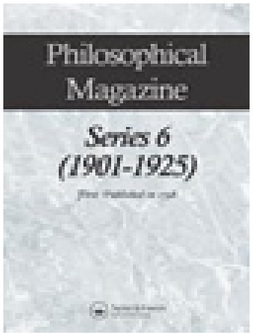

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XLIII. On restricted lines and planes of closest fit to systems of points in any number of dimensions

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XLIII. *On Restricted Lines and Planes of Closest Fit to Systems of Points in any Number of Dimensions.* By E. C. Snow, *M.A.**

Statement of the Problem.

1. THE theory of the lines and planes of closest fit to systems of points when no restriction is placed upon those lines and planes has been developed by Prof. Pearson in various papers† and is of frequent application. The connexion of these lines and planes with the formulæ of the theory of multiple correlation is indicated in those papers. If the criterion of "closest fit" is that the sum of the squares of the deviations from the line or plane *measured in the direction of the "dependent" variable* is to be a minimum, the equation of the line or plane is identical with the corresponding multiple correlation formula. If the sum of the squares of the deviations *measured at right angles to the line or plane* is a minimum (and this, from the purely geometrical point of view, is the more satisfactory criterion), the result is not of such a simple form, but the determinant from which the mean square residual is obtained is similar to the multiple correlation determinant.

While working on certain vital statistics, it was desired to obtain a formula connecting the "dependent" variable with the "independent" variables when the values of all the variables were known at the beginning and end of a certain range, and the correlation between "dependent" and "independent" variables at all intermediate points was a maximum. Thus, if x_0 denote the "dependent" variable, and $x_1, x_2, \ldots x_n$ the "independent" ones, we require to make the correlation of x_0 with $x_1, x_2, \ldots x_n$ a maximum, with the condition that when $x_1, x_2, \ldots x_n$ take up the values $p_{11}, p_{21}, \ldots p_{n1}$, $p_{12}, p_{22} \ldots p_{n2}$ respectively, x_0 is to take the values p_{01} and p_{02}.

A similar problem occurs in certain branches of Physics, especially in connexion with solutions and alloys. A property—*e. g.*, the freezing-point—of a pure substance may be definitely known, and it is required to investigate the behaviour of that property as certain amounts of some other substance or substances are added. Fixed conditions will be imposed upon the law which is to be investigated by the known properties of the pure substance. The law, then,

* Communicated by Prof. Karl Pearson, F.R.S.

† See Phil. Mag. Nov. 1901, pp. 559 *et seqq.*; Phil. Trans. vol. clxxxvii. A, pp. 301 *et seqq.*

has to give the best fit to the observations made of the property as definite quantities of the other substances are added. Two examples of such cases are given from the figures of certain alloys (§7 below).

The idea is capable of generalization, and the theory for the general case will be investigated. Looked at from the point of view of correlation, we shall require to assume a linear law connecting x_0 with $x_1, x_2, \ldots x_n$, and shall make the sum of the squares of the deviations of the actual observations from this linear law *measured in the direction of* x_0 a minimum, making use of the exact conditions which are imposed on the law. This will be first investigated. But a better geometrical fit to the observations will be obtained by measuring the deviations *perpendicular* to the "plane" given by the linear law, and this will be worked out subsequently.

Analytical Investigation.

First Method.

2. Let there be n "independent" variables and $(k+1)$ conditions connecting them with the "dependent" variable. $(k+1)$ is necessarily less than n. Measuring from one of these fixed conditions, we can assume our law is

$$x_0 = a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n, \quad \ldots\ldots \quad (1)$$

with the k conditions

$$\left.\begin{aligned} p_{01} &= a_1p_{11} + a_2p_{21} + \ldots\ldots + a_np_{n1}, \\ p_{02} &= a_1p_{12} + a_2p_{22} + \ldots\ldots + a_np_{n2}, \\ &\ldots\ldots\ldots\ldots \\ p_{0k} &= a_1p_{1k} + a_2p_{2k} + \ldots\ldots + a_np_{nk}. \end{aligned}\right\} \quad \ldots \quad (2)$$

Then we want to make

$$V = S(x_0 - a_1x_1 - a_2x_2 - \ldots\ldots - a_nx_n)^2,$$

the sum of the square of the deviations in the direction of x_0, a minimum, subject to the above conditions.

Hence we must have

$$0 = S(x_0 - a_1x_1 - a_2x_2 - \ldots\ldots - a_nx_n)(x_1.da_1 + x_2.da_2 + \ldots\ldots + x_n.da_n), \quad (3)$$

with the conditions

$$0 = p_{1s}.da_1 + p_{2s}.da_2 + \ldots\ldots + p_{ns}.da_n. \quad (s = 1, 2, \ldots\ldots k.) \quad (4)$$

Multiply equations (4) by λ_s $(s=1, 2, \ldots k)$, and add to (3). Then, by the ordinary theory of maximum and minimum values, we know that the coefficient of da_t $(t=1, 2, \ldots n)$ in the equation so obtained must vanish. This gives the n equations

$$\begin{aligned}\lambda_1 p_{t1}+\lambda_2 p_{t2}+ \ldots . +\lambda_k p_{tk} &= \mathrm{S}x_t(x_0-a_1x_1-a_2x_2- \ldots . -a_nx_n)\\ &= \mathrm{R}_{0t}-a_1\mathrm{R}_{1t}-a_2\mathrm{R}_{2t}- \ldots . -a_n\mathrm{R}_{nt}, \quad . \quad . \quad (5)\end{aligned}$$

if $\mathrm{R}_{uv}=\mathrm{S}(x_ux_v)=\mathrm{R}_{vu}$ for all positive integral values of u and v.

