each of which is, in fact,

$$= a^2c''^2 + 14aca''c'' + a''^2c^2 - 12a'^2cc'' - 4a'c'(ac'' + a''c) - 12c'^2aa'' + 16a'^2c'^2$$
$$- 12a''c''b^2 - (ac'' + a''c + 4a'c')(4bb'' + 8b'^2) - 12acb''^2$$
$$+ 24(a'c'' + a''c')bb' + 24(ac' + a'c)b'b'' + 16(bb'' - b'^2)^2.$$

The actual calculation of the other invariant J would be somewhat longer.

On Clifford's Graphs. *By* W. Spottiswoode, P.R.S.

[*Read June* 12*th*, 1879.]

In a very original paper, "On an Application of the new Atomic Theory to the Graphical Representation of the Invariants and Covariants of Binary Quantics," published in the "American Journal of Pure and Applied Mathematics," Vol. I., p. 64, Professor Sylvester states that he had "long been with a feeling of affinity, if not identity of object, between the inquiry into compound [chemical] radicals, and the search for 'Grundformen,' or irreducible invariants;" and that he was "agreeably surprised to find, of a sudden, distinctly pictured on my mental retina, a chemico-graphical image, serving to embody and illustrate the relations of those derived algebraical forms to their primitives, and to each other, which would perfectly accomplish the object which I had in view." "The factors of any algebraical form may be regarded as in some sense the analogues of the rays of atomicity in the equivalent chemical atom; these rays being what Dr. Frankland, according to his nomenclature, would have to designate as *free bonds*. Such rays between two consecutive atoms in a molecule are conceived as blending in some manner, so as to represent some unknown kind of special relation existing between them; they may then, with propriety, be called bonds, or lines of connection. An invariant of a form, or system of algebraical forms, must thus represent a saturated system of atoms, in which the rays of all the atoms are connected into bonds. Thus, *e.g.*, O_2 (oxygen combined with itself) will represent a quadric invariant of a quadric. Its graph is o-o. Potash, a combination of potassium, oxygen, and hydrogen, having for its graph ∧, will represent the invariant to a system of one quadratic and two linear forms. And, in general, the Jacobian to any two quantics will be completely expressed by their two corresponding atoms connected by a pair of bonds."

At the close of the paper, Prof. Sylvester adds, "The subjoined matter is so exceedingly interesting, and throws such a flood of light on the chemico-algebraical theory," that he decides to publish it. The matter in question is a letter from Prof. Clifford, written at Gibraltar; and it contains the only general statement of his views on the subject ever published. The following is the passage which particularly refers to Graphs:—

"Another [paper] was to be about the very thing you speak of, which was communicated to the British Association at Bristol. There is no question of reclamation, because the whole thing is really no more than a translation into other language of your own theories, published ages ago in the 'Cambridge Mathematical Journal.' I have a strong impression that you will there find the analogy of covariants and invariants to compound radicals and saturated molecules.

"I consider forms which are linear in a certain number of sets of k variables each. To fix the ideas, suppose $k=2$, and that I have altogether 6 sets of two variables each, namely

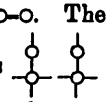
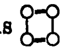
$$x_1x_2,\ y_1y_2,\ z_1z_2,\ u_1u_2,\ v_1v_2,\ w_1w_2.$$

Suppose the forms are

$$(xyzu),\ (yzvw),\ (xv),\ (uw)\ ;$$

viz., $(xyzu)$ means an expression separately linear and homogeneous in the x, the y, the z, and the u, and so on for the rest. I observe that in these four forms each set of variables occurs twice. This being so, there is one invariant of the four forms, which is invariant in regard to *independent* transformations of the six sets of variables. This you knew thirty years ago. All I add is: to obtain this invariant, regard the variables as alternate numbers, and simply multiply all the forms together. By alternate numbers, I mean those whose multiplication is polar ($xy=-yx$), and whose squares are zero. The product of the forms will then be equal to the invariant in question, multiplied by the product of all the variables. The quartic forms may be represented by the symbol , the quadrics by -o-. Thus the invariant

$$(xyzu)\,(yzvw)\,(xv)\,(uw)$$

will be represented by the figure ; whereas $(xyzu)\,(yzvw)\,(xu)\,(vw)$ is this form o-o-o-o. The former is clearly the product of the two quartic covariants  got by cutting it across the dotted lines; while the latter is the product of the quadric covariants o-o-, -o-o. A *bond* between two forms means a set of variables common to them. Of course we may regard two or more forms as identical, and so form invariants of a single form; thus  is the discriminant of a cubic.... Of course, the main thing is to pass from this system of separate variables to that in which the same variables occur to higher orders in the same form, or back again—what you call unravelment."

