and the equation is

$$(BCOIJ)^2 (IJ)^2 (A^2BCOIJ)^2 = \theta^2 . A^2 (ABCOIJ)^4,$$

that is

$$(IJ)^2 (ABCOIJ)^4 = \theta^2 . A^2 (ABCOIJ)^4;$$

so that the points are each quadruple.

The two Tetrahedra; A, B, C, D at P in ABC and A, B′, C′, D′ at P′ in A′B′C′ subtending equal angles.

I consider now the before-mentioned problem of the two tetrahedra; viz., on the two bases ABC and A′B′C′ respectively, letting fall the perpendiculars DK and D′K′, then first A, B, C, K at P and A′, B′, C′, K′ at P′ subtend equal angles; the locus of P is a cubic curve $ABCKOO_1O_2O_3IJ = 0$ through these ten points. ($O = ABC$ is derived from the points A, B, C; and in like manner $O_1 = BCK$, $O_2 = CAK$, $O_3 = ABK$.)

Next, B, C, K at P and B′, C′, K′ at P′ subtend equal angles, and moreover the distances KP and K′P′ are in a given ratio; the locus of P is a 12-thic curve $(BCKO_1IJ)^4 = 0$,
having each of these six points as a quadruple point. Hence among the 36 intersections of the two curves we have the points B, C, K, O_1, I, J each 4 times, and there remain $36-24, = 12$ intersections.

The conclusion is that A, B, C, D at a point P of ABC, and A′, B′, C′, D′ at a point P′ of A′B′C′, subtending equal angles, there are 12 positions of P, and of course 12 corresponding positions of P′.

Invariant Conditions for Three Conics having Common Points.

By J. J. Walker, M.A.

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1. The notation adopted in the discussion of the invariant conditions of the concurrence of three conics,

$$u = ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0,$$
$$v = a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy = 0,$$
$$w = a''x^2+b''y^2+c''z^2+2f''yz+2g''zx+2h''xy = 0,$$

is as follows. There are twelve independent invariants of the system, viz., the three invariants of the single conics,

$$\Delta = abc+2fgh-af^2-bg^2-ch^2,$$
$$\Delta' = a'b'c' + \dots\dots\dots\dots - c'h'^2,$$
$$\Delta'' = a''b''c'' + \dots\dots\dots\dots - c''h''^2;$$

the three pairs of invariants of the conics taken two and two,

$$\begin{aligned}
\Theta &= a''(b'c'-f'^2)+b''(c'a'-g'^2)+c''(a'b'-g'^2)\\
&\quad +2f''(g'h'-a'f')+2g''(h'f'-b'g')+2h''(f'g'-c'h')\\
\Theta' &= a'(b''c''-f''^2)+\ \ldots\ldots\\
&\quad +2f'(g''h''-a''f'')+\ \ldots\ldots\\
\Theta_1 &= a(b''c''-f''^2)+\ \ldots\ldots\\
&\quad +2f(g''h''-a''f'')+\ \ldots\ldots\\
\Theta_1' &= a''(bc-f^2)+\ \ldots\ldots\\
&\quad +2f''(gh-af)+\ \ldots\ldots\\
\Theta_2 &= a'(bc-f^2)+\ \ldots\ldots\\
&\quad +2f'(gh-af)+\ \ldots\ldots\\
\Theta_2' &= a(b'c'-f'^2)+\ \ldots\ldots\\
&\quad +2f(g'h'-a'f')+\ \ldots\ldots
\end{aligned}$$

lastly, the three invariants into which the coefficients of all three enter symmetrically, which may be considered as fundamental:

1°. Of the first order in the coefficients of each,

$$\begin{aligned}
\Phi &= a(b'c''+b''c'-2f'f'')+b(c'a''+c''a'-2g'g'')+c(a'b''+a''b'-2h'h'')\\
&\quad +2f(g'h''+g''h'-a'f''-a''f')+2g(h'f''+h''f'-b'g''-b''g')\\
&\quad +2h(f'g''+f''g'-c'h''-c''h')\\
&= a'(bc''+b''c-2f''f)+\ \ldots\ldots\\
&= a''(bc'+b'c-2ff')+\ \ldots\ldots
\end{aligned}$$