(5) gives n equations connecting the $(n+k)$ unknowns, $a_1, a_2, \ldots . a_n$, $\lambda_1, \lambda_2, \ldots . \lambda_k$. (2) gives k other equations between the a's, in which, however, the λ's are absent. Thus we have the following set of equations from which to determine the a's and the λ's :—

$$\left.\begin{aligned}a_1\mathrm{R}_{1t}+a_2\mathrm{R}_{2t}+ \ldots . +a_n\mathrm{R}_{nt}+\lambda_1 p_{t1}+\lambda_2 p_{t2}+ \ldots . +\lambda_k p_{tk} &= \mathrm{R}_{0t}.\\ (t=1, 2, \ldots . n.)&\\ a_1 p_{1s}+a_2 p_{2s}+ \ldots . +a_n p_{ns} &= p_{0s}.\\ (s=1, 2, \ldots . k.)&\end{aligned}\right\} (6)$$

Let Δ denote

$$\begin{vmatrix} \mathrm{R}_{00} & \mathrm{R}_{10} & \ldots . & \mathrm{R}_{n0} & p_{01} & p_{02} & \ldots . & p_{0k}\\ \mathrm{R}_{01} & \mathrm{R}_{11} & \ldots . & \mathrm{R}_{n1} & p_{11} & p_{12} & \ldots . & p_{1k}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ \mathrm{R}_{0n} & \mathrm{R}_{1n} & \ldots . & \mathrm{R}_{nn} & p_{n1} & p_{n2} & \ldots . & p_{nk}\\ p_{01} & p_{11} & \ldots . & p_{n1} & 0 & 0 & \ldots . & 0\\ p_{02} & p_{12} & \ldots . & p_{n2} & 0 & 0 & \ldots . & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ p_{0k} & p_{1k} & \ldots . & p_{nk} & 0 & 0 & \ldots . & 0 \end{vmatrix},$$

a determinant of the order $(n+k+1)$.

Then the solutions of equations (6) are :

$$a_t = -\frac{\Delta_{t0}}{\Delta_{00}} \qquad (t=1, 2, \ldots . n) \quad . \quad . \quad . \quad (7)$$

and

$$\lambda_s = -\frac{\Delta_{(n+s),0}}{\Delta_{00}} \quad (s=1, 2, \ldots . k) \quad . \quad . \quad . \quad (8)$$

Δ_{uv} being the first minor of the constituent of the $(u+1)$th column and $(v+1)$th row, and being positive or negative according as $(u+v)$ is even or odd.

Thus the coefficients in the required formula can be found at once by the evaluation of a number of determinants of order $(n+k)$*.

Particular Cases.

3. The simplification of the above results in a few particular cases will be useful.

(A) $k=0$.—In this case we have a plane passing through a single fixed point and closest fitting to a system of points. Here all the λ's disappear, and the equation of the plane becomes

$$x_0 = a_1x_1 + a_2x_2 + \ldots + a_nx_n,$$

where

$$a_t = -\frac{\Delta_{0t}}{\Delta_{00}},$$

Δ being

$$\begin{vmatrix} R_{00} & R_{10} & \ldots & R_{s0} & \ldots & R_{n0} \\ R_{01} & R_{11} & \ldots & R_{s1} & \ldots & R_{n1} \\ R_{02} & & & & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ R_{0n} & \ldots & \ldots & \ldots & \ldots & R_{nn} \end{vmatrix},$$

R_{uv} being equal to R_{vu}, and is the sum of the products $x_u \cdot x_v$ taken throughout the system of points.

If $\bar{x}_s$ and σ_s be the mean and standard deviation of x_s, and r_{uv} the correlation between the coordinates x_u and x_v, we have

$$R_{uv} = S(x_ux_v) = N\sigma_u\sigma_v r_{uv} + N\bar{x}_u\bar{x}_v \quad \text{if } u \neq v,$$

and $$R_{uv} = N\sigma_u^2 + N\bar{x}_u^2 \quad \text{if } u = v.$$

The analogy between Δ and the determinant used in the

* It is not difficult to show that, by first finding the a's in terms of the λ's from the first n of equations (6), and substituting these values in the last k of the same set—thus giving k equations for the λ's,—the a's can be found in a form involving only determinants of order n, though the number of determinants it is necessary to evaluate is increased. If k is large, this increase is considerable. The general result in this form will not be given, but it is exemplified in a particular case below (§ 3).

theory of multiple correlation is now clear, for a_t can be written $-\frac{\Delta'_{t0}}{\Delta'_{00}}$, where

$$\Delta' \equiv \begin{vmatrix} 1+\frac{\bar{x}_0^2}{\sigma_0^2}, & r_{01}+\frac{\bar{x}_0\bar{x}_1}{\sigma_0\sigma_1}, & \dots\dots\dots & r_{0n}+\frac{\bar{x}_0\bar{x}_n}{\sigma_0\sigma_n}, \\ r_{01}+\frac{\bar{x}_0\bar{x}_1}{\sigma_0\sigma_1}, & 1+\frac{\bar{x}_1^2}{\sigma_1^2}, & & \vdots \\ r_{02}+\frac{\bar{x}_0\bar{x}_2}{\sigma_0\sigma_2}, & \dots\dots\dots & 1+\frac{\bar{x}_2^2}{\sigma_2^2}, & \vdots \\ \vdots & & & \vdots \\ r_{0n}+\frac{\bar{x}_0\bar{x}_n}{\sigma_0\sigma_n}, & \dots\dots\dots & \dots\dots\dots & 1+\frac{\bar{x}_n^2}{\sigma_n^2}. \end{vmatrix}$$

Thus this determinant can be derived from the multiple correlation determinant by increasing r_{st} in the latter by $V_s . V_t$, and by increasing the constituents of the leading term by the corresponding V_s^2, where V_s and V_t are the coefficients of variation of the coordinates x_s and x_t. If the fixed point is at the mean of the system of given points, Δ' becomes at once the ordinary multiple correlation determinant.

