ON THE PROJECTIVE GEOMETRY OF A BINARY QUARTIC AND ITS HESSIAN

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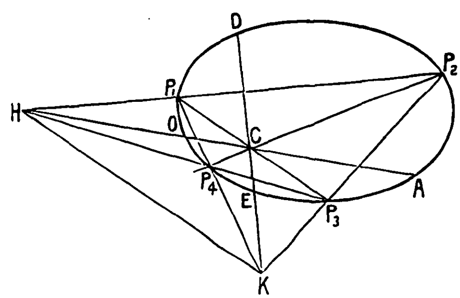
1. A conic drawn through the origin of coordinates x, y is chosen; and a quartic in x, y is represented by the quadrangle $P_1P_2P_3P_4$ of points in which the conic is met again by the four lines of which $u = 0$, the result of equating the quartic to zero, is the equation. Linear transformations of the quartic are obtained by taking any axes through any point of the conic, retaining the same quadrangle, and also by any projection of the figure. Real changes of origin and axes, and real projections, afford schemes of linear transformation with real coefficients.



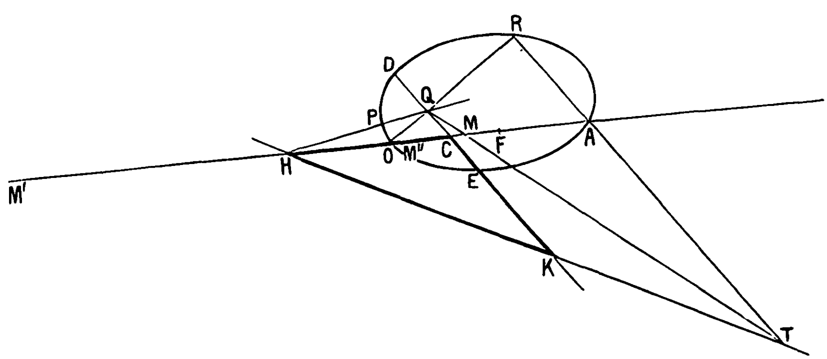
Let CHK be the harmonic triangle of the quadrangle. The sides of this are real if P_1, P_2, P_3, P_4 are all real or all imaginary in conjugate pairs, *i.e.*, if the quartic $u = 0$ with real coefficients has all its roots real or all imaginary. The sides of CHK cut the conic in three pairs of points, O and A, D and E, and a pair which in the case of a real CHK are imaginary, given by the equation $G = 0$, where G is the sextic covariant of u.

The quadrangle $u = 0$ is one of an infinity of quadrangles on the conic with the same harmonic triangle CHK. The vertices of any one are the points representative of some quartic of the pencil $\kappa u + \lambda H = 0$, where H is the Hessian of u. Given the triangle, one vertex P of any quadrangle of the pencil determines the other three as the second intersections of PC, PH, PK with the drawn conic.

To specify the particular quadrangle which gives the particular quartic of the pencil $H = 0$ itself is not so easy as the algebraic simplicity of

H in respect of u would lead us to expect, and I am not aware that any writer has taken the trouble to overcome the small difficulties of the specification of the Hessian by linear construction, though full attention has been paid to the expression of its properties in the language of polar pairs and triads. One difficulty arises from the fact that, if the roots of $u = 0$ are all real, those of $H = 0$ are all imaginary—it is easy to see that a binary quantic of any order with only real and unequal roots has a Hessian, a sum of squares, with no real roots—so that, starting from a quadrangle with real vertices as $u = 0$, we look for one with imaginary vertices as $H = 0$, a completed real exhibition of the H going with the u being thus an impossibility. There is also the complication that between quartics u and Hessians H there is not a one to one correspondence: a particular u has, of course, one Hessian H, but of u's which possess a particular H there are two. These are apolar with one another.