2°. Of the second order in the coefficients of each,

$$\begin{aligned}
X^* &= (ab'c'')^2+4\{(ab'f'')(ac'f'')+4(bc'g'')(b'a'g'')+4(ca'h'')(cb'h'')\}\\
&\quad +8\{(af'g'')(bf'g'')+(af'h'')(cf'h'')+(cg'h'')(bg'h'')-(ag'h'')(bc'f'')\\
&\quad -(cf'g'')(ab'h'')-(bh'f'')(ca'g'')\}+4\{(ab'c')-2(f'g'h')\}(fg'h'),
\end{aligned}$$

where $(ab'c'')$, $(ab'f'')$ ... $(ag'h'')$ ... stand for the determinants

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}, \quad \begin{vmatrix} a & b & f \\ a' & b' & f' \\ a'' & b'' & f'' \end{vmatrix} \ldots\ldots \begin{vmatrix} a & g & h \\ a' & g' & h' \\ a'' & g'' & h'' \end{vmatrix} \ldots\ldots$$

3°. Of the fourth order in the coefficients of each conic; viz., the quartic invariant 'S' (Salmon's "Higher Plane Curves," 2nd ed. p. 184) of the Jacobian of the system, multiplied by -81,

$$\begin{aligned}
\Psi &= -a_1b_2c_3m+b_2c_3a_2a_3+c_3a_1b_3b_1+a_1b_2c_1c_2+(a_1b_3c_2+b_2c_1a_3+c_3a_2b_1)m\\
&\quad -\{a_1(b_3^2c_1+b_1c_2^2)+b_2(c_1^2a_2+c_2a_3^2)+c_3(a_2^2b_3+a_3b_1^2)\}+m^4\\
&\quad -2(b_1c_1+c_2a_3+a_3b_3)m^2+3(a_2b_3c_1+a_3b_1c_2)m+b_1^2c_1^2+c_2^2a_2^2+a_3^2b_3^2\\
&\quad -(b_1c_1c_2a_2+c_2a_2a_3b_3+a_3b_3b_1c_1),
\end{aligned}$$

* Dr. Salmon's "Conics," p. 348, 5th ed., who, after mentioning its discovery by Mr. Sylvester, points out that the invariant may be obtained by operating on the Jacobian with the contravariant which is the condition that any transversal should be cut in involution by the three conics.

where $$a_1 = 3(ag'h''),\quad b_2 = 3(bh'f''),\quad c_3 = 3(cf'g''),$$
$$a_2 = -(ab'g'') - (ah'f''),\quad b_1 = -(ab'f'') - (bg'h''),$$
$$c_1 = -(ca'f'') - (cg'h''),\quad a_3 = -(ca'h'') - (af'g''),$$
$$b_3 = -(bc'h'') - (bf'g''),\quad c_2 = -(bc'g'') - (ch'f''),$$
$$2m = -(ab'c'') - 2(fg'h'').$$

This invariant is Dr. Salmon's 'M' ("Conics," p. 349).

2. I may remark—and the remark is new, as far as I am aware—that the sextic invariant of the Jacobian of three conics is not an independent invariant of the system, but calling the 'T' ("Higher Plane Curves," p. 185) of the Jacobian, multiplied by -729, Ω,

$$16\Omega - 48\Psi X + X^3 \equiv 0.$$

To prove this identity, suppose the triangle of reference to be the common self-conjugate triangle of v and w; then $f' = 0$, $g' = 0$, $h' = 0$, $f'' = 0$, $g'' = 0$, $h'' = 0$ simultaneously. The coefficients of three times the Jacobian reduce to

$$a_1 = 0,\quad b_2 = 0,\quad c_3 = 0,\quad 2m = -(ab'c''),$$
$$a_2 = (a'b'' - a''b')\,g,\quad b_3 = (b'c'' - b''c')\,h,\quad c_1 = (c'a'' - c''a')\,f,$$
$$a_3 = (c'a' - c''a')\,h,\quad b_1 = (a'b'' - a''b')\,f,\quad c_2 = (b'c'' - b''c')\,g,$$