Putting $n=1$, we derive the two-dimensional case of a line passing through the origin and giving closest fit (*measuring in the direction of y*) to a system of points. The equation of the line is easily seen to be

$$y = \frac{R_{10}}{R_{11}}x = \frac{S(x.y)}{S(x^2)}x. \quad \dots\dots \quad (\alpha)$$

Putting $n=2$, we reach the three-dimensional case of a plane passing through the origin and giving closest fit (*measured in the direction of z*) to a system of points. Its equation is

$$Z = \frac{R_{12}R_{02}-R_{01}R_{22}}{R_{11}R_{22}-R_{12}^2}.x + \frac{R_{12}R_{01}-R_{02}R_{11}}{R_{11}R_{22}-R_{12}^2}y$$

$$= \frac{S(xy).S(yz)-S(xz).S(y^2)}{S(x^2).S(y^2)-\{S(xy)\}^2}x + \frac{S(xy).S(xz)-S(yz).S(x^2)}{S(x^2).S(y^2)-\{S(xy)\}^2}y. \quad (\beta)$$

For values of $n>2$ it is more convenient to derive the coefficients direct from the determinant, and there is no need to write them in full.

(B) $k=1$.—Here we have the case of a plane—in n dimensions—passing through two fixed points and fitting most

closely a system of other points. We have in this case

$$\Delta = \begin{vmatrix} R_{00} & R_{10} & \dots & R_{n0} & p_0 \\ R_{01} & R_{11} & \dots & R_{n1} & p_1 \\ \vdots & \vdots & & \vdots & \vdots \\ R_{0n} & R_{1n} & \dots & R_{nn} & p_n \\ p_0 & p_1 & \dots & p_n & 0 \end{vmatrix},$$

$p_0, p_1, \dots p_n$ being the coordinates of one fixed point relative to the other, which is taken as origin.

The coefficients in the required equation can be found from the above in the form of determinants of order $(n+1)$.

But in this case the first n equations of (6) become

$$a_1 R_{1t} + \dots + a_n R_{nt} = R_{0t} - \lambda p_t \quad (t = 1, 2, \dots n).$$

Solving these for $a_1, a_2, \dots a_n$ in terms of λ, we have

$$a_t = -\frac{\delta_{0t}}{\delta_{00}},$$

where

$$\delta \equiv \begin{vmatrix} R_{00} & R_{01} & \dots & R_{0n} \\ R_{01} - \lambda p_1 & R_{11} & \dots & R_{1n} \\ \vdots & \vdots & & \vdots \\ R_{0n} - \lambda p_n & R_{1n} & \dots & R_{nn} \end{vmatrix}$$

$$= \delta' - \lambda \delta'',$$

where

$$\delta' \equiv \begin{vmatrix} R_{00} & R_{01} & \dots & R_{0n} \\ R_{01} & R_{11} & \dots & R_{1n} \\ \vdots & \vdots & & \vdots \\ R_{0n} & R_{1n} & \dots & R_{nn} \end{vmatrix}$$

and

$$\delta'' \equiv \begin{vmatrix} 1 & R_{01} & \dots & R_{0n} \\ p_1 & R_{11} & \dots & R_{1n} \\ p_2 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ p_n & R_{1n} & \dots & R_{nn} \end{vmatrix};$$

so that

$$a_t = -\frac{\delta'_{0t} - \lambda \delta''_{0t}}{\delta_{00}} \quad (t = 1, 2, \dots n).$$

When these values are substituted in the equation giving the fixed condition, viz.

$$p_0 = a_1 p_1 + \ldots\ldots + a_t p_t + \ldots\ldots + a_n p_n,$$

λ can be found, and from this the a's can be completely determined. This method will be used in an example below.

No other particular cases of this method need be worked out in detail. We see that it is always possible to obtain a plane in n-dimensional space to pass through any number (less than n) of fixed points and to be such that the sum of the squares of the deviations of any number of other points from the plane *measured in a fixed direction* is a minimum.

Second Method.

4. We have now to investigate the equation of the corresponding plane when the criterion for "closest fit" is that the sum of the squares of the deviations from the plane *measured at right angles to the plane* is the least possible. From a purely geometrical point of view this will give a closer plane to the system of points, but it will not give the regression of one variable on the others.

Let $l_1 \ldots . l_n$ be the generalized direction-cosines of the plane, and take one of the points through which the plane has to pass as origin.

Then the equation of the plane is

$$l_1 x_1 + \ldots\ldots + l_n x_n = 0, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the total number of variables being taken as n for convenience of notation. There will also be the conditions

$$l_1^2 + \ldots\ldots + l_n^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$\left.\begin{aligned} l_1 p_{11} + l_2 p_{21} + \ldots\ldots + l_n p_{n1} &= 0, \\ l_1 p_{12} + l_2 p_{22} + \ldots\ldots + l_n p_{n2} &= 0, \\ \ldots\ldots\ldots\ldots & \\ l_1 p_{1k} + l_2 p_{2k} + \ldots\ldots + l_n p_{nk} &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad (11)$$

$(k+1)$ being the total number of fixed conditions.

The criterion for closest fit is

$$V = S(l_1 x_1 + \ldots\ldots + l_n x_n)^2$$

to be minimum, subject to the above conditions.