2. Associate with each quadrangle, *i.e.*, each quartic of the pencil, a point M on CH as follows. Take P one vertex of the quadrangle, the other three being where PC, PH, PK cut the conic again. Let PH meet



KC in Q. Join OQ, and also join R, the point where OQ cuts the conic again, to A by a line meeting HK in T. QT cuts CH in the point M.

Now, take F the harmonic conjugate of H with regard to C and A. F has no reference to any particular quadrangle of the pencil.

Let M, M', M'' be the points of CH which are associated as above with the u-quadrangle, the u'-quadrangle of the pencil, taking u' apolar with u, and the H-quadrangle of u, respectively. The facts, to be established presently by aid of a canonical projection, are as follows.

(i.) OA, FH, MM' are pairs of an involution.

(ii.) CH, MM', OM'' are pairs of another involution.

In these there is complete symmetry as between M and M'; the H-quadrangle is that of u' as well as that of u.

3. This association by means of harmonic properties and involutions of three quartics of a pencil $\kappa u+\lambda H$, with the same G, of which two are apolar and the third their common Hessian, is general. The following statement of realities of geometrical construction applies only when the harmonic triangle CHK is real, and so has no reference to quartics u with two roots real and two imaginary.

The above construction of a real M (or M' or M'') from a real P (or P' or P'') is linear. To construct P (or P' or P'') from a given real M (or M' or M'') we may join points of HK to M and to A, letting the connectors with M meet CH in points q, and the connectors with A meet the conic again in points whose connectors with O meet CH in points q', and find, linearly by aid of the drawn conic, either double point Q of the involution of pairs qq', finally joining QH to meet the conic in P. According to the position of M on CH, Q may be between D and E, or real and outside the segment DE, or imaginary. Only in one of the two former cases are the points P real; in the other they are imaginary on real lines through H; in the third case they are imaginary on imaginary lines through H, but, as we shall see, on real lines through K.

Given either M or M', we are told by (i.) that the other can be linearly constructed, and then by (ii.) that M'' can be.

Conversely, given M'', M and M' constitute the common pair of two involutions, each given by two pairs, and can be linearly constructed by aid of the drawn conic. We shall see that a real M'' thus determines a real M and M'.

4. To justify the above statements, which are strictly projective in form, we have to prove them for a figure canonically simplified by any convenient projection. In the case, on which we are fixing attention, of a real CHK let us project the drawn conic into a circle in such a way that HK, the side of CHK which does not meet the conic in real points, goes to infinity. This is a real projection. We also choose for origin—which may be anywhere on the circle—an end O of the diameter OCA which is the projection of the side CH, and for axes OA and the tangent at O. The linear transformation which effects all this is a real one.

We are, in fact, thus led to the usual canonical form of a quartic. Slightly departing from the usual notation, we write for the canonical form of $u = 0$

$$4x^2y^2 = \mu\,(x^2+y^2)^2. \tag{1}$$

The sextic covariant $G = 0$ is

$$xy(x^4-y^4) = 0. \tag{2}$$

Every quartic of the pencil $\kappa u+\lambda H = 0$ has an equation of the form of (1). Take, in particular,

$$4x^2y^2 = \mu'(x^2+y^2)^2 \tag{3}$$

for that quartic $u' = 0$ of the pencil which is apolar with $u = 0$. The condition to be satisfied, found by equating to zero the lineo-linear invariant of (1) and (3), is

$$(2\mu-1)(2\mu'-1) = -3. \tag{4}$$

Using this relation, we at once find that the Hessian $H = 0$ of u may be written

$$\begin{aligned} 4x^2y^2 &= -\mu\mu'(x^2+y^2)^2 \\ &= \tfrac{1}{2}(2-\mu-\mu')(x^2+y^2)^2, \end{aligned} \tag{5}$$

the symmetry of which tells us that, as stated earlier, the two apolar quartics have the same Hessian.