Besides this passage, the only remains of Prof. Clifford's writings on this subject are contained in a few fragmentary jottings, made at irregular intervals during the last year or two of his life. He often spoke about his progress in the subject; but, knowing how little strength he had to spare, and hoping that he might have written some fuller and more connected account of it, his friends naturally abstained from asking particulars on a variety of points, which would have now been invaluable. The importance which he himself attached to the method, the amount of attention which he gave to it, and the power which it would manifestly give to any one capable of wielding the master's weapon, will perhaps be sufficient apology for offering to mathematicians the very incomplete notes here collected.

The subject of Prof. Clifford's investigations was that of binary quantics, so that the number of variables in each of his sets, viz., the number of x's, y's, &c., is two; say, x_1, x_2; y_1, y_2, &c. A linear form will then, when written in full, be represented thus:

$$a_1x_1 + a_2x_2;$$

a quadric form thus:

$$a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2;$$

and so on. Between any two of the sets, *e.g.*, between the x's and the y's, we may form the determinant $x_1y_2 - x_2y_1$, an expression which will be of frequent occurrence.

To find the product of a form by the determinant of some of its variables. If the form be linear, we have

$$(a_1x_1 + a_2x_2)(x_1y_2 - x_2y_1) = (x)\,|x, y|, \text{ suppose};$$

and effecting the multiplication, while bearing in mind that the variables are to combine as alternate numbers, the product will be found to be

$$= -x_1x_2\,(a_1y_1 + a_2y_2),$$

and if we put $x_1x_2 = y_1y_2 = \ldots 1$, the result of multiplying the form by the determinant will be the same as making x identical with $-y$.

If the form be quadric, we have

$$(xy)\,|x, y| = -(a_{12} - a_{21}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

i.e., the effect of the multiplication will be, just as in the case of the linear form, the same as making x identical with $-y$. Similarly, for a cubic $(xyz) = a_{111}x_1y_1z_1 + \ldots$, we should find

$$(xyz)\,|\,x,\,y\,| = (a_{211} - a_{121})\,z_1 + (a_{212} - a_{122})\,z_2 \quad \ldots\ldots\ldots\ldots(3);$$

and the same is obviously true with forms of any order.

It is further to be observed that the effect of multiplication of a symmetric form by the determinant of any two sets of its variables will be to reduce the expression to zero. Thus, if the quadric be symmetrical, $a_{12} = a_{21}$, and the expression (2) vanishes. If the cubic be symmetrical, $a_{211} = a_{121}$, $a_{212} = a_{122}$, and the product (3) will then vanish. And, generally, if $((x, y, z, \ldots))$ represent the symmetrical form of which $(x, y, z, \ldots)$ is the general form, the same form written as a quadric $((x, y))$, in which the coefficients involve the other variables, will be symmetrical; and consequently

$$((x, y, z, \ldots))\,|\,x,\,y\,| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Lastly, it will be convenient to remark that the square of any form of an odd order will vanish; for any such form may be written as a linear form in respect of one of its sets of variables, thus

$$(x, y, \ldots v, w) = (x, y, \ldots v)_1 w_1 + (x, y, \ldots v)_2 w_2,$$

in which $(x, y, \ldots v)_1$, $(x, y, \ldots v)_2$ are forms of an even order. Hence, by known properties of alternate numbers,

$$(x, y, \ldots v, w)^2 = (x, y, \ldots v)_1 (x, y, \ldots v)_2 (w_1w_2 + w_2w_1) = 0 \ldots(5).$$

We now come to the forms themselves. In what follows, the main object has been to collect together the fragments which remain of Prof. Clifford's "Graphs"; and this seems to be all that can be done for the present purpose. But, with a view to a better understanding of the very novel method, I have ventured to prefix to each section an algebraical statement of the question.

Linear Forms.