Hence

$$0 = S(l_1 x_1 + \ldots\ldots + l_n x_n)(x_1 dl_1 + \ldots\ldots + x_n dl_n),$$

together with

$$\begin{aligned} l_1 dl_1 + \ldots\ldots + l_n dl_n &= 0, \\ p_{11} dl_1 + \ldots\ldots + p_{n1} dl_n &= 0, \\ \ldots\ldots\ldots & \\ p_{1k} dl_1 + \ldots\ldots + p_{nk} dl_n &= 0. \end{aligned}$$

The conditions for a minimum give

$$\left.\begin{aligned} X_1+\mu l_1+\mu_1 p_{11}+\mu_2 p_{12}+ \ldots\ldots +\mu_k p_{1k} &= 0,\\ \ldots\ldots\ldots\ldots\ldots & \\ X_n+\mu l_n+\mu_1 p_{n1}+\mu_2 p_{n2}+ \ldots\ldots +\mu_k p_{nk} &= 0,\end{aligned}\right\} \quad . \quad (12)$$

where $\mu, \mu_1, \ldots\ldots \mu_k$ are undetermined constants, and

$$\begin{aligned} X_s &= S(l_1x_1+ \ldots\ldots +l_nx_n)x_s\\ &= l_1R_{1s}+ \ldots\ldots +l_nR_{ns}, \end{aligned}$$

where, as before,

$$R_{uv} = S(x_u . x_v) = R_{v . u}.$$

Multiply equations (12) by $l_1, l_2, \ldots\ldots l_n$ respectively, and add. Remembering the conditions (10) and (11), we at once obtain

$$\mu+l_1X_1+l_2X_2+ \ldots\ldots +l_nX_n = 0,$$

where $l_1 \ldots\ldots l_n$ have the values which make V a minimum. But then $l_1X_1+l_2X_2+ \ldots\ldots +l_nX_n$ becomes V_m, the minimum value of V.

It follows that

$$\mu = -V_m.$$

Substituting this value of μ in (12), we shall have with (11) $(n+k)$ equations between $l_1 \ldots\ldots l_n, \mu_1 \ldots\ldots \mu_k$, and V_m. Hence we can eliminate the l's and the μ's and obtain an equation to determine V_m.

Since

$$X_s = l_1R_{1s}+ \ldots\ldots +l_nR_{ns},$$

this eliminant can be written in the determinantal form

$$D \equiv \begin{vmatrix} R_{11}-V_m & R_{21} & \ldots\ldots & R_{n1} & p_{11} & p_{12} & \ldots\ldots & p_{1k}\\ R_{12} & R_{22}-V_m & \ldots\ldots & R_{n2} & p_{21} & p_{22} & \ldots\ldots & p_{2k}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ R_{1n} & R_{2n} & \ldots\ldots & R_{nn}-V_m & p_{n1} & p_{n2} & \ldots\ldots & p_{nk}\\ p_{11} & p_{21} & \ldots\ldots & p_{n1} & 0 & 0 & \ldots\ldots & 0\\ p_{12} & p_{22} & \ldots\ldots & p_{n2} & 0 & 0 & \ldots\ldots & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ p_{1k} & p_{2k} & \ldots\ldots & p_{nk} & 0 & 0 & \ldots\ldots & 0 \end{vmatrix} = 0.$$

This is an equation of the $(n-k)$th degree in V_m. Its roots are necessarily positive (being the sum of a number of

squares of real quantities), and the smallest of these must be taken. When this value of V_m is substituted in (11) and (12), $(n+k)$ homogeneous equations in $l_1\, l_2 \ldots . l_n,\ \mu_1\, \mu_2 \ldots . \mu_k$ are obtained, $(n+k-1)$ of which, together with (10), suffice to determine all the l's and the μ's. As before, this can be done in determinant form, the order of the determinants involved being $(n+k-1)$.

Particular Cases.

5. Useful particular cases of the general formula are obtained by taking $k=0$ and $k=1$.

(A′) If $k=0$, we derive the case analogous to (A) above. The equation to determine V_m takes the well-known form

$$\begin{vmatrix} R_{11}-V_m & R_{21} & \ldots . & R_{n1} \\ R_{12} & R_{22}-V_m & \ldots . & R_{n2} \\ \vdots & \vdots & & \vdots \\ R_{1n} & R_{2n} & \ldots . & R_{nn}-V_m \end{vmatrix} = 0.$$

Putting $n=2$, we have the two-dimensional case, and V_m is the least root of the quadratic

$$\begin{vmatrix} R_{11}-V_m & R_{21} \\ R_{12} & R_{22}-V_m \end{vmatrix} = 0\ ;$$

so that

$$2V_m = (R_{11}+R_{22})-\{(R_{11}-R_{22})^2+4R_{12}^2\}^{1/2},$$

since $R_{12} = R_{21}$,

and

$$2(R_{11}-V_m) = (R_{11}-R_{22})+\{(R_{11}-R_{22})^2+4R_{12}^2\}^{1/2}.$$

Put

$$\frac{R_{11}-R_{22}}{\cos\theta} = \frac{2R_{12}}{\sin\theta} = \rho.$$

Then

$$2(R_{11}-V_m) = \rho(1+\cos\theta) = 2\rho\cos^2\frac{\theta}{2}$$

and

$$2R_{12} = 2\rho\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

The first of equations (12) now gives

$$l_1\cos\frac{\theta}{2}+l_2\sin\frac{\theta}{2}=0,$$

since the p's vanish when $k=0$.

The equation of the line, therefore, is

$$x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2} = 0,$$

where

$$\tan \theta = \frac{2R_{12}}{R_{11} - R_{22}}$$

$$= \frac{2S(xy)}{S(x^2) - S(y^2)},$$

and the notation is altered to agree with the usual form. This value of $\tan \theta$ gives rise to two values of $\frac{\theta}{2}$, each less than 180°. In a particular numerical example, however, it is not difficult to pick out the value required ; while it can be verified that the other value corresponds to the " worst-fitting " line.

In cases of $n > 2$ it is better to substitute the values of the R's direct in the determinant above, and to find V_m by the usual methods of approximating to the roots of an equation.

