planes parallel to $\lambda x+\mu y+z=0$ in curves of that order, viz.,

(2.1)	(3.1)	(4.1)	(5.1)	(6.1)	(7.1)	(8.1)	(9.1)	(10.1)	(11.1)	...
(3.1)	(4.1)	(5.1)	(6.1)	(7.1)	(8.1)	(9.1)	(10.1)	(11.1)	(12.1)	...
(3.2)	(4.2)	(5.2)	(6.2)	(7.2)	(8.2)	(9.2)	(10.2)	(11.2)	(12.2)	...
(4.1)	(5.1)	(6.1)	(7.1)	(8.1)	(9.1)	(10.1)	(11.1)	(12.1)	(13.1)	...
(4.2)	(5.2)	(6.2)	(7.2)	(8.2)	(9.2)	(10.2)	(11.2)	(12.2)	(13.2)	...
(4.3)	(5.3)	(6.3)	(7.3)	(8.3)	(9.3)	(10.3)	(11.3)	(12.3)	(13.3)	...
(5.1)	(6.1)	(7.1)	(8.1)	(9.1)	(10.1)	(11.1)	(12.1)	(13.1)	(14.1)	...
(5.2)	(6.2)	(7.2)	(8.2)	(9.2)	(10.2)	(11.2)	(12.2)	(13.2)	(14.2)	...
5.3)	(6.3)	(7.3)	(8.3)	(9.3)	(10.3)	(11.3)	(12.3)	(13.3)	(14.3)	...
5.4)	(6.4)	(7.4)	(8.4)	(9.4)	(10.4)	(11.4)	(12.4)	(13.4)	(14.4)	...
...	...	...	...	...	...	...	...	...	...	...

.....................(98),

to $\frac{1}{2}n(n+1)$ rows and columns, where $(m.\mu)$ denotes the multiplier of k^m in the expansion of

$$(F_2k^2+F_3k^3+F_4k^4+\ldots)^\mu.$$

On the Figures formed by the Intercepts of a System of Straight Lines in a Plane, and on analogous relations in Space of Three Dimensions. By SAMUEL ROBERTS.

[*Read May 10th*, 1888.]

I. *Plane Space.*

1. In studying some questions relating to the closed branches of curves, I was led to consider the clear spaces enclosed by the finite segments determined by the intersections of straight lines in a plane. By "clear spaces" I mean those not cut by any of the lines, and it will be convenient to call them simply "spaces." I have since found that, long ago, Steiner treated of the subject, in consequence of his

finding formulated in certain geometrical text-books connected with the Pestalozzian system the following proposition, viz.:—"To determine how many parts of the plane can be marked off by means of a given number of straight lines and circles altogether finite." Accordingly in an early paper entitled, "Einige Gesetze über die Theilung der Ebene und des Raumes" (*Crelle's Journal*, B. I., § 349—364), Steiner determines the number of parts in various cases, taking systems of straight lines with parallel groups, and of circles with concentric groups, afterwards proceeding to the solution of similar questions relating to planes and spheres. He assumes that no more than two lines intersect in the same point finitely situate, and imposes similar conditions on the circles, planes, and spheres, so that the final formulæ exhibit the number of parts "at most."

In the present paper, I study in somewhat more detail the nature of these figures. The determination of the number of parts cut off is plainly only one of many problems which arise in connection with such systems. For the figures formed by a system of straight lines in a plane are not only finite in number, but definite in form. Thus three straight lines not meeting in the same point finitely situate form by their finite segments a triangle; four straight lines, of which no three meet in the same point, make by their finite segments two triangles and a quadrilateral, and, although for higher numbers the general configuration is variable, it is so within limits.

I shall confine myself in what follows to the consideration of systems of straight lines and planes.

2. Let n straight lines in one plane intersect in points finitely situate, no three of the lines meeting in the same point. Several numerical relations are matter of immediate inference.

The number of points of intersection is $\frac{n \,.\, n-1}{2}$; that of the finite segments (which form the sides of the finite spaces) is $n\,(n-2)$; that of the segments unlimited in one direction (which I shall call "prolongations") is $2n$, the sum of the two sets being n^2.

If now an additional transversal be applied to the system, $n-1$ new finite spaces will be added, and, corresponding to the numbers of lines $3, 4 \dots n$, the numbers of the finite spaces are $1, 3, 6 \dots \frac{n-1 \,.\, n-2}{2}$. The number of open spaces is $2n$. Relatively to the finite figure the intersections may be distributed in four classes—(1) apices, (2) neutral or level points, (3) reentrant points, (4) interior points, altogether surrounded by the external contour.