[whence $a_2b_3c_1 - a_3b_1c_2 \equiv 0$], while in this case

$$X = 4m^2 - 4(b_1c_1 + c_2a_2 + a_3b_3),$$
$$\Psi = m^4 - 2(b_1c_1 + c_2a_2 + a_3b_3)\,m^2 + 3(a_2b_3c_1 + a_3b_1c_2)\,m + b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2$$
$$- (c_2a_2a_3b_3 + a_3b_3b_1c_1 + b_1c_1c_2a_2),$$
$$\Omega = 8m^6 - 24(b_1c_1 + \ldots)\,m^4 + 36(a_2b_3c_1 + a_3b_1c_2)\,m^3 + (12\,c_2a_2a_3b_3 + \ldots)\,m^2$$
$$+ 24(b_1^2c_1^2 + \ldots)\,m^2 - 36(a_2b_3c_1 + a_3b_1c_2)(b_1c_1 + \ldots)$$
$$- 8(b_1^3c_1^3 + \ldots) + 27(a_2^2b_3^2c_1^2 + a_3^2b_1^2c_2^2) + 6\,b_1c_1c_2a_2a_3b_3$$
$$+ 12\{b_1^2c_1^2(c_2a_2 + a_3b_3) + c_2^2a_2^2(a_3b_3 + b_1c_1) + a_3^2b_3^2(b_1c_1 + c_2a_2)\}.$$

Hence $$\Omega - \tfrac{1}{8}X^3 = 36\{(a_2b_3c_1 + a_3b_1c_2)\,m^3 - (c_2a_2a_3b_3 + \ldots)\,m^2$$
$$- (a_2b_3c_1 + a_3b_1c_2)(b_1c_1 + \ldots)\,m + b_1^2c_1^2(c_2a_2 + a_3b_3) + \ldots\}$$
$$+ 27(a_2^2b_3^2c_1^2 + a_3^2b_1^2c_2^2) + 54\,b_1c_1c_2a_2a_3b_3.$$

Also $$\Psi - \tfrac{1}{16}X^2 = 3\{(a_2b_3c_1 + a_3b_1c_2)\,m - (c_2a_2a_3b_3 + \ldots)\},$$

whence

$$3X(\Psi - \tfrac{1}{16}X^2) = 36\{(a_2b_3c_1 + a_3b_1c_2)\,m^3 - (c_2a_2a_3b_3 + \ldots)\,m^2$$
$$- (a_2b_3c_1 + a_3b_1c_2)(b_1c_1 + \ldots)\,m + b_1^2c_1^2(c_2a_2 + a_3b_3) + \ldots$$
$$+ 3\,b_1c_1c_2a_2a_3b_3\}$$
$$= \Omega - \tfrac{1}{8}X^3 - 27(a_2^2b_3^2c_1^2 + a_3^2b_1^2c_2^2) + 54\,b_1c_1c_2a_2a_3b_3$$
$$= \Omega - \tfrac{1}{8}X^3 - 27(a_2b_3c_1 - a_3b_1c_2)^2.$$

It has been shown above that $a_2b_3c_1 - a_3b_1c_2$ vanishes in this case, consequently, clearing of fractions,

$$3X(16\Psi - X^2) = 16\Omega - 2X^3,$$

or

$$16\Omega - 48\Psi X + X^3 = 0.$$

3. Dr. Salmon has discussed the condition for the three conics having one point in common, or a single concurrence, and shown it to be ("Conics," p. 349) $X^2 = 64\Psi$.

In fact, taking the common point as one of the corners of the triangle of reference, in this case

$$a = 0, \quad a' = 0, \quad a'' = 0$$

simultaneously, whence X reduces to

$$X = 8\{(bg'h'')(cg'h'') - (f'g'h'')^2\}.$$

But in this case, $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, and consequently Ψ reduces to

$$\Psi = m^4 - 2b_1c_1m^2 + b_1^2c_1^2 = (m^2 - b_1c_1)^2.$$

Now, however, b_1, c_1, m reduce to

$$b_1 = -(bg'h''), \quad c_1 = -(cg'h''), \quad m = -(fg'h''),$$

consequently,

$$X - 8(b_1c_1 - m^2),$$

and

$$X^2 = 64(b_1c_1 - m^2)^2 = 64\Psi.$$

The form of the Jacobian, wanting the terms x^3, x^2y, x^2z, shows, that the point common to the three conics is a double point on it. Conversely, it may be shown, that if the Jacobian have a double point, the three conics must pass through this point, or else the relation $X^2 = 16\Psi$ must hold.

For, generally, the discriminant of the Jacobian is proportional to

$$\Omega^2 - 64\Psi^3.$$

Equating this with zero, and eliminating Ω between the resulting equation and the identity $16\Omega = (48\Psi - X^2)X$,
as the condition for a double point,

$$16384\Psi^3 - (48\Psi - X^2)^2 X^2 = 0,$$

which may be transformed into

$$(16\Psi - X^2)^2(64\Psi - X^2) = 0.$$

Consequently the Jacobian will have a double point if either

$$64\Psi = X^2,$$

which is the condition for the three conics having a common point of intersection, or if $16\Psi = X^2$.