(B′) If $k=1$, the equation to determine V_m is

$$d = \begin{vmatrix} R_{11}-V_m & R_{21} & \dots & R_{n1}, & p_1 \\ R_{12} & (R_{22}-V_m) & \dots & R_{n2}, & p_2 \\ \vdots & \vdots & & \vdots & \vdots \\ R_{1n} & R_{2n} & \dots & (R_{nn}-V_m), & p_n \\ p_1 & p_2 & \dots & p_n, & 0 \end{vmatrix} = 0,$$

the origin and the point $(p_1 p_2 \dots p_n)$ being the fixed points.

In this case, equations (12) take the form

$$l_1(R_{11}-V_m) + l_2 R_{21} + \dots + l_n R_{n1} = -\lambda p_1,$$

$$\dots\dots\dots\dots\dots$$

$$l_1 R_{1n} + l_2 R_{2n} + \dots + l_n(R_{nn}-V_m) = -\lambda p_n.$$

Hence l_t is proportional to d_t, the first minor of the constituent in the tth column and bottom row of d, and V_m is given the value which is the least root of $d=0$. Using (10), the actual values of the l's can be found.

6. It will be seen from the foregoing analysis that the work involved in determining the " closest fitting " plane by

the second criterion is much greater, at any rate in all cases of $n > 2$, than that necessitated by the first criterion. In the second method, if $(n-k)$ is three or more, the smallest root of an equation of the third or higher degree has to be approximated to. This in itself is no light task, and is not necessary in the first method. The methods do not necessarily lead to results at all alike (see Example 3, below), and only the terms of the particular question in hand can decide which method is to be used. The second gives the best geometrical fit, considered in a direction perpendicular to the plane. The first gives the "regression" plane—*i. e.*, the most probable value of one variable in terms of the others. This is the most frequently needed in practical cases, as is exemplified in Examples 1 and 2.

7. Illustrations.

I. The second column in Table I.* gives the temperatures (Centigrade) at solidification of a series of alloys of iron and

Table I.

x. Percentage of Carbon present.	*y.* Temperature at end of solidification.	Temperature by 1st method to nearest degree.	Temperature by 2nd method to nearest degree.
·02	1470	1501	1501
·12	1470	1483	1483
·16	1465	1476	1476
·17	1450	1474	1474
·24	1448	1461	1461
·38	1416	1436	1436
·53	1404	1409	1409
·61	1394	1395	1394
·80	1351	1360	1359
·81	1351	1358	1357
1·31	1286	1268	1267
1·51	1244	1231	1230
1·85	1179	1171	1169
2·12	1110	1122	1120
2·21	1107	1105	1103

carbon. The percentage of carbon in the various alloys is given in the first column. The solidifying temperature of pure iron is 1505° C. Any curve, therefore, which attempts

* The table is taken from a paper on "The Range of Solidification and the Critical Ranges of Iron-Carbon Alloys," by H. C. H. Carpenter, M.A., and B. F. E. Keeling, B.A., in the Journal of the Iron and Steel Institute, No. 1, for 1904.

to express a relationship between the percentage of carbon present and the solidifying point of the alloy should pass through the fixed point (0 %, 1505° C.). Up to 2 % of carbon a straight line seems to be the most likely fit. The two methods will therefore be applied to get a line to pass through the point (0 %, 1505° C.) and to fit the series of observations. It will be seen from the figure that up to a percentage of carbon of 0·5 % the results are irregular, but from that point up to 2 % the irregularities are small and within the limits of experimental error.

Measuring x positively from zero and y negatively from 1505, we find

$$\begin{aligned} S(x^2) &= 19{\cdot}0176, \\ S(xy) &= 3438{\cdot}41, \\ S(y^2) &= 624986. \end{aligned}$$

By the first method the equation of the line (measured from 0 % and 1505° C.) is

$$y = \frac{S(x \cdot y)}{S(x^2)} \cdot x \quad \text{or} \quad y = 180{\cdot}801\,x.$$

The relationship between the temperature of solidification and percentage of carbon present is therefore

$$T = 1505 - 180{\cdot}801\,x.$$

If we apply the second method, we find

$$\tan\theta = -{\cdot}0110035,$$

whence

$$\frac{\theta}{2} = 89^\circ\ 41'\ 5''{\cdot}23$$

and

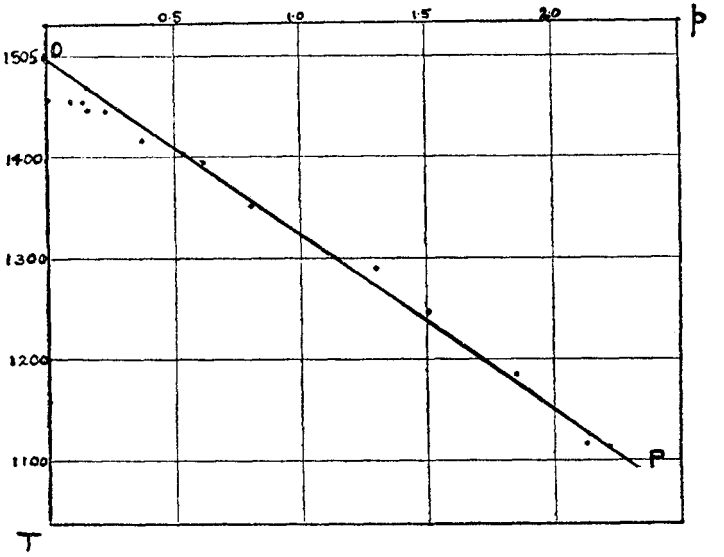
$$\tan\frac{\theta}{2} = 181{\cdot}80843,$$

which gives the relationship

$$T = 1505 - 181{\cdot}808\,x.$$

The actual temperatures obtained from the two formulæ are given in columns 3 and 4 of Table I. We see that, to the nearest degree, there is no difference in the results up to 0·5 % of carbon, and the difference beyond that point is small. The two lines cannot be distinguished on a diagram of the size shown. The line OP in the diagram represents,

therefore, the best fit to the system of points of a line through O by both methods.