Let the numbers of each class in the same order be a_1, a_2, a_3, a_4, then

$$a_1+a_2+a_3+a_4 = \frac{n(n-1)}{2}$$

If we take into account for a moment the prolongations, it appears that to an apex belong two prolongations, to a neutral point belongs one, the reentrant and interior points are not immediately connected with any prolongation. Hence

$$2a_1+a_2 = 2n.$$

Further an apex terminates two finite segments; a neutral point, three; a reentrant or interior point, four; therefore

$$\left.\begin{aligned} 2a_1+3a_2+4a_3+4a_4 &= 2n(n-2) \\ \text{or} \qquad a_2+2a_3+2a_4 &= n(n-3) \\ -2a_1+2a_3+2a_4 &= n(n-5) \end{aligned}\right\}$$

If K, L, M denote respectively the numbers of interior segments, of finite contour segments, and the sum of the numbers of the sides bounding the finite spaces, then

$$K = \frac{n(n-3)}{2}+a_4, \quad L = \frac{n(n-1)}{2}-a_4, \quad M = \frac{3n^2-7n}{2}+a_4.$$

For the number of contour sides is

$$a_1+a_2+a_3, \text{ and } L+M = 2n(n-2).$$

The maximum and minimum values of a_4 determine therefore the maximum and minimum values of K, M, and the minimum and maximum values of L.

If N is the number of right angles which make up the sum of the angles of the finite spaces,

$$N = n(n-1)+2a_4-4.$$

Let A_p denote the number of p-agons contained among the finite spaces, then

$$\left.\begin{aligned} A_n+A_{n-1}+\ldots+A_3 &= \frac{n-1\,.\,n-2}{2} \\ nA_n+(n-1)A_{n-1}+\ldots+3A_3 = M &= \frac{3n^2-7n}{2}+a_4 \end{aligned}\right\} \ldots\ldots(\text{A}).$$

Taking account only of the sides of the open spaces, and denoting by

B_p the number of such spaces having p sides, we have

$$\left.\begin{aligned} B_n+B_{n-1}+\ldots+B_3 &= 2n \\ nB_n+(n-1)B_{n-1}+\ldots+2B_3 &= \frac{n(n+7)}{2}-a_4 \end{aligned}\right\} \quad \ldots\ldots\ldots(B).$$

The value of B_3 is a_1. But the possible forms fall short of the integer and positive solutions of these equations except when $n = 3$ or 4.

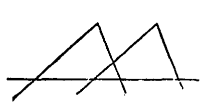
3. Still considering the finite figure, the maximum value of a_1 is n, if n is odd. For no line can contain more than two apices. If n be odd and the lines be numbered consecutively, we can arrange the cycle (1, 2), (2, 3) ... $(n-1, n)(n, 1)$, so that each line contains two apices.

When n is even, we cannot form a figure having n apices, since, if the lines be numbered as before and arranged in cycle, an evenly numbered line must, when we set out from an apex on it, cut all the oddly numbered lines previous to it in order, before the second apex is arrived at. Hence we cannot form the apex $(n, 1)$ in the cycle.

The maximum number of apices is consequently $n-1$. The minimum number of apices is in both cases 3, and any intermediate number can be given to the figure so that, for n odd, a_1 ranges from 3 to n, for n even, from 3 to $n-1$. It follows that a_2 (always even) ranges from 0 to $2n-6$ when n is odd, from 2 to $2n-6$ when n is even.

4. There must be at least one reentrant point between each pair of apices, except when a contour line contains no reentrant point. When $n = 3$, there are three such contour lines, and when $n = 4$ there are two; but when n is greater than 4 we can only have one such line, except in the case of $a_1 = 3$, when we may have two. For it will be observed that, given a figure for 4 lines, we can add as many transversals as we please, terminated at both ends by neutral points, that is to say, not containing apices. Hence, except in the case of $a_1 = 3$, we must have at least $a_3 = a_1$ or a_1-1.

We reduce the reentrant points between a pair of apices to a single one by aggregating them thus



On the other hand, by segregating them thus

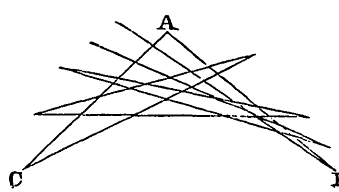
we get two reentrant points between the pair of apices; and we shall get the maximum value of a_3 by providing as many as possible of such pairs together, with as many as possible of reentrant points not immediately depending on apices.

We may set out with two lines, viz., the contour line without reentrants, and another beyond which we can, at most, place, if n is odd, $\frac{n-1}{2}$ aggregated apices. Add to these the two apices at the extremities of the contour line free from reentrants, and the number is $\frac{n+3}{2}$. If n is even, we can make at most, $\frac{n-2}{2}$ such apices, and, adding the two apices on the contour line free from reentrants, we have $\frac{n}{2}+1$. It follows that, up to and inclusive of $a_1 = \frac{n+3}{2}$ (n odd), and up to and inclusive of $a_1 = \frac{n}{2}+1$ (n even), we have (except in the special case of $a_1 = 3$) $a_3 = a_1 - 1$ for a minimum; for higher values the minimum value of a_3 is a_1.