4. Suppose, now, the three conics to have two points in common,

and let these be taken as two corners of the triangle of reference; then u, v, w simultaneously reduce to

$$u \equiv ax^2+2fyz+2gzx+2hxy = 0,$$
$$v \equiv a'x^2+2f'yz+2g'zx+2h'xy = 0,$$
$$w \equiv a''x^2+2f''yz+2g''zx+2h''xy = 0,$$

whence $\Delta = 2fgh-af^2, \quad \Delta' = 2f'g'h'-a'f'^2, \quad \Delta'' = 2f''g''h''-a''f''^2,$

$$\Theta = -a''f'^2-2f''a'f'+2(f''g'h'+g''h'f'+h''f'g'),$$
$$\Theta' = -a'f''^2-2f'a''f''+2(f'g''h''+g'h''f''+h'f''g''),$$
$$\Theta_1 = -af''^2-2fa''f''+2(fg''h''+gh''f''+hf''g''),$$
$$\Theta_1' = -a''f^2-2f''af+2(f''gh+g''hf+h''fg),$$
$$\Theta_2 = -a'f^2-2f'af+2(f'gh-g'hf+h'fg),$$
$$\Theta_2' = -af'^2-2fa'f'+2(fg'h'+gh'f'+hf'g'),$$
$$\Phi = -2(af'f''+a'f''f+a''ff')+2f(g'h''+g''h')$$
$$+2g(h'f''+h''f')+2h(f'g''+f''g').$$

From these values there follow, identically,

$$\Delta'f''^3-\Theta f''^2f'+\Theta'f'f''^2-\Delta''f'^3 = 0,$$
$$\Delta''f^3-\Theta_1f^2f''+\Theta_1'ff''^2-\Delta f''^3 = 0,$$
$$\Delta f'^3-\Theta_2f'^2f+\Theta_2'f'f^2-\Delta'f^3 = 0,$$
$$(\Theta f''-\Theta'f')f^2+(\Theta_1f-\Theta_1'f'')f'^2+(\Theta_2f'-\Theta_2'f)f''^2 = 0,$$

which last may be derived from the first three by multiplying them by f^3, f'^3, f''^3 respectively, adding and dividing by $-ff'f''$.

The resultant of the first three of the above four equations, it has already been pointed out (Salmon's "Conics," 5th ed., p. 345), is the condition that u, v, w should have double contact with the same conic; and accordingly the "point-pair" common to the three conics in this case form the vertices of a degenerate conic which is touched by every line drawn through them. This resultant, however, has never yet, as far as I am aware, been exhibited. I obtain it as follows:

Multiply the first of the above system of four equations by f^2, $f'f''$, $f''f$, ff' successively; the second by $f'^2, f'f'', f''f, ff'$ successively; the third by $f''^2, f'f'', f''f, ff'$ successively, and the fourth by $f^2, f'^2, f''^2, f'f'', f''f, ff'$ successively. Thus will be formed a system of 18 equations, from which the 18 quantities

$$f'^4f'', f''^4f', f''^4f, f^4f'', f^4f', f'^4f, f'^3f''^2, f''^3f'^2, f''^3f^2, f^3f''^2, f^3f'^2, f'^3f^2,$$
$$f^3f'f'', f'^3f''f, f''^3ff', f'^2f''^2f, f''^2f^2f', f^2f'^2f'',$$

may be eliminated linearly; so that the resultant may be exhibited in the form of a determinant of the 18th degree:

$$\left|\begin{array}{cccccccccccccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \Delta & \cdots & \cdots & -\Delta' & \cdots & \cdots & \cdots & \cdots & \cdots & -\Theta_2 & \Theta_2' & \cdots \\
\Delta & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -\Delta' & -\Theta_2 & \cdots & \cdots & \cdots & \Theta_2' \\
\cdots & \cdots & \cdots & -\Delta' & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \Theta_2' & \Delta & \cdots & \cdots & \cdots & -\Theta_2 \\
\cdots & \cdots & \cdots & \cdots & -\Delta' & \Delta & \cdots & \cdots & \cdots & \cdots & \Theta_2' & -\Theta_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -\Delta & \cdots & \cdots & \Delta'' & \cdots & \cdots & \cdots & \cdots & \Theta_1' & \cdots & -\Theta_1 \\
\cdots & -\Delta' & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \Delta'' & \cdots & \Theta_1' & \cdots & -\Theta_1 & \cdots \\
\cdots & \cdots & -\Delta & \Delta'' & \cdots & \cdots & \cdots & \cdots & \Theta_1' & -\Theta_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \Delta'' & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -\Theta_1 & \cdots & -\Delta & \cdots & \Theta_1' & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \Delta' & \cdots & \cdots & -\Delta'' & \cdots & \cdots & \cdots & \cdots & -\Theta & \Theta' \\
-\Delta'' & \Delta' & \cdots & \cdots & \cdots & \cdots & \Theta' & -\Theta & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \Delta' & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -\Delta'' & -\Theta & \Theta' & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & -\Delta'' & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \Theta' & \Delta' & -\Theta & \cdots & \cdots \\
\cdots & \cdots & \cdots & -\Theta & \Theta' & \cdots & \cdots & \cdots & \cdots & \Theta_2' & -\Theta_1 & \cdots & \cdots & \cdots & \cdots & \cdots & -\Theta_2 & \Theta_1' \\
\Theta_1' & \cdots & \cdots & \cdots & \cdots & -\Theta_1 & -\Theta_2 & \cdots & \cdots & \cdots & \cdots & \Theta' & \cdots & \cdots & \cdots & \Theta_1' & \cdots & -\Theta \\
\cdots & -\Theta_2 & \Theta_2' & \cdots & \cdots & \cdots & \cdots & \Theta_1' & -\Theta & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \Theta_1 & \Theta' & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \Theta_1' & -\Theta_2 & \cdots & \cdots & \cdots & \cdots & \cdots & -\Theta_1 & \Theta_2' & \cdots & -\Theta & \Theta' \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \Theta_2' & -\Theta & \cdots & \cdots & \Theta' & \cdots & -\Theta_2 & \Theta_1' & \cdots & -\Theta_1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \Theta' & -\Theta_1 & -\Theta & \Theta_1' & \cdots & -\Theta_2 & \Theta_2' & \cdots
\end{array}\right|$$

The resultant of the three equations

$$\Delta' f''^3 - \Theta f''^2 f' + \Theta' f'' f'^2 - \Delta'' f'^3 = 0,$$
$$\Delta'' f^3 - \Theta_1 f^2 f'' + \Theta_1' f f''^2 - \Delta f''^3 = 0,$$
$$\Delta f'^3 - \Theta_2 f'^2 f + \Theta_2' f' f^2 - \Delta' f^3 = 0,$$

may be reduced to that of a pair of cubics thus: since

$$\frac{f'}{f''} = \frac{f'}{f} \div \frac{f''}{f},$$

the equation $\Delta'' x^3 - \Theta' x^2 + \Theta x - \Delta' = 0$

must have a common root with the equation (of the ninth degree) whose roots are the quotients of those of

$$\Delta x^3 - \Theta_2 x^2 + \Theta_2' x - \Delta' = 0$$

divided by those of $\Delta x^3 - \Theta_1' x^2 + \Theta_1 x - \Delta'' = 0.$

This equation of the ninth degree is

$$\begin{aligned}
&\Delta^2\Delta''^3 x^9 - \Delta\Delta''^2\Theta_1\Theta_2 x^8 + \Delta''(\Delta\Theta_1^2\Theta_2' + \Delta''\Theta_2^2\Theta_1' - 2\Delta\Delta''\Theta_1'\Theta_2')\, x^7 \\
&- \{\Delta''^2(\Theta_2^3 - 3\Delta\Theta_2\Theta_2' + 3\Delta^2\Delta') + \Theta_1(\Delta''\Theta_2\Theta_1'\Theta_2' + \Delta\Delta'\Theta_1^2 - 3\Delta\Delta'\Delta''\Theta_1')\}\, x^6 \\
&+ \{\Delta''\Theta_1(\Theta_2^2\Theta_2' - 2\Delta\Theta_2'^2 - \Delta\Delta'\Theta_2) + \Delta'\Theta_2\Theta_1'(\Theta_1^2 - 2\Delta''\Theta_1') + \Delta''\Theta_1'^2\Theta_2'^2\}\, x^5 \\
&- \{\Delta'\Theta_2'(\Theta_1'^2\Theta_1 - 2\Delta\Theta_1^2 - \Delta\Delta''\Theta_1') + \Delta''\Theta_1'\Theta_2(\Theta_2'^2 - 2\Delta'\Theta_2) + \Delta'\Theta_2^2\Theta_1^2\}\, x^4 \\
&+ \{\Delta'^2(\Theta_1'^2 - 3\Delta\Theta_1'\Theta_1 + 3\Delta^2\Delta'') + \Theta_2'(\Delta'\Theta_1'\Theta_2\Theta_1 + \Delta\Delta''\Theta_2'^2 - 3\Delta\Delta'\Delta''\Theta_2\}\, x^3 \\
&- \Delta'(\Delta\Theta_2'^2\Theta_1 + \Delta'\Theta_1'^2\Theta_2 - 2\Delta\Delta'\Theta_2\Theta_1)\, x^2 + \Delta\Delta'^2\Theta_2\Theta_1' x - \Delta^2\Delta'^3 = 0.
\end{aligned}$$