II.* In this case we will take the figures of an alloy of three metals—copper, aluminium, and manganese. The percentage of manganese present varies from 0 to 10·24, and of aluminium from 0 to 7·40. It is not possible to tell from the mere figures if the distribution is approximately coplanar or not, but the material seemed good enough to work upon. The freezing-point of pure copper is 1084° C.; in this case, therefore, we require to find a relationship of the form

$$T - 1084 = p \,.\, x + q \,.\, y,$$

where T is the temperature at solidification of an alloy containing x % of aluminium and y % of manganese. Taking our origin at zero percentages of aluminium and manganese and 1084° C., we require to obtain a plane through the origin and fitting most closely the observations in the second, third, and fourth columns of Table II.

* The figures for this example were taken from Table 43 (p. 229) of the "Ninth Report to the Alloys Research Committee" to the Inst. of Mechanical Engineers, by Dr. W. Rosenhain and Mr. F. C. A. H. Lantsberry.

TABLE II.

Actual Freezing-Point. ° C.	Deviation of Temp. from 1084° C.	Percentage of Aluminium present.	Percentage of Manganese present.	Temp. by Formula (γ).	Deviation from Actual Temp.	Temp. by Formula (ϵ).	Deviation from Actual Temp.
1077	7	1·19	1·14	1067·8	− 9·2	1067·7	− 9·3
1045	39	1·37	2·75	1053·2	+ 8·2	1052·9	+ 7·9
1051	33	1·04	4·92	1036·6	−14·4	1036·6	−14·4
1023	61	1·42	5·38	1030·7	+ 7·7	1030·2	+ 7·2
1015	69	0·94	6·48	1024·0	+ 9·0	1023·4	+ 8·4
1007	77	1·43	7·72	1010·8	+ 3·8	1010·8	+ 3·8
1011	73	0·91	8·16	1012·1	+ 1·1	1009·2	− 1·8
985	99	1·91	10·24	987·2	+ 2·2	985·9	+ 0·9
1075	9	2·69	.....	1069·3	− 5·7	1069·2	− 5·8
1057	27	2·36	1·95	1054·6	− 2·4	1054·4	− 2·6
1045	39	2·31	3·82	1039·0	− 6·0	1038·6	− 6·4
1010	74	2·27	7·80	1006·5	− 3·5	1004·2	− 5·2
996	88	2·37	9·76	988·4	− 7·6	987·5	− 8·5
1059	25	3·26	0·97	1058 0	− 1·0	1057·8	− 1·2
1042	42	3·71	2·97	1038·6	− 3·4	1038·2	− 3·8
1043	41	3·93	2·99	1037·2	− 5·8	1036·8	− 6·2
1024	60	3 29	4·80	1025·4	+ 1·4	1025·1	+ 0·9
1022	62	3·57	5·64	1016·7	− 5·3	1016·1	− 5·9
1015	69	3·08	6·88	1008·9	− 6·1	1008·2	− 6·8
995	89	3·95	7·95	995·1	+ 0·1	994·3	− 0·7
1067	17	4·67	.....	1058·5	− 8·5	1058·4	− 8·6
1054	30	4·14	1·77	1046·4	− 7·6	1046·1	− 7·9
1031	53	4·62	3·26	1031·1	+ 0·1	1030·7	− 0·3
1036	48	4·48	3·86	1026·8	− 9·2	1027·4	− 9·6
987	97	4·51	7·74	993·8	+ 6·8	993·0	+ 6·0
975	109	4·12	9·60	980·2	+ 5·2	979·2	+ 4·2
1050	34	5·21	0·98	1047·3	− 2·7	1047·0	− 3·0
1035	49	5·66	2·58	1031·2	− 3·8	1030·8	− 4·2
1018	66	5·21	4·86	1014·4	− 3·6	1013·8	− 4·2
1001	83	5·86	6·00	1001·2	+ 0·2	1005·5	− 0·5
998	86	5·19	6·72	998·7	+ 0·7	998·0	0·0
986	98	5·62	8·50	982·7	− 3·3	980·4	− 5·6
976	108	5·99	9·50	970·8	− 5·2	969·8	− 6·2
1025	59	6·88	0·98	1038·1	+13·1	1037·8	+12·8
1030	54	6·54	1·98	1031·5	+ 1·5	1031·1	+ 1·1
1022	62	6·29	3·50	1020 0	− 2·0	1019·5	− 2·5
995	89	6·62	4 82	1007·0	+12·0	1006·4	+11·4
978	106	6·26	8·02	981·9	+ 3·9	981·0	+ 3·0
957	127	6·91	9·06	969·5	+12·5	968·5	+11·5
1042	42	7·40	.....	1043 6	+ 1·6	1043·4	+ 1·4

If z denote deviation of the temperature from 1084° C., we find

$$S(x^2) = 779{\cdot}1338, \qquad S(yz) = 15421{\cdot}230,$$
$$S(y^2) = 1336{\cdot}0155, \qquad S(zx) = 10617{\cdot}960,$$
$$S(z^2) = 190296{\cdot}0, \qquad S(xy) = 751{\cdot}017.$$

Using equation (β) above, we quickly reach

$$z = 5{\cdot}4605x + 8{\cdot}4732y,$$

and therefore

$$T = 1084 - 5{\cdot}4605x - 8{\cdot}4732y. \quad . \quad . \quad . \quad (\gamma)$$

The values of T obtained by this formula are given in column 5 of Table II. The differences between these values and the experimental results are shown in the next column. It will be seen from the figures that the fit is a good one except at the ends of the range. Had the last seven observations been omitted, *i. e.* had the amount of aluminium present in the alloy been less than 6 %, a linear law such as the above one would have agreed quite well with the observed results. As the authors of the original paper state that "the precise temperatures given in the table possess no very great significance," it seems quite reasonable to assume that the observations, up to 6 % of aluminium, follow a linear law.