Let $(x) = a_1x_1 + a_2x_2$; and, when there are two such forms with different coefficients, let

$$(x)_1 = a_{11}x_1 + a_{21}x_2,$$
$$(x)_2 = a_{12}x_1 + a_{22}x_2.$$

Then the cases which we have to consider are

$$(x)^2,\ (x)(y),\ (x)_1(x)_2.$$

Of these, the first $(x)^2 = 0$, in virtue of the general remark made above. The second

$$(x)(y) = a_1^2\,x_1y_1 + a_1a_2\,(x_1y_2 + x_2y_1) + a_2^2\,x_2y_2,$$

in which the coefficients a_1, a_2 may represent either ordinary quantities, or forms of alternate numbers of even orders. If, however, the coefficients represent forms of odd orders, then

$$(x)(y) = -a_1 a_2 (x_1 y_2 + x_2 y_1).$$

Again, $\qquad (x)_1 (x)_2 = a_{11} a_{22} - a_{21} a_{12} = D$, suppose,

or it will $= -D$, if the coefficients represent forms of odd orders.

Lastly, as shown in the preceding section,

$$(x) \mid x, y \mid = -(y).$$

In graphical language, (x) will be represented by the symbol ○–; and the square of it by ○–○; so that we shall have the relation

○–○ $= 0.$

And if $(x)_1 =$ ①–, and $(x)_2 =$ ②–, then

①–② $= D,$

where D represents the determinant of the two linear forms.

Again, if the determinant $\mid x, y \mid$, be represented by $\substack{x \\ y}$), or by) where no ambiguity can arise, then we have the relation

○–x) $=$ –○–y.

Quadric Forms.

Algebraically, the form will be represented by (x, y); and the combinations which we have to consider are

$$(x, y) \mid x, y \mid,\ (x, y) \mid y, z \mid,\ (x, y)^2,\ (x, y)(x, z).$$

In virtue of the general remark I made above, in order to evaluate the first of these, we have only to replace y by $-x$; the result will then easily be seen to be

$$(x, y) \mid x, y \mid = a_{21} - a_{12}, = c \text{ suppose.}$$

Again, in virtue of the general theorem II.,

$$(x, y) \mid y, z \mid = -(x, z).$$

Also $\qquad (x, y) = (x)_1 y_1 + (x)_2 y_2$,

in which $(x)_1$, $(x)_2$, are linear functions of x; hence

$$(x, y)^2 = -2\,(x)_1 (x)_2 = -2D.$$

Again, $\qquad (x, y)(x, z) = \{(x)_1 y_1 + (x)_2 y_2\} \{(x)_1 z_1 + (x)_2 z_2\}$

$$= -(x)_1 (x)_2 \mid y, z \mid$$

$$= -D \mid y, z \mid.$$

Proceeding further, we might form combinations, such as

$$(x, y)\,(x, z)\,(y, z) = -D\,|\,y, z\,|\,(y, z)$$
$$= -CD.$$

Or, again, $(x, y)\,(x, z)\,(u, y)\,(u, z) = (x, y)(x, z)\,.\,(u, y)\,(u, z)$
$$= 2D^2.$$

From which it appears that no new forms are to be obtained.

To these may be added, if

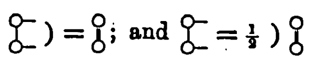
$$(x, y)_1 = a_{111}x_1y_1 + a_{121}x_1y_2 + a_{211}x_2y_1 + a_{221}x_2y_2,$$
$$(x, y)_2 = a_{112}x_1y_1 + a_{122}x_1y_2 + a_{212}x_2y_1 + a_{222}x_2y_2\,;$$

then $(x, y)_1\,(x, y)_2 = -a_{111}a_{222} + a_{121}a_{212} + a_{211}a_{122} - a_{221}a_{112}$
$$= -D_{12},\ \text{suppose}.$$

Graphically, the form itself will be represented by –o– ; –o–), or oO, will then represent the invariant $a_{21} - a_{12}$, which vanishes when the form is symmetrical. o–o will then represent the discriminant, say

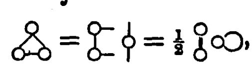
o–o = – 2D.

Again, –o–o– will be a quadri-covariant. But, multiplying this by), and remembering that the square of the determinant) is equal to 2, it follows that



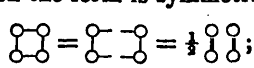
From the last equation it appears that this invariant vanishes when the variables are made identical.

Proceeding further, we may add



which also vanishes when the form is symmetrical.

Again,



and so on. Hence these and all other ulterior derived forms are only products and powers of the form itself and its discriminant.

Cubic Forms.