5. In order to get the maximum value of a_3, we place, if a_1 is even, $\frac{a_1-2}{2}$ apices beyond one of two fundamental lines, say AB, and $\frac{a_1-4}{2}$ beyond AC, the other fundamental line. There are thus a_1-3 apices, each accompanied by two reentrant points, and we can get

$$n-3-\frac{a_1-2}{2}$$

other reentrant points at most. The following figure is a typical form for 8 apices and 9 lines:



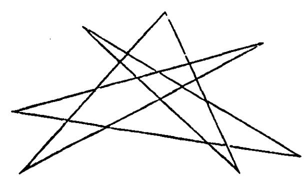
The maximum value of a_3 is for a_1 even,

$$n-3-\frac{a_1-2}{2}+2(a_1-3)=n+\frac{3a_1}{2}-8,$$

If a_1 is odd, the maximum of a_3 is

$$n-3-\frac{a_1-3}{2}+2(a_1-3)=n+\frac{3a_1}{2}-\frac{15}{2}.$$

The typical figure for 7 apices and 7 lines is



We can have any number of reentrant points between these limits and unity inclusive. Certain classes of figures (finite) can now be indicated in a tabular form (Table I.). The numbers under the respective letters at the heads of the column denote their corresponding values, the system in each row belonging to the same class.

To obtain the number of classes for each value of $n>4$, we observe that for n odd, and assuming in the first instance $a_3=a_1$ for the lowest value of a_3, we have

$$2\left\{(n-5)+(n-4)+\ldots+\frac{3n-15}{2}\right\}+\frac{3n-13}{2},$$

and for n even

$$2\left\{(n-5)+(n-4)+\ldots+\frac{3n-16}{2}\right\}+\frac{3n-14}{2}.$$

But for the exceptional cases in which the minimum value of a_3 is $< a_1$, we must add in the first case $\frac{n+1}{2}$, and in the second $\frac{n}{2}$. Therefore the number of classes is $\frac{1}{4}(5n^2-32n+51)$ for n odd, and $\frac{1}{4}(5n^2-38n+76)$ for n even.

The minimum value of a_4 (interior points) obtains when a_3-a_1 is a maximum, and is $\frac{n^2-8n+15}{2}$ for n odd, and $\frac{n^2-8n+16}{2}$ for n even. The maximum value is $\frac{n(n-5)}{2}+2$ $(n>3)$.

I.

n	a_1	a_2	a_3	a_4	M	N
3	3				3	2
4	3	2	1		10	8
5	3	4	1	2	22	20
	3	4	2	1	21	18
	4	2	3	1	21	18
	5		5		20	16
6	3	6	1	5	38	36
	3	6	2	4	37	34
	3	6	3	3	36	32
	4	4	3	4	37	34
	4	4	4	3	36	32
	5	2	5	3	36	32
	5	2	6	2	35	30
7	3	8	1	9	58	56
	3	8	2	8	57	54
	3	8	3	7	56	52
	3	8	4	6	55	50
	4	6	3	8	57	54
	4	6	4	7	56	52
	4	6	5	6	55	50
	5	4	4	8	57	54
	5	4	5	7	56	52
	5	4	6	6	55	50
	5	4	7	5	54	48
	6	2	6	7	56	52
	6	2	7	6	55	50
	6	2	8	5	54	48
	7		7	7	56	52
	7		8	6	55	50
	7		9	5	54	48
	7		10	4	52	46

n	a_1	a_2	a_3	a_4	M	N
8	3	10	1	14	82	80
	3	10	2	13	81	78
	3	10	3	12	80	76
	3	10	4	11	79	74
	3	10	5	10	78	72
	4	8	3	13	81	78
	4	8	4	12	80	76
	4	8	5	11	79	74
	4	8	6	10	78	72
	5	6	4	13	81	78
	5	6	5	12	80	76
	5	6	6	11	79	74
	5	6	7	10	78	72
	5	6	8	9	77	70
	6	4	6	12	80	76
	6	4	7	11	79	74
	6	4	8	10	78	72
	6	4	9	9	77	70
	7	2	7	12	80	76
	7	2	8	11	79	74
	7	2	9	10	78	72
	7	2	10	9	77	70
	7	2	11	8	76	68

6. As I have said, the equations (A) and (B) which must be satisfied give also inadmissible solutions. Some of the limitations on these general expressions can be immediately inferred. Thus, relative to the equations (A), the first number A_n must be unity or zero, since n lines can at most make one n-agon. It is moreover found, by actual inspection of the figure, when $n = 5$, that we cannot by an additional transversal create a hexagon and a pentagon. It follows that, for values of $n > 5$, $A_{n-1} = 0$ if $A_n = 1$.

Again, A_3 cannot be less than $n-2$. Suppose this is so up to $n-1$. In such a system, the removal of a line diminishes the number of triangles by one. Now take an n^{th} transversal not forming a divided triangle with at least one of the triangles of the $(n-1)$ system. That triangle is lost and not replaced by the removal of an original line. If the transversal makes divided triangles with all the triangles of the $(n-1)$ system, a triangle is still lost by the removal of an extreme line. The transversal must therefore make an additional triangle, and the (n) system has $(n-2)$ triangles, since three lines give one triangle, four lines give two triangles, &c. By "divided triangle" I mean a triangle divided by a line into a triangle and quadrilateral. The number of triangles cannot be diminished by adding transversely.