The sum of the squares of the deviations from the observed temperatures in this case is 1641·0210, and the "root mean square" is 6·41 *. The sum of the squares of the deviations measured perpendicular to the plane can be obtained from the above figure by dividing by $\{(5{\cdot}186)^2 + (8{\cdot}392)^2 + 1\}$, *i. e.* 102·6120. It is found to be 15·9924.

When the second method is used, the equation in V_m is

$$\begin{vmatrix} 779{\cdot}134 - V_m & 751{\cdot}017 & 10617{\cdot}96 \\ 751{\cdot}017 & 1336{\cdot}015 - V_m & 15421{\cdot}23 \\ 10617{\cdot}96 & 15421{\cdot}23 & 190296 - V_m \end{vmatrix} = 0.$$

This when expanded becomes

$$V_m^3 - 192411\,V_m^2 + 52425892\,V_m - 786607941 = 0.$$

We want the least root of this cubic. It is quickly seen to be in the neighbourhood of 15, and by successive approximations is found to be

$$V_m = 15{\cdot}9362,$$

very nearly.

* The second decimal place was taken into account in finding this figure. This was done in order to compare with the results of (ϵ), which do not greatly differ from (γ).

If $$lx+my+nz=0$$

is the equation of required plane, the equations to find the ratios of l, m, and n are

$$763{\cdot}198l+\ 751{\cdot}017m+10617{\cdot}960n=0,$$
$$751{\cdot}017l+1320{\cdot}079m+15421{\cdot}230n=0.$$

From these we find

$$\frac{l}{5{\cdot}4908}=\frac{m}{8{\cdot}5582}=\frac{n}{-1}.$$

The equation of the plane is

$$z=5{\cdot}4908x+8{\cdot}5582y$$

and

$$\mathrm{T}=1084-5{\cdot}4908x-8{\cdot}5582y. \quad . \quad . \quad . \quad . \quad (\epsilon)$$

The temperatures given by this formula are shown in column 7 of the table, and the deviations from the observed values in column 8. They do not differ greatly from the results given by (γ). The sum of the squares of the deviations in the table is 1658·9718, which is, of course, greater than the corresponding number given by the first method. The "root mean square" is 6·44, not greatly different from the first method value. Also $l^2+m^2+n^2$ becomes 104·3923. The actual sum of the squares of the deviations perpendicular to the plane is therefore 15·8917, which is less than the value given by the first method, as it should be, but is not a very great improvement on it. Thus in this example, as in the last, the two methods lead to very similar results.

III. For a third example we will take the case of a plane in three dimensions to pass through two fixed points and to be closest fitting to a series of other points. The data for this case are taken from a railway time-table. The two fixed points are two terminal stations, and the variables are x, the distance (in miles) from one of these stations to some other station; y, the scheduled time (in minutes) allowed for a train between those stations; and z, the first-class single fare (in pence) between the stations. The figures are:—

x.	y.	z.
30	49	69
52	80	117
60	97	135
69	115	156
81	136	182
100	164	224

the corresponding figures up to the other terminus being 114, 187, and 244 respectively. The figures should be expected to be fairly coplanar, and any formula obtained to represent them ought to give a good "fit." Four cases can be worked out here, viz. those obtained by making the sum of the square of the deviations in the directions of x, y, z and perpendicular to the plane respectively a minimum. We find :

$$\begin{aligned} S(x^2) &= 28526, & S(yz) &= 105264, \\ S(y^2) &= 76827, & S(zx) &= 64160, \\ S(z^2) &= 144311, & S(xy) &= 46801. \end{aligned}$$

For the best fit in the direction of z, the equation of the plane will be

$$z = ax + by,$$

with the condition

$$224 = 114a + 187b. \quad . \quad . \quad . \quad . \quad . \quad (\eta)$$

In this case we have

$$\Delta = \begin{vmatrix} 1 & 64160-114\lambda & 105264-187\lambda \\ 64160-114\lambda & 28526 & 46801 \\ 105264-187\lambda & 46801 & 76827 \end{vmatrix}.$$

Using the relations

$$a = -\frac{\Delta_{10}}{\Delta_{00}}, \quad b = -\frac{\Delta_{20}}{\Delta_{00}},$$

we obtain

$$\begin{aligned} a &= 2{\cdot}2376 + {\cdot}00526\,\lambda, \\ b &= {\cdot}00706 - {\cdot}00077\,\lambda. \end{aligned}$$

Substituting in (η), λ becomes $-27{\cdot}2290$, and therefore

$$\begin{aligned} a &= 2{\cdot}0943, \\ b &= {\cdot}0281, \end{aligned}$$

and the best fitting plane in the direction of z is

$$z = 2{\cdot}0943x + {\cdot}0281y. \quad . \quad . \quad . \quad . \quad . \quad (\theta)$$

In a similar manner we find the best fitting plane in the direction of y is

$$y = 1{\cdot}6327x + {\cdot}0036z, \quad . \quad . \quad . \quad . \quad . \quad (\zeta)$$

and in the direction of x it is

$$x = {\cdot}5031y + {\cdot}0817z. \quad . \quad . \quad . \quad . \quad . \quad (\xi)$$