It is interesting to notice incidentally that another statement of these facts is that the quartic equations in $\tan\theta$,

$$\cos 4\theta = \cos 4\alpha, \quad \cos 4\theta = \cos 4\beta, \quad \cos 4\theta = \cos 4\gamma,$$

where $\cos 4\alpha \cos 4\beta = -3$ and $\cos 4\alpha+\cos 4\beta+2\cos 4\gamma = 0$,

are canonical forms of $u = 0$, $u' = 0$, $H = 0$ respectively, arrived at by real transformation, when $u = 0$ has only real or only imaginary roots. It is here clear that α and β cannot be both real, and readily shown that if one of the two, say α, is real, γ cannot be, except in the extreme cases of $\cos 4\alpha = \cos 4\gamma = \pm 1$, $\cos 4\beta = \mp 3$, in which cases u and H are identical, but for a constant factor, and perfect squares.

Returning to the notation of (1) to (5), the conic by which we cut the pencil of lines is the circle

$$x^2+y^2 = 2x. \tag{6}$$

The vertices of the u-, the u'-, and the H-quadrangles are then by (1), (3), and (5) respectively on the line pairs

$$y^2 = \mu, \quad y^2 = \mu', \quad y^2 = -\mu\mu', \text{ subject to (4).} \tag{7}$$

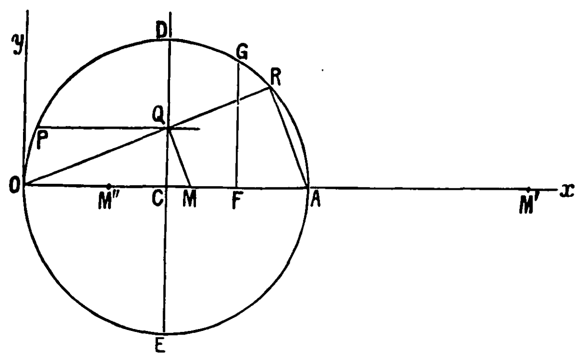
The quadrangles are now rectangles symmetrical about the lines $y = 0$ and $x = 1$. One vertex or side of a rectangle determines it completely.

The six points $G = 0$ are the ends of the rectangular diameters OCA, DCE and the circular points at infinity.

If we start from a real u-rectangle, all we can hope for in the way of real construction is, in virtue of a remark made above, to obtain real sides of the associated u'- and H-rectangles, whose vertices are imaginary except in the extreme cases of $\mu = 0, 1$.

Take P a vertex of the u-rectangle, and let a side $y = \sqrt{\mu}$ through it meet DE in Q. Join OQ, and let QM at right angles to it meet OA in M; then

$$CM = CQ^2/OC = CQ^2 = \mu,$$



so that, if F is the middle point of CA, $FM = \mu - \frac{1}{2}$, in sign and magnitude. If, then, M' corresponds to vertices P' of the u'-rectangle, just as M does to P, we have, by (4), that

$$FM \, . \, FM' = -\tfrac{3}{4} = FO \, . \, FA,$$

i.e.,

$$MF \, . \, FM' = FG^2,$$

so that MGM' is a right angle.

The constructions, or real steps towards construction, of the other three from a given one of P, P', M, M' are most naturally performed by drawing circles. But the effectiveness of methods which yield on projection the general linear ones expressed in §§ 2, 3 is clear. Given P, and so Q, a parallel through Q to RA cuts OA in M; and M' is the conjugate of M in the involution of which F is centre and OA a pair [cf. § 2, (i.)]. M' stands to Q', where DE (produced) meets a $y = \sqrt{\mu'}$ through a u'-point P', just as M does to Q, *i.e.*, OM' subtends a right angle at Q'. Given M (or M'), Q (or Q') is either double point of the involution on DE of which lines through M (or M') and the perpendiculars to them through O, *i.e.*, the lines from O to the points where parallels to them through A cut the circle, determine pairs.