We can determine various general solutions. Thus a figure can be obtained $\frac{n-2\,.\,n-3}{2}$ quadrilaterals and $n-2$ triangles. By adjusting the angle of intersection we can draw a line through a point on an interior segment so as to add two triangles, $n-3$ quadrilaterals, and two sides, one to each of two spaces, and one of them may be a triangle, in which case the transversal must make two triangles. Through a point on a contour segment we can draw a line adding two triangles, $n-3$ quadrilaterals, and one side to a space. Through a point on a prolongation we can draw a line adding one triangle and $n-2$ quadrilaterals. Any one of the numbers A_n, A_{n-1}, ... A_4 may vanish. Similarly other results applicable to the general number n can be obtained. But I have not succeeded in finding an exhaustive method of determining all the admissible solutions of the equations.

The accompanying scheme shows the admissible forms for $n = 5, 6$. I denote as before by P_q a q-agon (Table II.). The forms marked with an asterisk are inadmissible. All but six of these are excluded by the preceding considerations.

7. The general expressions of § 2 may be extended to the case in which the system contains groups of lines passing through one point. If p lines cointersect in one point, it has absorbed all the spaces, the finite edges, and all the points due to the intersection of p lines. If therefore, in a system of n lines, p pass through one point, q through another, r through another, and so on, the number of spaces is

$$\frac{n-1\,.\,n-2}{2}-\frac{p-1\,.\,p-2}{2}-\frac{q-1\,.\,q-2}{2}-\frac{r-1\,.\,r-2}{2}-\&c.,$$

and the number of finite edges is

$$n\,(n-2)-p\,(p-2)-q\,(q-2)-r\,(r-2)-\&c.$$

In the latter case we must take p, q, r, &c. $=$ or >2.

The points, on this general supposition, may be described as terminating a certain number of finite segments and a certain number of prolongations. If a point terminates $2a_1$ segments in all, of which a_1 are prolongations, this is, in fact, an apex; if a_1-1 are prolongations, it is a level point; if there are no prolongations, it is an interior point. If there are α_1 prolongations where α_1 is $< a_1-1$, it is a re-entrant point. Including these in one class, let there be p points of the orders $a_1, a_2 \dots a_p$ terminating respectively $\alpha_1, \alpha_2 \dots \alpha_p$ prolongations; then the sum of the sides of the faces is

$$\Sigma_1^p a_p - \Sigma_1^p \alpha_p - C,$$

where C is the number of contour points, and $\Sigma_1^p a_p = 2n$.

II.

$n = 5$

	P_5	P_4	P_3
	1	2	3
	1	1	4
	1		5
		3	3
*		4	2
*		2	4

$n = 6$

	P_6	P_5	P_4	P_3
*	1	2	1	6
*	1	1	3	5
	1		5	4
*	1	1	2	6
*	1	2		7
	1		4	5
*	1	1	1	7
	1		3	6
*	1	1		8
*	1		2	7
*		1	6	3
	2		4	4
*	3		2	5
	1		5	4
	2		3	5
*	3		1	6
	1		4	5
	2		2	6
	3			7
	1		3	6
*	2		1	7
*			8	2
*			7	3
			6	4
*			5	5

8. If a group of p lines is a parallel one, we must further deduct p from the number of the edges, and $p-1$ from the number of spaces, and so for other groups of parallels. In his paper, Steiner does not consider intersections finitely situate of a higher order than 2, but only parallel groups. He gives the number of spaces in the form

$$1-U+A+\frac{a \,.\, a-1}{2},$$

where a is the number of single lines, U is the sum of the orders of the groups, and A is the sum of their products in pairs. This form gives a very symmetrical expression when circles also are involved. Putting aside for a moment the case of single lines, we may write U for n, and our expression becomes

$$\frac{(p+q+r+\&c.-1)(p+q+r+\&c.-2)}{2}$$

$$-\frac{p-1\,.\,p-2}{2}-\frac{q-1\,.\,q-2}{2}-\frac{r-1\,.\,r-2}{2}-\&c.,$$

or $\Sigma pq-k+1$, where k is the number of groups; but, since the groups are parallel, we must deduct

$$(p-1)+(q-1)+(r-1)+\&c. \quad \text{or} \quad p+q+r+\&c.-k,$$

giving $$\Sigma pq-\Sigma p+1.$$

If now we suppose one of the groups, say the p group, to consist of single lines differently directed, we have deducted too much by

$$\frac{p-1\,.\,p-2}{2}+p-1 \quad \text{or} \quad \frac{p\,.\,p-1}{2},$$

so that Steiner's formula results.

9. If we take generally a system of points at which respectively $a_1, a_2 \ldots a_p$ finite segments terminate, the number of segments is $\frac{a_1+\ldots+a_p}{2}$, and the number of spaces is $\frac{a_1+a_2+\ldots+a_p}{2}-p+1$. For, assuming the formula, if we add a point a_{p+1}, we increase the number of segments by a_{p+1}, and the number of spaces by $a_{p+1}-1$, and we have

$$\frac{a_1+a_2+\ldots+a_p}{2}+a_{p+1}-p=\frac{a_1+a_2+\ldots+a_{p+1}+a_{p+1}}{2}-(p+1)-1,$$

which is the same form, since the original system of points contains a_{p+1} points to which a segment has been added. The formula is true for $p=3$, 4, &c. Let μ be the number of contour points, then the sum of the sides of the corresponding spaces is $a_1+a_2+\ldots+a_p-\mu$. The sum of the angles is equivalent to $2\mu-4+4(p-\mu)$ or $4p-2\mu-4$ right angles.