Algebraically, the form will be represented by (x, y, z). In this we may change one, two, or three letters; hence we have the following four forms :—

$$(x, y, z),$$
$$(x, y, w),$$
$$(u, v, z),$$
$$(u, v, w).$$

The form being of an odd order, its square will vanish, viz.,

$$(x, y, z)^2 = 0.$$

The product of the form by the determinant of two sets of its variables gives a linear covariant, as mentioned above, viz.,

$$(x, y, z)\,|x, y| = (a_{311} - a_{131})\,z_1 + (a_{312} - a_{132})\,z_2,$$

which vanishes when the form is symmetrical.

Of binary products we have three forms, viz.,

$$\begin{aligned}(x, y, z)(x, y, w) &= (u, v, z)(u, v, w),\\ (x, y, z)(u, v, z) &= (x, y, w)(u, v, z),\\ (x, y, z)(u, v, w) &= (x, y, w)(u, v, z).\end{aligned}$$

Of these, we may write the first thus,

$$\begin{aligned}(x, y, z) &= (x, y)_1 z_1 + (x, y)_2 z_2,\\ (x, y, u) &= (x, y)_1 u_1 + (x, y)_2 u_2.\end{aligned}$$

Hence

$$\begin{aligned}(x, y, z)(x, y, u) &= (x, y)_1^2 z_1 u_1 + (x, y)_1 (x, y)_2 (z_1 u_2 + z_2 u_1) + (x, y)_2^2 z_2 u_2\\ &= -\{2D_{11} z_1 u_1 + D_{12}(z_1 u_2 + z_2 u_1) + 2D_{22} z_2 u_2\}\\ &= -2H.\end{aligned}$$

Again, $(x, y, z)(x, u, v) = \{(y, z)_1 x_1 + (y, z)_2 x_2\}\{(u, v)_1 x_1 + (u, v)_2 x_2\}$

$$= (y, z)_1 (u, v)_2 - (y, z)_2 (u, v)_1.$$

But

$$\begin{aligned}-2\,(y, z)_1 &= (y, z)_1 \,|y, z|^2 = (a_{131} - a_{113})\,|y, z|,\\ -2\,(y, z)_2 &= (y, z)_2 \,|y, z|^2 = (a_{231} - a_{213})\,|y, z|.\end{aligned}$$

Hence

$$\begin{aligned}4\,(x, y, z)(x, u, v) &= (a_{131} - a_{113})(a_{231} - a_{213})\{\,|y, z|\,|u, v| - |y, z|\,|u, v|\,\}\\ &= 0.\end{aligned}$$

Of ternary products we have four forms, viz.,

$$\begin{array}{cccc} \cdot & (x, y, w) & (u, v, z) & (u, v, w),\\ (x, y, z) & \cdot & (u, v, z) & (u, v, w),\\ (x, y, z) & (x, y, w) & \cdot & (u, v, w),\\ (x, y, z) & (x, y, w) & (u, v, z) & \cdot \end{array}$$

These are all really the same, as each involves three of the variables twice, and three of them once. They therefore represent the cubi-covariant.

The cubi-covariant is symmetrical, for it is reduced to zero by the factor, *e.g.*, $|u, w|$. In fact, taking the last form

$$(x, y, z)(x, y, w)(u, v, z)\,|u, w| = (x, y, z)\,.\,(x, y, w)(w, v, z),$$

which is equal to the product of the cubic itself into a quadri-covariant, which, as was already shown, vanishes.

Of quaternary products we have only one form, viz.,

$$(x, y, z)\ (x, y, w)\ (u, v, z)\ (u, v, w),$$

which involves each of the variables twice, and is consequently an invariant; viz., it is the discriminant.

The above written forms give an immediate proof of the following theorems. The Hessian of the cubi-covariant may be formed by multiplying together two of the four forms; but

$$\begin{aligned} &(x, y, w)\ (u, v, z)\ (u, v, w) \,.\, (x, y, z)\ (u, v, z)\ (u, v, w) \\ &= -\,(x, y, z)\ (x, y, w)\ (u, v, z)\ (u, v, w) \,.\, (u, v, z)\ (u, v, w). \end{aligned}$$

That is to say, the Hessian of the cubi-covariant is equal to the discriminant of the cubic multiplied by the Hessian of the cubic. Again, to form the discriminant of the cubi-covariant, we must multiply all the four forms together; but this product

$$= \{(x, y, z)\ (x, y, w)\ (u, v, z)\ (u, v, w)\}^3.$$

That is to say, the discriminant of the cubi-covariant is equal to the cube of the discriminant of the cubic. And again, to form the cubi-covariant of the cubi-covariant, we must multiply together three of the four forms; but, taking the last three, this product

$$= (x, y, z)\ \{(x, y, z)\ (x, y, w)\ (u, v, z)\ (u, v, w)\}^2.$$

That is to say, the cubi-covariant of the cubi-covariant is equal to the product of the cubic itself into the square of the discriminant of the cubic.