Again, take M'' on OA (produced if necessary) such that

$$CM'' = -\mu\mu',$$

i.e., that $$CO \,.\, CM'' = CM \,.\, CM',$$

so that, in language suitable for projection, OM'', MM' are pairs of an involution with C for centre [cf. § 2, (ii.)]. M'' goes with a Q'' (imaginary in the figure) on DE, and vertices P'' of the H-rectangle, just as M goes with Q and P.

5. The statements of §§ 2, 3 have now been justified, except that there remain for consideration questions of reality and imaginariness.

We are supposed to have started from a quartic $u = 0$ with real coefficients, and have seen that when it has only real or only imaginary roots a real linear transformation gives it the form $4x^2y^2 = \mu\,(x^2+y^2)^2$. In these cases then μ is real, and consequently by (4) so are μ' and $-\mu\mu'$. The points M, M', M'' are real points on OA. Moreover, any real M (or M') on OA, infinitely produced, determines a real M' (or M) and a real M''. Conversely, any real M'' determines a real M and M': for, having $-\mu\mu'$ real, $= m$ say, (4) tells us that $\mu+\mu' = 2\,(1-m)$, so that $(\mu-\mu')^2 = 4\,(1-m)^2+4m = (1-2m)^2+3$, and consequently $\mu-\mu'$ as well as $\mu+\mu'$ is real.

As M covers the whole line OA produced, proceeding from left to right and starting from $-\infty$, M' also covers it all from left to right, starting from F, while M'' covers it twice from right to left. We have, in fact, by (4), the following correspondence of parts of the ranges of μ, μ', $-\mu\mu'$.

μ or μ'...	$-\infty$ to -1	-1 to 0	0 to $\frac{1}{2}$	$\frac{1}{2}$ to 1	1 to 2	2 to ∞
μ' or μ...	$\frac{1}{2}$ to 1	1 to 2	2 to ∞	$-\infty$ to -1	-1 to 0	0 to $\frac{1}{2}$
$-\mu\mu'$ ...	∞ to 1	1 to 0	0 to $-\infty$	∞ to 1	1 to 0	0 to $-\infty$

Half the table gives the whole, as it is symmetrical in its reference to μ, μ'. If B is the reflexion of C in A, we may also express it: according as

M or M' is to the left of O, between O and C, or between C and F,

M' or M is between F and A, between A and B, or to the right of B,

and M'' is to the right of A, between A and C, or to the left of C.

The speciality of the separating cases $-\mu\mu' = 0,\ 1,\ \infty$ will be noticed.

Now, if by μ we mean either of μ, μ', $-\mu\mu'$, the sides $y^2 = \mu$ of the associated rectangle are real only if μ is not negative, *i.e.*, if the associated M is not to the left of C, and the vertices of the rectangle are real only if further μ does not exceed 1, *i.e.*, if M is not outside the segment CA. If, then, the u- (or the u'-) rectangle is one of distinct real vertices, the u'- (or the u-) rectangle has imaginary vertices, which lie on real or on imaginary parallels to the diameter OA according as M (or M') is between C and F or between F and A, and the H-rectangle has also imaginary vertices, on imaginary or real parallels to the diameter according to the same circumstances; and, if the H-rectangle is one of distinct real vertices, both the u- and the u'-rectangles have distinct imaginary vertices, those of one being on real and those of the other on imaginary parallels to the diameter. In the special separating cases of $-\mu\mu' = 0,\ 1$, *i.e.*, of M'' and either M or M' at C or A, the H-rectangle is identical with one of the u- and u'-rectangles, having real vertices united in pairs. In the third separating case of $-\mu\mu'$ infinite, the H-rectangle is also identical with one of the u- and u'-rectangles, having imaginary vertices united in pairs.

The summary of results as to reality is then as follows :—

(i.) If the roots of u (or of u') are all real, the roots of u' (or of u) are all imaginary; and so are those of H, except that when u (or u') is the square of a quadratic with real roots H is the same square but for a constant factor.