The formulæ of § 2 are, in fact, independent of the linear relations which reduce the number of admissible figures in the case of systems of lines. Disregarding linear relations, we can with 10 points and 15 segments, no more than 4 segments meeting in a point, construct 4 quadrilaterals and 2 triangles, or with 15 points and 24 segments, no more than 4 meeting in a point, we can construct 3 pentagons, 1 quadrilateral, and 6 triangles. These are inadmissible forms when the parts and segments are those due to a system of straight lines.

II. *Space of Three Dimensions.*

10. Let us now take a system of n planes, of which no more than three meet in one point, and no more than two have a common line, and no two are parallel. Moreover the points of intersection (triple points) are supposed to be finitely situate.

If we add one more plane to the system, it is cut in n lines which give $\frac{n-1\,.\,n-2}{2}$ new finite spaces, to each of which belongs an additional clear space or volume.

If u is the number of finite volumes of the system of n planes, we may write

$$\Delta u = \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2} - n + 1,$$

whence

$$u = \frac{n(n-1)(n-2)}{2\,.\,3} - \frac{n(n-1)}{2} + n - 1 = \frac{(n-1)(n-2)(n-3)}{2\,.\,3}.$$

because u must vanish for $n=1$.

When we include the open volumes, and write v for the corresponding number, we have

$$\Delta v = \frac{n(n-1)}{2} + n + 1,$$

and $$v = \frac{n(n-1)(n-2)}{2\,.\,3} + \frac{n(n-1)}{2} + n + 1 = \frac{n^3+5n+6}{6}.*$$

because $n=3$ gives two spaces.

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* This and some other particular cases will be found given as examples in the text-books, *e.g.*, in Mr. C. Smith's *Treatise on Algebra*, recently published, Examples XXIII.

The number of finite faces is $\frac{n(n-2)(n-3)}{2}$ since each plane is cut by $n-1$ planes, giving $\frac{n-2\,.\,n-3}{2}$ plane spaces. Including open spaces, the number is

$$\frac{n\left[(n-1)^2+(n-1)+2\right]}{2} \text{ or } \frac{n(n^2-n+2)}{2}.$$

The number of finite edges is $\frac{n(n-1)(n-3)}{2}$, or, the prolongations being included, $\frac{n(n-1)^2}{2}$.

When p planes meet in one and the same point, but no more than two have a common line, the volumes, faces, and edges, due to a system of p planes, are absorbed in the common point. If, therefore, such groups of $p_1, p_2, \ldots p_m$ planes exist in the system of n planes, the number of finite volumes is

$$\frac{(n-1)(n-2)(n-3)}{2\,.\,3} - \sum_{l=1}^{l=m} \frac{(p_l-1)(p_l-2)(p_l-3)}{2\,.\,3},$$

that of the finite faces is

$$\frac{n(n-2)(n-3)}{2} - \sum_{l=1}^{l=m} \frac{p_l(p_l-2)(p_l-3)}{2},$$

and that of the finite edges is

$$\frac{n(n-1)(n-3)}{2} - \sum_{l=1}^{l=m} \frac{p_l(p_l-1)(p_l-3)}{2}.$$

11. We will next suppose that the system of n planes contains certain groups of planes having a common line, but that the several multiple lines do not intersect.

Let there be one such group of a_1 planes. If w is the number of finite volumes, we have

$$\Delta w = \frac{(n-1)(n-2)}{2} - \frac{(a_1-1)(a_1-2)}{2},$$

and

$$w = \frac{(n-1)(n-2)(n-3)}{2\,.\,3} - (n-1)\frac{(a_1-1)(a_1-2)}{2} + \frac{a_1(a_1-1)(a_1-2)}{3},$$

for w must be zero, for $n = a_1$.

If now we cut the system by another group of a_2 planes having a common line, the increment of volumes is

$$\frac{(n-1)(n-2)}{2} - \frac{(a_1-1)(a_1-2)}{2} + (a_2-1)\left\{\frac{n(n-1)}{2} - \frac{(a_1-1)(a_1-2)}{2}\right\},$$

and, changing n into $n-a_2$, we get

$$w=\frac{(n-1)(n-2)(n-3)}{2.3}-(n-1)\left\{\frac{(a_1-1)(a_1-2)}{2}+\frac{(a_2-1)(a_2-2)}{2}\right\}$$

$$+\frac{a_1(a_1-1)(a_1-2)}{3}+\frac{a_2(a_2-1)(a_2-2)}{3};$$

and the result will be of similar form when we include groups of a_3, $a_4 \dots a_q$ planes having common lines. In fact, assuming the general expression to be

$$\frac{(n-1)(n-2)(n-3)}{2.3}-(n-1)\left\{\frac{(a_1-1)(a_1-2)}{2}+\dots+\frac{(a_q-1)(a_q-2)}{2}\right\}$$

$$+\frac{a_1(a_1-1)(a_1-2)}{3}+\dots+\frac{a_q(a_q-1)(a_q-2)}{3}.$$

add another group of a_{q+1} planes having a common line. The increment of finite volumes is

$$\frac{(n-1)(n-2)}{2}-\left\{\frac{(a_1-1)(a_1-2)}{2}+\dots+\frac{(a_q-1)(a_q-2)}{2}\right\}$$

$$+(a_{q+1}-1)\left\{\frac{n(n-1)}{2}-\left[\frac{(a_1-1)(a_1-2)}{2}+\dots+\frac{(a_q-1)(a_q-2)}{2}\right]\right\}$$