We may consider also the case of

$$(y, z, u)\ (z, x, v)\ (x, y, w).$$

But $(y, z, u)\,|y, z| = (a_{211}-a_{121})\,u_1 + (a_{212}-a_{122})\,u_2 = (u)$, suppose; with similar transformations of the other factors. Hence, operating a second time with $|y, z|$, $|z, x|$, $|x, y|$, on the three factors respectively, we obtain

$$-8\,(y, z, u)\ (z, x, v)\ (x, y, w) = (u)\ (v)\ (w)\ |y, z|\,|z, x|\,|x, y|.$$

But the product of the last three factors $= -2$;* hence

$$4\,(y, z, u)\ (z, x, v)\ (x, y, w) = (u)\ (v)\ (w).$$

And consequently the covariant vanishes when the original cubic is symmetrical. The covariant itself is also obviously symmetrical.

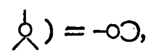
* See Spottiswoode on "Determinants of Alternate Numbers," Proceedings of the London Mathematical Society, Vol. VII., Nos. 94 and 95.

Again, the form

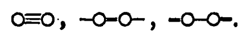
$$4\,(y, z, u)\,(z, x, v)\,(x, y, v) = (u)\,(v)^2 = 0.$$

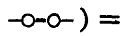
Hence this covariant vanishes.

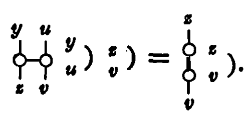
Graphically, the cubic form will be represented by ; and then we shall have



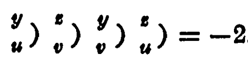
representing a linear covariant which vanishes when the form is symmetrical. The quadric combinations will be represented by

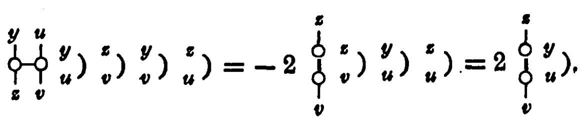


Of these the first vanishes. The second represents the Hessian, or $= -2H$. And since  $= 0$, it follows that the Hessian is symmetrical. As to the third, writing it explicitly, we have



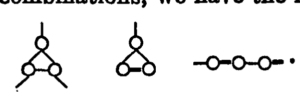
But



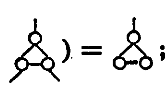
Hence 

Hence, when the variables are made identical, this vanishes.

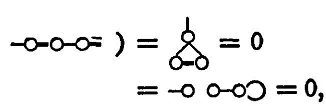
Passing to ternary combinations, we have the forms



The first of these is obviously symmetrical; and it is therefore reduced to zero by). But

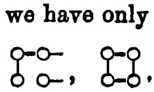


and consequently the second form vanishes. Moreover, since

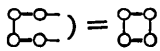
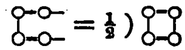


it follows that this form is symmetrical. It is the cubi-covariant Φ.

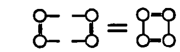
For combinations of four we have only the two forms



As to the first of these, since

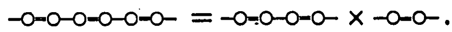
 and consequently 

it follows that there is no quarto-quadri-covariant. And as to the second, since

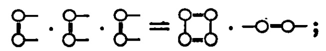


it follows that the discriminant of the Hessian is the discriminant of the cubic.

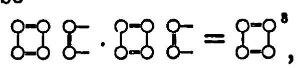
If we take the cubi-covariant as a form, its Hessian will be



But

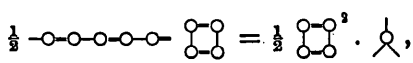


in other words, as shown algebraically, the Hessian of the cubi-covariant is the Hessian of the cubic multiplied by the discriminant of the cubic. Its discriminant will be



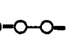
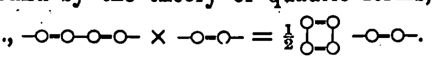
as proved algebraically.