When the deviations are measured in a direction perpendicular to the plane, the equation to determine V_m (the least sum of the squares of these deviations) is, by § 5 above,

$$\begin{vmatrix} 0 & 114 & 187 & 244 \\ 114 & 28526-V_m & 46801 & 64160 \\ 187 & 46801 & 76827-V_m & 105264 \\ 244 & 64160 & 105264 & 144311-V_m \end{vmatrix} = 0,$$

the determinant being reversed for convenience in evaluation, *i. e.*

$$107501V_m^2 - 19413930V_m + 191936824 = 0,$$

giving

$$V_m = 10{\cdot}49664.$$

Then

$$d\propto \begin{vmatrix} 1 & 114\mu & 187\mu & 244\mu \\ 114\mu & 28515{\cdot}5 & 46801 & 64160 \\ 187\mu & 46801 & 76816{\cdot}5 & 105264 \\ 244\mu & 64160 & 105264 & 144301{\cdot}5 \end{vmatrix}$$

and

$$l_1 = -\frac{d_{10}}{d_{00}}, \quad l_2 = -\frac{d_{20}}{d_{00}}, \quad l_3 = -\frac{d_{03}}{d_{00}},$$

where

$$l_1x + l_2y + l_3z = 0$$

is the equation of the required plane. Since only the ratios of l_1, l_2, and l_3 are required, it is sufficient to find d_{10}, d_{20}, and d_{30} (each of which contains μ as a factor). When we find these ratios we must divide each by $\{l_1^2+l_2^2+l_3^2\}^{\frac{1}{2}}$ in order to have the sum of their squares unity. In this way we find the equation of the plane is

$$\cdot 8843x - \cdot 4632y - \cdot 0582z = 0. \quad . \quad . \quad . \quad (\phi)$$

The deviations of the results given by the formulæ (θ), (ζ), (ξ), and (ϕ) from the actual values are (the deviation being positive when the formula gives a value greater than the actual value) :—

(θ) Deviation in direction of z*.	(ζ) Deviation in direction of y.	(ξ) Deviation in direction of x.	(ϕ) Deviation perpendicular to plane.
−4·795	+ ·227	+ ·285	− ·181
−5·850	+5·319	−2·200	+2·122
−6·619	+1·445	− ·178	+ ·275
−8·265	−1·787	+1·592	−1·325
−8·544	−3·101	+2·281	−1·952
−9·966	+ ·070	+ ·796	− ·563

The sum of the squares of these deviations are

(θ) 341·6570,

(ζ) 43·2458,

(ξ) 13·3240,

(ϕ) 10·4942 (the exact value here should be 10·4966, the value of V_m above).

The sum of the squares of the deviations given by (θ), (ζ), and (ξ) in directions perpendicular to those planes are found to be (by dividing the above values by the sum of the squares of the coefficients of the various equations) 63·4239, 11·7969, and 10·5767 respectively, all these, of course, being greater than the corresponding value given by (ϕ).

Equation (ϕ) can be written in the three forms :

$$z = 15{\cdot}2048x - 7{\cdot}9644y \quad . \quad . \quad . \quad . \quad (\theta')$$

$$y = 1{\cdot}9091x - {\cdot}1256z \quad . \quad . \quad . \quad . \quad (\zeta')$$

$$x = {\cdot}5238y + {\cdot}0658z \quad . \quad . \quad . \quad . \quad (\xi')$$

These equations should be compared with (θ), (ζ), and (ξ) respectively. The sum of the squares of the deviations

* At first sight it seems remarkable that all the deviations given by (θ) are of the same sign, but a moment's consideration will show that this is quite possible. For a line in the plane is fixed, and the plane can only swing about this line. All the points may be on one side of the plane, but on either side of the line. Swinging the plane about the line to become closer (measured in a particular direction) to some of the points, therefore, may take it farther from some others. To verify (θ) the results given by the planes $z = 2{\cdot}1404x$ and $z = 2x + {\cdot}0856$, one on either side of (θ), were found. The sum of the squares of the deviations given by these formulæ were 341·6991 and 341·8112, both greater than the corresponding number for (θ). Thus (θ) gives a true minimum.

given by these equations in the directions of z, y, and x respectively are

3102·9568, 48·9069, and 13·4189.

Comparing these with the results given by (θ), (ζ), and (ξ) above brings out clearly the fact that the plane which satisfies one criterion for closest fit may give a very bad fit if measured by another criterion. This is particularly the case with (θ) and (θ'), though (ξ') is not greatly inferior to (ξ) as the best fitting plane in the direction of x.

The Sir John Cass Technical Institute,
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XLIV. *A Method of Calibrating Fine Capillary Tubes.* *By* Thomas Ralph Merton, *B.Sc. (Oxon.)**.

THE methods commonly used in the determination of the bore of capillary tubes are direct optical measurement of the bore at the orifice, or weighing a drop of mercury which occupies a known length in the capillary. When very fine capillaries, having an internal diameter of the order of ·1 mm., are to be measured, these methods present serious difficulties.

For many purposes it is necessary to obtain a value of the mean bore, and as no glass capillary is uniform for any considerable length, a measurement of the bore at the orifices is liable to serious error. The weight of a column of mercury 10 cm. long contained in a capillary tube of ·1 mm. bore is about 0·01 grm., so that to obtain an accuracy of 1 per cent. the weighing must be correct to 0·1 mgrm.

The following experiments were performed with the object of investigating the accuracy with which a measurement of the electrical resistance of a fine glass capillary filled with mercury can be made. From this a mean value of r^2 (where r is the internal radius) can be calculated.

The first series of experiments was conducted in a large water-bath, containing about 30 litres, kept at 18° C. by an electric-filament lamp which was governed by a large spiral toluene regulator; and other experiments were performed in a bath kept at 25° by a small gas-flame governed by a fluted toluene regulator. In both baths the temperature could be kept constant to 0°·01 C.

* Communicated by the Author.