(ii.) If the roots of u (or of u') are all imaginary, either the roots of the other are all real and those of H all imaginary, or *vice versa*, except that in two separating cases the other and H are identical and squares of quadratics with real roots.

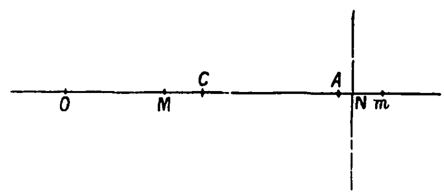
(iii.) If the roots of H are all real and different, those of both u and u' are imaginary; and, if those of H are real and equal in pairs, those of one of u, u' are the same equal pairs and those of the other are imaginary.

(iv.) If the roots of H are all imaginary, those of one of u, u' are real and those of the other imaginary; and in particular, if those of H are imaginary and equal in pairs, those of the one of u, u' which are imaginary are the same equal pairs.

6. It has been stated (in § 3) that in a case when the vertices of one of the triangles lie on imaginary parallels to OA, *i.e.*, when the construction of Q (or Q' or Q'') from M (or M' or M'') is devoid of reality, they lie on real perpendiculars to OA. This is clear; for the points of the circle which lie on $y^2 = \mu$ also lie on $(x-1)^2 = 1-\mu$. The intersections N with OA of the required perpendiculars to OA are then given by

$$CN^2 = CA \,.\, MA = CA \,.\, Cm,$$

where
$$Am = MC.$$



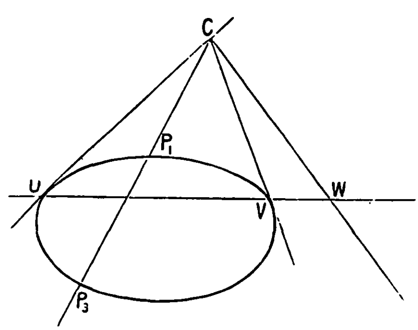
The construction of the real sides of the quadrangle in the general figure of § 2 may then be stated. Find, linearly, m, the harmonic conjugate of M with regard to FH, and then find, linearly by aid of the drawn conic, the double points of the involution of which Am, CH are pairs: the connectors of these with K are the real sides required.

7. The canonical projection and reference of the above do not apply when u is a fourth power, or when it is the product of a square and a non-square. These cases are so simple and well known that they will not be dwelt upon.

Excluding these singular cases, it has been observed in § 3 that the association in projective terms of three allied quartics u, u', H of a pencil $\kappa u+\lambda H$ with a given G is general. But the realities of construction detailed above have no applicability in cases when a chosen one of u, u', H with real coefficients has two real and two imaginary roots. In such cases the projection adopted above is imaginary, or, in other words, the linear transformation by which it is effected involves imaginary coefficients.

We have already the ground for asserting that, if one of u, u', H has two real and two imaginary roots, so also have the other two. For, if either had not, (i.) to (iv.) above would tell us that u had not.

For guidance to realities of construction in cases of two real and two imaginary roots a different canonical projection and reference are desirable, and to these we now proceed.



8. In a case when $u=0$ has two real and two (conjugate) imaginary roots, two vertices P_1, P_3 of its representative quadrangle on the chosen conic are real, and the other two are imaginary on a real line CW. One vertex C, and one side, the polar UVW of C, of the harmonic triangle are real, and the other two are the imaginary points on UV where it is met by the common pair of harmonic conjugates of CU, CV and CP_1, CW.