If this is subtracted from $\Delta'' x^3 - \Theta' x^2 + \Theta x - \Delta' = 0$, multiplied by

$$\Delta^2\Delta''^2 x^6 + \Delta\Delta''(\Delta\Theta' - \Theta_1\Theta_2)\,x^5$$
$$+\{\Delta^2(-\Delta''\Theta+\Theta'^2)+\Delta\Theta_2'(-\Delta''\Theta_1'+\Theta_1^2)+\Delta''\Theta_1'(-\Delta\Theta_2'+\Theta_2^2)-\Delta\Theta_1\Theta_2\Theta'\}\,x^4$$
$$+\Delta^2(2\Delta'\Delta''-\Theta\Theta')\,x^3$$
$$+\{\Delta^2(\Theta^2-\Delta'\Theta')+\Delta\Theta_1(\Theta_2'^2-\Delta'\Theta_2)+\Delta'\Theta_2(\Theta_1'^2-\Delta\Theta_1)-\Delta\Theta_2'\Theta_1'\Theta\}\,x^2$$
$$+\Delta\Delta'(\Delta\Theta-\Theta_2'\Theta_1')\,x+\Delta^2\Delta'^2,$$

the result is the cubic

$$\{3\Delta\Delta''(\Delta\Theta\Theta'-\Delta'\Theta_1\Theta_1'-\Delta''\Theta_2\Theta_2')+(\Delta\Theta_1\Theta'-\Delta''\Theta_2\Theta_1')(\Theta_2\Theta'-\Theta_1\Theta_2')$$
$$+\Delta''^2\Theta_2^3-\Delta^2\Theta'^3+\Delta\Delta''(2\Theta'\Theta_1'\Theta_2'-\Theta\Theta_1\Theta_2)+\Delta\Delta'\Theta_1^3\}\,x^3$$
$$-\{\Delta''(3\Delta\Theta_2'-\Delta\Theta_2^2)(\Theta\Theta_1'-\Theta_1\Theta_2')-3\Delta'\Delta''\Theta_2\Theta_1'^2+\Delta''\Theta_1'^2\Theta_2'^2$$
$$+\Delta\Theta\Theta_1(\Theta_2\Theta'-\Theta_1\Theta_2')+\Delta'\Theta_1^2\Theta_2\Theta_1'\}\,x^2$$
$$+\{\Delta'(3\Delta\Theta_1-\Theta_1'^2)(\Theta'\Theta_2-\Theta_1\Theta_2')-3\Delta'\Delta''\Theta_2^2\Theta_1'+\Delta'\Theta_1^2\Theta_2^2$$
$$+\Delta\Theta'\Theta_2'(\Theta\Theta_1'-\Theta_1\Theta_2')+\Delta''\Theta_2'^2\Theta_1'\Theta_2\}\,x$$
$$-\{3\Delta\Delta'(\Delta\Theta\Theta'-\Delta''\Theta_2\Theta_2'-\Delta'\Theta_1\Theta_1'+(\Delta\Theta_2'\Theta-\Delta'\Theta_2\Theta_1')(\Theta\Theta_1'-\Theta_1\Theta_2')$$
$$+\Delta'^2\Theta_1'^3-\Delta^2\Theta^3+\Delta\Delta'(2\Theta\Theta_1\Theta_2-\Theta'\Theta_1'\Theta_2')+\Delta\Delta''\Theta_2'^3\}=0.$$

The resultant of this cubic and

$$\Delta''x^3-\Theta'x^2+\Theta x-\Delta'=0$$

may be calculated by the known formula, and will evidently be of the 18th degree in $\Delta, \Delta', \Delta''\Theta \ldots \Theta_2'$.