Writing now $n-a_{q+1}$ for n, and observing that

$$\frac{(n-a_{q+1}-1)(n-a_{q+1}-2)(n-a_{q+1}-3)}{2.3}+\frac{(n-a_{q+1}-1)(n-a_{q+1}-2)}{2}$$

$$+(a_{q+1}-1)\frac{(n-a_{q+1})(n-a_{q+1}-1)}{2},$$

is reducible to

$$\frac{1}{2.3}\{(n-1)(n-2)(n-3)-3a_{q+1}^2n+2a_{q+1}^3$$

$$+9a_{q+1}n-3a_{q+1}^2-5a_{q+1}-6n+6\},$$

we have finally,

$$\frac{(n-1)(n-2)(n-3)}{2.3}-(n-1)\left\{\frac{(a_1-1)(a_1-2)}{2}+\dots\right.$$

$$\left.+\frac{(a_q-1)(a_q-2)}{2}+\frac{(a_{q+1}-1)(a_{q+1}-2)}{2}\right\}+\frac{a_1(a_1-1)(a_1-2)}{3}+\dots$$

$$+\frac{a_q(a_q-1)(a_q-2)}{3}+\frac{a_{q+1}(a_{q+1}-1)(a_{q+1}-2)}{3},$$

which verifies the form generally.

We can deal similarly with the open spaces. For if we suppose the finite figure constituted by n planes to be surrounded by a superficies, say, a sphere, the number of spaces will be the same as that of the parts into which the superficies is divided by the planes. Let groups of a_1, $a_2 \dots a_p$ planes have common lines respectively. The effect of the multiple lines is to produce pairs of multiple points among the intersections of the arcs determined by the planes on the sphere, and each of these absorbs the same number of superficial spaces as if the arcs were straight lines. Hence the number of open spaces or regions is

$$n^2-n+2-(a_1-1)(a_1-2)-(a_2-1)(a_2-2)-(a_p-1)(a_p-2);$$

and, if $n=\Sigma a_1$, this result is

$$2\Sigma a_1 a_2+2\Sigma a_1-(p-1)\,2.$$

For brevity's sake, and because the finite figure possesses more interest, I concern myself chiefly with the finite volumes, &c.

12. If, however, some of the multiple lines intersect, the above determinations become incorrect. Suppose that a number of multiple lines of various orders meet in one point. The intersection has absorbed the volumes, faces, and edges due to a system made up of the same number of multiple lines of the same orders, and constituted by the same number of planes, but not co-intersecting. Hence, the foregoing expression gives us the form of the correction.

Let the system of n planes contain multiple lines of the orders $k_1, k_2 \dots k_s$ meeting at a multiple poi..t of the order m_1, of the orders $l_1, l_2 \dots l_t$ meeting at a multiple point of the order m_2, of the orders $\kappa_1, \kappa_2 \dots \kappa_o$, meeting at a multiple point of the order m_3, of the orders $\lambda_1, \lambda_2 \dots \lambda_r$, meeting at a multiple point of the order m_4, and so on.

The expression for the number of finite volumes is

$$\frac{(n-1)(n-2)(n-3)}{2\,.\,3}-(n-1)\left\{\frac{(a_1-1)(a_2-2)}{2}+\dots+\frac{(a_q-1)(a_q-2)}{2}\right\}$$
$$+\frac{a_1(a_1-1)(a_1-2)}{2}+\dots+\frac{a_q(a_q-1)(a_q-2)}{3}$$
$$-\frac{(m_1-1)(m_1-2)(m_1-3)}{2\,.\,3}+(m_1-1)\left\{\frac{(k_1-1)(k_1-2)}{2}+\dots\right.$$
$$\left.+\frac{(k_s-1)(k_s-2)}{2}\right\}-\left[\frac{k_1(k_1-1)(k_1-2)}{3}+\dots+\frac{k_s(k_s-1)(k_s-2)}{3}\right]$$
$$-\frac{(m_2-1)(m_2-2)(m_2-3)}{2\,.\,3}+(m_2-1)\left\{\frac{l_1-1)(l_1-2)}{2}+\dots\right.$$
$$\left.+\frac{(l_t-1)(l_t-2)}{2}\right\}-\left\{\frac{l_1(l_1-1)(l_1-2)}{3}+\dots+\frac{l_t(l_t-1)(l_t-2)}{3}\right\}$$
$$-\&c.,$$