The best canonical projection now is the real one by which UVW is sent to infinity, and the angles UCV, P_1CW are made right angles. The conic becomes a rectangular hyperbola with C for centre. Take the transverse axis, and O one of its ends, for axis of x and origin. The equation of the hyperbola, say

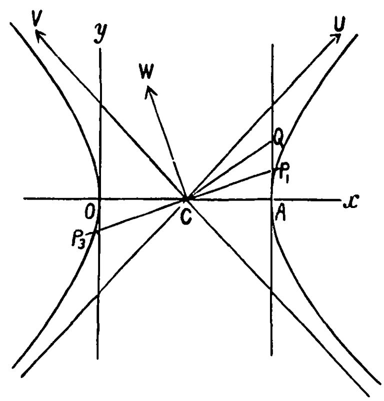
$$(x-1)^2-y^2 = 1, \qquad (8)$$

and that of P_1CP_3 and CW, say

$$\mu[(x-1)^2-y^2] = 2(x-1)y, \qquad (9)$$

give the vertices of the quadrangle. The lines to them from O are at once seen to be

$$4xy(x^2+y^2) = \mu(x^2-y^2)^2; \qquad (10)$$



and this is accordingly the canonical form of $u = 0$ to which we are led by a linear transformation with real coefficients.

Every quartic of the pencil $\kappa u+\lambda H = 0$ derived from this is of the same form, with some μ or other. In particular the apolar $u' = 0$ of the pencil is

$$4xy(x^2+y^2) = \mu'(x^2-y^2)^2 \qquad (11)$$

where

$$\mu\mu' = 3, \qquad (12)$$

and the Hessian $H = 0$ is

$$4xy(x^2+y^2) = -\tfrac{1}{2}(\mu+\mu')(x^2-y^2)^2. \qquad (13)$$

The sides of the common harmonic triangle of the quadrangles representative of the pencil are the line at infinity and the lines from C to the circular points, *i.e.*, the lines $(x-1)^2+y^2 = 0$; and the sextic covariant $G = 0$, the equation of the lines from O to the intersections of these sides with the hyperbola, is

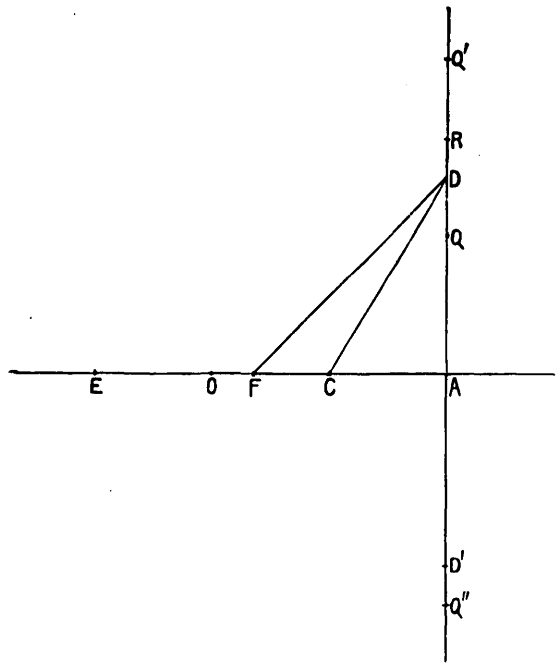
$$(x^2-y^2)(x^4+6x^2y^2+y^4) = 0. \qquad (14)$$

9. Now (9) gives that, if P_1 is a u-point on the hyperbola,

$$\tan 2ACP_1 = \mu. \qquad (15)$$

Hence, if on the tangent at A we take an ordinate $AQ = \mu$, the two sides P_1CP_3, CW through C of the u-quadrangle are the bisectors of the angle ACQ.

We are thus led to exhibit the relative positions of the real vertices P_1 and P_3, P_1' and P_3', P_1'' and P_3'' of the u-, the u', and the H-quadrangles

