the thicker line. The figure as a whole shows the deformed shape corresponding to the second root. In this case the sign of θ changes at a distance almost exactly three-eighths of the total length from the fixed end, while y is of the same sign throughout. The portion of the cantilever between the point at which θ is zero and the free end is unstable in the mode corresponding to the first root of equation (6). In this case θ and y are each of one sign throughout.

III. Another set of end conditions will express the case of a beam supported and prevented from turning about its axis at its ends and loaded with a single transverse load at its middle point.

If $2l$ be the length of the beam, $2N$ the applied force, the other symbols having the same meanings as before, θ is to be made zero when x is zero, and $\frac{d\theta}{dx}$ zero when x is equal to l (the middle point).

Hence in equation (5) $A=0$ and either l is zero or

$$1 - \frac{R}{4} + \frac{R^2}{8 \cdot 5 \cdot 4} - \dots \pm \frac{R^n}{4n(4n-3) \dots 5 \cdot 4} \mp \dots = 0. \quad (7)$$

The first root of this equation is

$$R=4.4817\dots$$

IV. To obtain the condition of critical stability for the case of a beam fixed in azimuth at its ends, it is necessary to modify the equations of equilibrium by inserting in them the couples necessary to maintain the ends in their original positions.

If at the origin a couple Q in the plane xy , and a couple S in the plane yz , be applied to the beam, the equations (3) above must be replaced by

$$\left. \begin{aligned} \frac{\beta_1 d^2 z}{N dx^2} &= x, \\ \frac{\beta_2 d^2 y}{N dx^2} &= \frac{\beta_1 - \beta_2}{\beta_1} x \theta + \frac{Q}{N}, \\ \frac{\gamma}{N} \frac{d\theta}{dx} &= -x \frac{dy}{dx} + (y - y_1) + \frac{S}{N} \end{aligned} \right\}$$

Hence

$$\frac{\gamma}{N} \frac{d^2 \theta}{dx^2} = -\frac{N(\beta_1 - \beta_2)}{\beta_1 \beta_2} x^2 \theta - \frac{Qx}{\beta_2}.$$

or

$$J \frac{d^2 \theta}{dx^2} + x^2 \theta + Cx = 0,$$

where

$$C = \frac{\beta_1 Q}{N(\beta_1 - \beta_2)}.$$

The solution in this case, θ being zero when x is zero, is, in the same manner as before, easily found to be

$$\theta = \frac{Cx^3}{J \cdot 3 \cdot 2} \left\{ 1 - \frac{x^4}{J \cdot 7 \cdot 6} + \frac{x^8}{J \cdot 11 \cdot 10 \cdot 7 \cdot 6} - \dots \right. \\ \left. \pm \frac{x^{4n}}{J^n (4n+3)(4n+2)(4n-1)(4n-2) \dots 7 \cdot 6} \mp \dots \right\};$$

and θ being zero when $x=l$, the condition of critical stability for a cantilever when the free end is prevented from rotating about the axis is

$$1 - \frac{R}{7 \cdot 6} + \frac{R^2}{11 \cdot 10 \cdot 7 \cdot 6} - \dots \pm \frac{R^n}{(4n+3)(4n+2) \dots 7 \cdot 6} \mp \dots = 0.$$

The first root of this equation is

$$R=101.23\dots$$

V. If instead of a single transverse load a uniformly distributed load be applied, the equations are still simple. Consider the case of the cantilever as in the first case treated. With all the other symbols as before, let p be the load per unit length. Then

$$B_1 = -\frac{p}{2}x^2,$$

$$B_2 = +\frac{p}{2}x^2\theta;$$

whence

$$\left. \begin{aligned} \frac{dB_1}{dx} &= -px, \\ \frac{dB_2}{dx} &= px\theta + \frac{p}{2}x^2\frac{d\theta}{dx}, \\ \frac{dG}{dx} &= -\frac{p}{2}x^2\frac{d^2y}{dx^2}. \end{aligned} \right\} \text{also}$$

Thus

$$\gamma \frac{d^2\theta}{dx^2} = \frac{dG}{dx} = B_1 \frac{d^2y}{dx^2};$$

but as before

$$\frac{d^2y}{dx^2} = \frac{B_2}{\beta_2} + \frac{B_1}{\beta_1}\theta = -B_1 \frac{\beta_1 - \beta_2}{\beta_1\beta_2}\theta.$$

Therefore

$$\gamma \frac{d^2\theta}{dx^2} + B_1^2 \frac{\beta_1 - \beta_2}{\beta_1\beta_2}\theta = 0,$$

or

$$\frac{d^2\theta}{dx^2} + \frac{p^2}{4} \frac{\beta_1 - \beta_2}{\beta_1\beta_2\gamma} x^4\theta = 0.$$

If for p be substituted P/l , P being the total load on the cantilever, $\frac{\beta_1\beta_2\gamma}{P^2(\beta_1 + \beta_2)}$ is a quantity of the same kind as J , and may be denoted by H .