or as we may write it

$$F(n, a_1 \dots a_q)-F(m_1, k_1 \dots k_s)-F(m_2, l_1 \dots l_t)$$
$$-F(m_3, \kappa_1 \dots \kappa_o)-F(m_4, \lambda_1 \dots \lambda_r)-\&c.,$$

where the symbol F denotes similar functions, except as to the number of the letters involved. It is to be remembered also that the letters k_1, &c., l_1, &c., and so on, really represent orders comprised in a_1, &c., and may be repeated in different functions. In this way we include cases in which the same multiple line intersects several others. If no multiple line meets another more than once (excepting at the usual triple points), the expression is simplified and becomes

$$\frac{(n-1)(n-2)(n-3)}{2.3}-(n-m_1)\left\{\frac{(k_1-1)(k_1-2)}{2}+\ldots+\frac{(k_s-1)(k_s-2)}{2}\right\}$$
$$-\frac{(m_1-1)(m_1-2)(m_1-3)}{2.3}$$
$$-(n-m_2)\left\{\frac{(l_1-1)(l_1-2)}{2}+\ldots+\frac{(l_t-1)(l_t-2)}{2}\right\}$$
$$-\frac{(m_2-1)(m_2-2)(m_2-3)}{2}-\&c.$$

The general formula includes the case of multiple points finitely situate, since we may consider these as constituted by double lines meeting together.

13. If we add to the system p parallel planes, the increment is

$$p\left[\frac{(n-1)(n-2)}{2}-\frac{(a_1-1)(a_1-2)}{2}-\frac{(a_2-1)(a_2-2)}{2}-\ldots\right.$$
$$\left.-\frac{(a_q-1)(a_q-2)}{2}\right].$$

A further addition of q parallel planes of different direction gives an increment

$$q\left[\frac{(n+p-1)(n+p-2)}{2}-\frac{(a_1-1)(a_1-2)}{2}-\frac{(a_2-1)(a_2-2)}{2}-\ldots\right.$$
$$\left.\ldots-\frac{(a_q-1)(a_q-2)}{2}\right]-\frac{p-1\,.\,p-2}{2}-p+1.$$

and so on.

If in the resulting expression we put $n=0$, and therefore dismiss $a_1, a_2 \ldots a_q$, we have

$$-1+p+q\left[\frac{(p-1)(p-2)}{2}-\frac{p-1\,.\,p-2}{2}-p+1\right]$$
$$-r\left[\frac{(p+q-1)(p+q-2)}{2}-\frac{(p-1)(p-2)}{2}-\frac{(q-1)(q-2)}{2}-p-q+2\right]$$
$$-\&c.$$

or $$-1+p-\Sigma pq+\Sigma pqr-\&c.,$$

which is the form in which Steiner gives the result. It includes the case in which single planes differently directed enter, for it is permissible to suppose the value of any of the letters p, q, r, &c. to be unity.

14. Similar considerations enable us to determine the number of faces and edges.

In the first instance, we take a system of n planes containing multiple lines not co-intersecting, and of the orders $a_1, a_2 \ldots a_q$.

There are $n-a_1+a_2-\ldots-a_q$ planes containing $n-1$ lines each, a_1 planes containing $n-a_1+1$ lines, and so on; but it must be remembered that in this way we count the multiple lines $a_1, a_2 \ldots a_q$ times respectively, whereas the other lines are counted only twice. Hence, for edges, we have to make a final deduction of

$$(a_1-2)(n-a_1-1)+\ldots+(a_q-2)(n-a_q-1).$$

Thus the formula for the number of edges will be

$$\frac{1}{2}\left\{\begin{array}{r} (n-a_1-a_2-\ldots-a_q)\,\{(n-1)(n-3)-a_1(a_1-2)-\ldots \\ \ldots-a_q(a_q-2)\} \\ +a_1\{(n-a_1+1)(n-a_1-1)-a_2(a_2-2)-\ldots \\ \ldots-a_q(a_q-2)\} \\ +\ldots+a_q\{(n-a_q+1)(n-a_q-1)-a_1(a_1-2)-\ldots \\ \ldots-a_{q-1}(a_{q-1}-2)\} \\ -(a_1-2)(n-a_1-1)-\ldots-(a_q-2)(n-a_q-1) \end{array}\right\}$$
$$= F_1(n, a_1, a_2 \ldots a_q).$$

And if a certain number of these multiple lines intersect in one point, let these lines be of the orders $k_1, k_2 \ldots k_s$. We must deduct for the edges lost according to the number of planes constituting the intersection. For m_1 planes the deduction will be $F_1(m_1, k_1, k_2 \ldots k_s)$, and so for any number of such intersections; *i.e.*, the general formula will be

$$F_1(n, a_1, a_2 \ldots a_q)-F_1(m_1, k_1, k_2 \ldots k_s)-F_1(m_2, l_1, l_2 \ldots l_t)-\&c.,$$

the symbol F_1 being interpreted in the same way as F. In like manner, we get, for the number of faces when the multiple lines do not intersect,