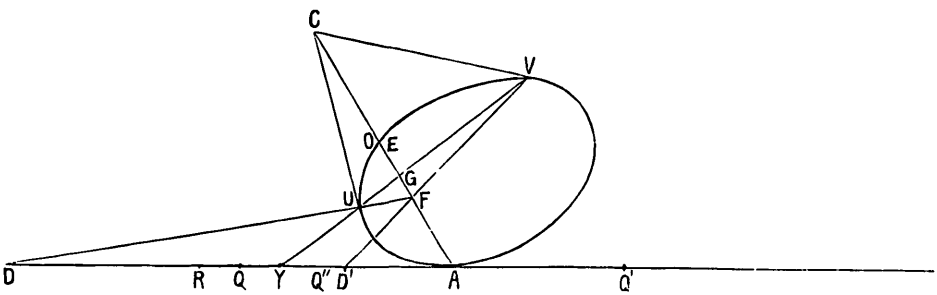


respectively, by means of corresponding points Q, Q', Q'' on the tangent at A. Taking $ACD = 60°$, Q and Q' are such that $AQ \,.\, AQ' = AD^2$; and Q'', on the opposite side of A from Q and Q', is such that $Q''A = AR$, where R is the middle point of QQ'. The lines P_1CP_3, $P_1'CP_3'$, $P_1''CP_3''$ are respectively the internal bisectors of the angles ACQ, ACQ', ACQ''.

It remains to express in terms suitable for projection ways of finding the two others of Q, Q', Q'' when one of the three is given. First we must specify D in projective language. In imaginary terms we may say that CD forms with the perpendicular through C to CA and the lines from C to the circular points an equi-anharmonic pencil. But a real construction for D of the kind we require is obtained by taking $OE = CO$ on CO produced, finding F such that $AF^2 = AC \,.\, AE$, *i.e.*, finding a double point of the involution of which A is centre and C, E a pair, and drawing FD (or FD') parallel to an asymptote. The determination of Q', and then of Q'', when Q is given, is now immediate, the constructions being in projective language that of fourth harmonic points. When it is Q'' that we have given, its reflection R is found as a harmonic conjugate, and then Q and Q' as harmonically conjugate both with regard to D, D' and with regard to R and the point at infinity on AR.

Projecting the figure back into the general one in which CP_1P_3 and C_1W are two lines of which only the first meets the conic of reference in real points, we may now state as follows.

CP_1P_3 and CW, $CP_1'P_3'$ and CW', $CP_1''P_3''$ and CW'' are pairs of an overlapping involution to which every quartic of the pencil $\kappa u+\lambda H$ contributes a pair, and of which CU and CV are one pair. This pair and any other determine it. Take COA the one which cuts the conic of the common pair of this involution and the, also overlapping, involution of pairs of conjugate lines with regard to the conic through C. Let Q on the tangent at A be where that tangent is met by the harmonic conjugate of CA with regard to CP_1, CW. Points Q', Q'' on the tangent are similarly associated with the CP_1', CW' and CP_1'', CW'' of u' and H.



Construct two points D, D' on the tangent at A, having no connection with the particular CP_1 or CP_1' or CP_1'' as follows. On COA, which meets UV in G, take E the harmonic conjugate of C with regard to O, G, and F either double point of the involution of which C, E and A, G are pairs. D and D' are where FU and FV meet the tangent.

Now, having Q on ADD', take Q' the harmonic conjugate of Q with regard to D and D'. This is the point associated with the $CP_1'P_3'$ of the apolar u' just as Q is with the CP_1P_3 of u.

Again, take R the harmonic conjugate of Y, where UV cuts ADD', with regard to Q and Q', and then take Q'' the harmonic conjugate of R with regard to A and Y. Q'' has the same association with the $CP_1''P_3''$ of the Hessian of u, and of u'.

All the above constructions can be performed linearly by aid of the drawn conic. So can that of Q, Q' from a given Q'', by finding R, the harmonic conjugate of Q'' with regard to A, Y, and then constructing the common pair of harmonic conjugates of D, D' and R, Y. So, finally, can that of CP_1P_3 and CW (or $CP_1'P_3'$ and CW' or $CP_1''P_3''$ and CW'') from CQ (or CQ' or CQ''), by finding the common pair of the involution of which CA, CQ are the double lines, and that of which CU, CV and CA, CY are pairs.