The differential equation is then

$$\frac{d^2\theta}{dx^2} + \frac{x^4\theta}{4l^2H} = 0,$$

and the solution in series

$$\theta = A \left\{ 1 - \frac{x^6}{(4Hl^2)6.5} + \frac{x^{12}}{(4Hl^2)^2.12.11.6.5} - \dots \right. \\ \left. \pm \frac{x^{6n}}{(4Hl^2)^n 6n(6n-1)(6n-6)(6n-7) \dots 6.5} \mp \dots \right\}$$

$$+ Bx \left\{ 1 - \frac{x^6}{(4Hl^2)7.6} + \frac{x'^2}{(4Hl^2)^2.13.12.7.6} - \dots \right. \\ \left. \pm \frac{x^{6n}}{(4Hl^2)^n(6n+1)6n\dots 7.6} \mp \dots \right\} = 0.$$

As in the case of the cantilever with single load, make $\frac{d\theta}{dx}$ equal to zero at the origin, and θ zero when x is equal to l ; then $B=0$, and the condition for critical stability, if R be put for $\frac{l^4}{4H}$, is

$$1 - \frac{R}{6.5} + \frac{R^2}{12.11.6.5} - \dots \\ \pm \frac{R^n}{6n(6n-1)(6n-6)(6n-7)\dots 6.5} \mp \dots = 0 \dots$$

The first root of this equation is

$$R = 41.305 \dots$$

VI. Another mode of loading, which is interesting as leading to solutions in finite terms and as reducing to a well-known result in a particular case, is that in which equal and opposite forces parallel to the axis are applied at symmetrical points in the plane of greatest rigidity.

Let forces $-P$, $+P$ be applied at the points (l, o, h) , $(-l, o, h)$ respectively; then

$$\left. \begin{aligned} B_1 &= -Ph, \\ B_2 &= -Py + Ph\theta, \\ G &= -Ph \frac{dy}{dx}. \end{aligned} \right\}$$

Thus

$$\frac{d\theta}{dx} = -\frac{Ph}{\gamma} \frac{dy}{dx},$$

and

$$\frac{d^2\theta}{dx^2} = -\frac{Ph}{\gamma} \frac{d^2y}{dx^2};$$

also

$$\frac{d^2y}{dx^2} = \frac{B_2}{\beta_2} + \frac{B_1}{\beta_1} \theta$$

therefore

$$\frac{d^2\theta}{dx^2} = -(Ph)^2 \frac{\beta_1 - \beta_2}{\beta_1 \beta_2 \gamma} \theta + \frac{P^2 h y}{\gamma \beta_2}$$

and

$$\frac{d^3\theta}{dx^3} + \left(P^2 h^2 \frac{\beta_1 - \beta_2}{\beta_1 \beta_2 \gamma} + \frac{P}{\beta_2} \right) \frac{d\theta}{dx} = 0.$$

Of which the solution is

$$\frac{d\theta}{dx} = D \sin \frac{x}{c} + E \cos \frac{x}{c},$$

where D and E are arbitrary constants, or

$$\theta = A \cos \frac{x}{c} + B \sin \frac{x}{c} + C. \quad . \quad . \quad . \quad (8)$$

If the beam is to be deformed symmetrically on the two sides of the origin, B must be zero. If also, as previously assumed, θ is zero when x is equal to $\pm l$,

$$C = -A \cos \frac{l}{c},$$

so that

$$\theta = A \left(1 - \cos \frac{l}{c} \right) \cos \frac{x}{c},$$

Also since y and θ are both zero when x is equal to l , $\frac{d^2\theta}{dx^2}$ is also zero, at that point, and the condition of critical stability is

$$\cos \frac{l}{c} = 0, \text{ or}$$

$$\frac{l}{c} = \frac{n\pi}{2}, \text{ where } n \text{ is any integer and}$$

$$\frac{1}{c^2} = P^2 h^2 \frac{\beta_1 - \beta_2}{\beta_1 \beta_2 \gamma} + \frac{P}{\beta_2}.$$

In this case the number of loops in the beam between the points $+l$ and $-l$ is odd.

If θ is zero at the origin, in equation (8)

$$C = -A, \text{ and } \theta = B \sin \frac{x}{c},$$

so that θ is also zero at $+l$ and $-l$.

$$\frac{l}{c} = n\pi \text{ is the condition of critical stability.}$$

In this case the unstable form of the beam consists of an even number of loops

If in these equations h is made very large, each of the single forces becomes equivalent to a couple applied at the end of the beam of magnitude

$$Ph = L.$$

In this case

$$\frac{1}{c^2} = P^2 h^2 \frac{\beta_1 - \beta_2}{\beta_1 \beta_2 \gamma} = \frac{L^2 (\beta_1 - \beta_2)}{\beta_1 \beta_2 \gamma}.$$

If, on the other hand, h becomes vanishingly small

$$\frac{1}{c^2} = \frac{P}{\beta_2},$$

and the condition of instability in the first case becomes

$$l^2 = \frac{n^2 \pi^2 \beta_2}{4P},$$

or

$$P = \frac{n^2 \pi^2 \beta_2}{4l^2};$$

or for the first mode of instability n is equal to 1, and

$$P = \frac{\pi^2 \beta_2}{4l^2},$$

which is Euler's well-known result for the unstable load, P_1 , of a column of length $2l$.