$$\frac{1}{2}\left\{\begin{array}{r} (n-a_1-a_2\ldots-a_q)\,[(n-2)(n-3)-(a_1-1)(a_1-2)-\ldots \\ \ldots-(a_q-1)(a_q-2)] \\ +a_1[(n-a_1)(n-a_1-1)-(a_2-1)(a_2-2)-\ldots \\ \ldots-(a_q-1)(a_q-2)] \\ +\ldots+a_q[(n-a_q)(n-a_q-1)-(a_1-1)(a_1-2)-\ldots \\ \ldots-(a_{q-1}-1)(a_{q-1}-2)] \end{array}\right\}$$
$$= F_2(n a_1 a_2 \ldots a_q),$$

and the correction for intersecting lines will be as before

$$-\left[F_2(m_1k_1k_2\ldots k_s)+F_2(m_2l_1l_2\ldots l_t)+\&c.\right],$$

a similar meaning being attached to the symbol F_3.

If certain groups of planes through a line intersect at infinity so that the multiple lines are parallel, the multiple point is the limit of an apex.

An apex subtends the same number of superficial spaces bounded by its lines as it would do if cut by a single plane; so that, when the apex is infinitely distant, we must deduct the number of volumes due to the planes through it, augmented by another plane. In addition, therefore, to the usual deduction of $F(m_1k_1k_2\ldots k_s)$, where m_1 is the number of planes, and $k_1, k_2\ldots k_s$ are the orders of the lines, we have to deduct

$$\frac{(m_1-1)(m_1-2)}{2}-\frac{(k_1-1)(k_1-2)}{2}-\ldots-\frac{(k_s-1)(k_s-2)}{2}.$$

It will be observed that an apex may subtend more faces of the figure than the number mentioned, but the number of superficies subtended will be as stated, by the principles of perspective.

In the like case, we can determine the correction for the numbers of the faces and edges considered as finite and the foregoing formulæ can be adapted to other cases which I do not treat of at length,—for example, to the case in which some of the parallel planes pass through a multiple point, and so forth. So that we may conclude that the numbers of the volumes, faces, and edges can be generally determined for systems of planes, consisting partly of single planes, of groups of planes having a common line finitely situate, or on the infinitely distant plane and multiple points formed by the intersection of multiple lines, and finitely or infinitely distant.

15. The numbers which occur relative to systems of a very moderate number of planes are large. Applying, for instance, the formula to the 45 real triple tangent planes of a cubic surface, we have to consider that they pass, five together, through 27 lines, and these intersect in 135 points in pairs, each point being constituted by 9 planes, since the two intersecting lines belong to a common plane of the system. The formulæ give therefore, for the numbers of the volumes or completely enclosed cells, the faces and the edges respectively

$$\frac{44.43.42}{2.3}-135\frac{8.7.6}{2.3}-27.44.\frac{4.3}{2}+27.\frac{5.4.3}{3}$$

$$+135.8.2.\frac{4.3}{2}-135.2.\frac{5.4.3}{3}=6656,$$

$$\tfrac{1}{2}\left\{\begin{matrix}(45-5.27)(43.42-27.4.3)+5.27(40.39-26.4.3)\\ -135\,(9-10)(7.6-2.4.3)-2.5\,(4.3-4.3)\end{matrix}\right\} = 18765,$$

$$\tfrac{1}{2}\left\{\begin{matrix}(45-5.27)(44.42-27.5.3)-5.27\,(41.39-26.5.3)\\ -27.3.39-135\left[(9-10)(8.6-2.5.3)\right.\\ \left.+5.2\,(5.3-5.3)-2.3.3\right]\end{matrix}\right\} = 17523,$$

and by the formula of § 11 there are 1658 open regions.

When we attempt to determine more particularly the forms of the volumes involved, the difficulty which we already encountered in the analogous plane problem is much intensified.

The edges of the finite figure made by n planes, no more than three meeting in one point, and no more than two having a common line, and no two being parallel, may be divided into four classes.

The edges may be (1) convex, (2) level, (3) re-entrant, (4) interior. Let a_1, a_2, a_3, a_4 be the numbers of the four kinds respectively. A convex edge belongs to two faces and one volume, a level to three faces and two volumes, a re-entrant to four faces and three volumes, an interior to four faces and four volumes. Hence, if we put F for the number of all the edges of all the faces taken separately, V for the number of all the edges of all the volumes taken separately, and E for the number of edges taken once only, we have

$$\left.\begin{aligned} 2a_1+3a_2+4a_3+4a_4 &= F\\ a_1+2a_2+3a_3+4a_4 &= V\\ a_1+a_2+a_3+a_4 &= E = \frac{n\,(n-1)(n-3)}{2}\end{aligned}\right\}\;\;\ldots\ldots\ldots(C),$$

and therefore $$a_4 = E+V-F.$$

These relations and others similarly obtainable are quite insufficient for a solution of the main question.

If we refer the letters a_1, a_2, a_3, a_4 back to the plane case, so that a_4 means the number of interior points, and represent by F, V, E, respectively, the number of the extremities of the finite edges taken separately, the sum of the number of the sides of the faces, and the number of intersections, the equations (C) hold in the same form, so that the value of a_4 is in the same form.