Lastly, the cubi-covariant of the cubi-covariant will be



as before.

The only irreducible form of the third degree being the cubi-covariant, the forms of the fourth degree are its alliances with the cubic; of these the only irreducible one is an invariant, and there are therefore no more irreducible forms.

If we wish to find the Hessian of the compound form $\kappa f+\lambda\Phi$, we shall have to join κ  $+\lambda$  to itself by two bonds. And bearing in mind that $f\Phi = -\Phi f$, we have, in fact, only to find the Hessian of Φ. This is found by the theory of quadric forms, applied to Δ, to be $\frac{1}{2}H\Delta$; viz., . Therefore the Hessian in question, say $H_{\kappa,\lambda} = \kappa^2 H+\frac{1}{2}\lambda^2\nabla H = H(\kappa^2+\frac{1}{2}\lambda^2\nabla) = H\Theta$, suppose. To find $\Phi_{\kappa,\lambda}$, we must join $\kappa f+\lambda\Phi$ to the above by one bond. The result is

$$(\kappa\Phi-\tfrac{1}{2}\lambda\nabla f) = \tfrac{1}{2}\Theta\,(\Phi\vartheta_\kappa - f\vartheta_\lambda)\,\Theta.$$

Quartic Forms.

The form itself will be represented by (x, y, z, t). Then

$$(x, y, z, t)^2 = \text{the quadric invariant} = I.$$

Again, $(x, y, z, t)\,(x, y, z, s) = (x, y, z)'\,(x, y, z)''\,|\,t, s\,|$;

and consequently this covariant vanishes when the variables are made identical. Next,

$$(x, y, z, t)\,(x, y, w, s) = (z, t, w, s)_1 = \text{Hessian} = H,$$
$$(x, y, z, t)\,(x, v, w, s) = x_1 x_2 \mid (y, z, t)_1\,(v, w, s) \mid;$$

and consequently this covariant also vanishes when the variables are made identical.

Next, for cubic covariants,

$$(x, y, z, t)\,(x, y, w, s)\,(z, t, w, s) = \text{cubic invariant} = J,$$
$$(x, y, z, t)\,(x, y, w, s)\,(z, v, w, s) = (z, t, w, s)_1\,(z, v, w, s) = (t)_1\,(v)$$
$$= (x, y, z, v)_1\,(x, y, z, t) = (v)_1\,(t).$$

But if $$(w) = b_1 v_1 + b_2 v_2,\quad (v)_1 = c_1 v_1 + c_2 v_2,$$

the condition $(t)_1\,(v) = (v)_1\,(t)$ will give

$$t_1 v_2 - v_1 t_2 = 0,\ \text{ or }\ t_1 v_2 + t_2 v_1 = 0;$$

and consequently $(t)_1\,(v)$, or $(v)_1\,(t)$, will contain $|t, v|$ as a factor, and the covariant will consequently vanish when the variables are made identical.

Again,

$$(x,y,z,t)\,(x,y,w,s)\,(u,v,w,s) = (z,t,w,s)_1\,(u,v,w,s) = (z,t)_1'\,(u,v)'$$
$$= (u, v, x, y)_1\,(z, t, x, y) = (u, v)_1'\,(z, t)$$

contains factors $|z, u|$, $|t, v|$; or $|z, v|$, $|t, u|$; and consequently vanishes when the variables are made identical.

Next, $$(x, y, z, t)\,(x, y, w, s)\,(u, v, k, s);$$

which does not admit of any such double form as the two preceding products. This last is the sextic covariant, Φ.

Binary Forms of Alternate Variables. By the late Prof. W. K. Clifford, F.R.S.

[*Read June* 12*th*, 1879.]

Introduction.

1. Alternate numbers are such that $\alpha\beta = -\beta\alpha$, $\alpha^2 = 0$, $\beta^2 = 0$. It is easily shown that linear functions of them possess the same properties; *i.e.*, if $\bar{\alpha} = a_1\alpha_1 + a_2\alpha_2 + \ldots$, $\bar{\beta} = b_1\beta_1 + b_2\beta_2 + \ldots$, where the a, b are scalars and the α, β alternate numbers, then we shall have $\bar{\alpha}\,\bar{\beta} = -\bar{\beta}\,\bar{\alpha}$, and $\bar{\alpha}^2 = 0 = \bar{\beta}^2$. If M, N are homogeneous functions of alternate numbers of degrees m, n respectively, the number of interchanges of