Experimental Verification.

VII. The conditions for the instability of beams in the modes discussed above probably admit of more accurate realization than those for any other known instabilities of elastic solids.

I have made the attempt to verify one or two of the results given by the theory.

The test-beam experimented upon was one of Chesterman's engineer's steel straight-edges, from which the feathered edge was removed by planing. The tests were made on the middle portion of a bar 4 feet long in all. The tested portion had a mean width of 4.367 cms., and mean thickness of .2591 cm. over a length of 110 cms.

The elastic constants of the material were determined by bending and twisting the specimen itself. The deflexions in the bending experiments were measured by a screw-micrometer

with an error of about 1 part in 2000. The angles of twist in the torsion experiments were determined by setting a small mirror fixed to the vertical circle of a theodolite normal to a line of sight attached to the specimen. The average error in these angles was about 1/600 part.

The values found for the bending and torsional rigidities were

$$\beta_2 = 1.382 \times 10^7 \text{ grammes weight cm.}^2$$

$$\gamma = 2.174 \times 10^7 \quad \text{,,} \quad \text{,,} \quad \text{,,}$$

β_1 was throughout treated as infinite in comparison with β_2 , being in fact about 300 times greater.

The results which it was attempted to verify were (6) and (7) of this paper, as well as Euler's result for a long column.

The specimen was placed in a vertical plane with its axis horizontal, and supported in such a manner as to very approximately realize the conditions postulated in each case.

In all the experiments counterpoises were used to apply forces directed vertically upwards at points 10 cms. apart along the axis, each force being equal to the weight of 10 cms. of the specimen. By this means the effects of the weight of the specimen were almost eliminated. These forces, as well as the test-loads, were applied by means of steel hooks fitting in small double-counter-sunk holes drilled through the specimen. The verticality of the upward directed forces was ensured by the following arrangement. The balance-beam from which each hook and the corresponding counterpoise were suspended, was itself hung from one end of a second balance-beam at right-angles to the first, and both balance-beams were free to turn in azimuth. This arrangement served its purpose very satisfactorily.

The specimen was adjusted before each experiment so that no lateral deflexion occurred with a moderate load. The test-load was then gradually increased until the point of application would remain at rest in contact with either of two stops placed about 1 cm. on each side of the initial position. The smallest load which would produce this effect was adopted as the critical load. In every case a marked movement to one side or the other occurred with weights 1 or 2 per cent. less than the least which would maintain a deflexion to either side. It is probable, therefore, that the critical load adopted as the result of experiment was slightly in excess of the true critical load.

The specimen was tested as a cantilever (formula 6) in four positions, being twice inverted and once reversed end for end.

The critical loads observed were—

5885, 6015, 5897, and 5800 grammes,

Mean 5899 grammes.

The critical load calculated by (6), the length l being 110 cms., is 5732 grammes.

Tested as the beam considered in (7), with a single central load, and the ends free to rotate in azimuth, the observed critical loads, the beam being inverted for the second experiment, were

24,098 and 24,303 grammes,

Mean 24,200 grammes.

The critical load calculated from (7), l being 55 cms., is 24,258 grammes. As a check on the methods and apparatus, the loads producing Euler's instability of a long column were observed for two positions of the specimen. The results were

11,570 and 11,470 grammes.

The theoretical critical load is 11,270 grammes.

The chief source of error in the experiments was the want of uniformity in the thickness of the specimen, this dimension varying from .2550 cm. to .2628 cm. in different parts. No method of completely eliminating this cause of error suggested itself; and since the stiffness varies as the cube of the thickness, the divergence of the observed from the calculated results is not greater than might be expected to arise from this irregularity.

Melbourne, May 1899.

XXXIII. *Notices respecting New Books.*

Electromagnetic Theory. Vol. II. By OLIVER HEAVISIDE.
(The Electrician Series.)

THIS second volume of Mr. Heaviside's 'Electromagnetic Theory' contains four chapters, the first being a very short one on the Age of the Earth, a subject suggested to the Author by Professor Perry. Mr. Heaviside attacks the well-known differential equations relating to the flow of heat in spheres and solid blocks by his *operational method*, on which, indeed, the whole of this volume is founded, and of which (apart from physical conceptions) it may be said to be a continuous illustration. No definite result as to the age of the Earth can be deduced, owing, of course, to the uncertainty of the various thermal constants (or, rather, parameters) involved, and the widest (or wildest?) estimates are still possible. For example, M. Resal gives in his *Physique Mathématique* an investigation which leads to the result that from the Coal