5. The condition just discussed does not involve Φ; but when the coefficients of y^2, z^2 are zero in the equations of the three conics simultaneously, any one of the following system of six equations may be verified to hold identically, and the other five to subsist in virtue of symmetry, viz.:

$$\Phi f'^2f''+3\Delta'f''^2f-2\Theta ff'f''-\Theta_2'f''^2f'+\Theta'f'^2f-\Theta_1f'^3=0,$$
$$\Phi f''^2f'+3\Delta''f^2f-2\Theta'ff'f''-\Theta_1f'^2f''+\Theta f''^2f-\Theta_2'f''^3=0,$$
$$\Phi f''^2f+3\Delta''f^2f'-2\Theta_1ff'f''-\Theta'f^2f''+\Theta_1'f''^2f'-\Theta_2f''^3=0,$$
$$\Phi f^2f''+3\Delta f''^2f'-2\Theta_1'ff'f''-\Theta_2f''^2f+\Theta_1f^2f'-\Theta'f^3=0,$$
$$\Phi f^2f'+3\Delta f'^2f''-2\Theta_2ff'f''-\Theta_1'f'^2f+\Theta_2'f^2f''-\Theta_4f^3=0,$$
$$\Phi f'^2f+3\Delta'f^2f''-2\Theta_2'ff'f''-\Theta f^2f'+\Theta_2f'^2f''-\Theta_2f'^3=0.$$

Among these six and the four equations first deduced the ten quantities $f^3, f'^3, f''^3, f'^2, f''$, ... $ff'f''$ may be eliminated linearly, and the resultant is the following determinant of the tenth order, in which the highest power of Φ is Φ^4:

$$\begin{vmatrix} -\Delta' & \Delta & 0 & 0 & 0 & 0 & 0 & \Theta_2' & -\Theta_2 & 0 \\ \Delta'' & 0 & -\Delta & 0 & 0 & \Theta_1' & -\Theta_1 & 0 & 0 & 0 \\ 0 & -\Delta'' & \Delta' & \Theta' & -\Theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Theta_1' & -\Theta_2 & \Theta_2' & -\Theta & \Theta' & -\Theta_1 & 0 \\ 0 & -\Theta_1 & 0 & \Phi & -\Theta_2' & 3\Delta' & 0 & 0 & \Theta' & \Theta \\ 0 & 0 & -\Theta_2' & -\Theta_1 & \Phi & \Theta & 0 & 0 & 3\Delta'' & \Theta' \\ 0 & 0 & -\Theta_2 & 0 & \Theta_1' & \Phi & -\Theta' & 3\Delta'' & 0 & \Theta_1 \\ -\Theta' & 0 & 0 & 0 & 3\Delta & -\Theta_2 & \Phi & \Theta_1 & 0 & \Theta_1' \\ -\Theta & 0 & 0 & 3\Delta & 0 & 0 & \Theta_2' & \Phi & -\Theta_1' & \Theta_2 \\ 0 & -\Theta_1' & 0 & \Theta_2 & 0 & 0 & 3\Delta' & -\Theta & \Phi & \Theta_2' \end{vmatrix} = 0.$$

6. It has been pointed out by Dr. Salmon that when three conics have two points in common, their Jacobian breaks up into a conic and the right line joining those points. The cubic

$$\Delta x^3 + \Delta' y^3 + \Delta'' z^3 + \Theta y^2 z + \Theta' y z^2 + \Theta_1 z^2 x + \Theta_1' z x^2 + \Theta_2 x^2 y + \Theta_2' x y^2 + \Phi xyz$$

likewise breaks up in this case into a right line and conic. For the invariants Δ, Δ', ... Φ being the coefficients of λ^3, μ^3, ... $\lambda\mu\nu$ in the discriminant of $\lambda u + \mu v + \nu w$ in general, this discriminant, when the coefficients of y^2, z^2 can be made to vanish simultaneously in u, v, w, is

$$(\lambda f + \mu f' + \nu f'')\{2(\lambda g + \mu g' + \nu g'')(\lambda h + \mu h' + \nu h'') \\ -(\lambda a + \mu a' + \nu a'')(\lambda f + \mu f' + \nu f'')\};$$

consequently, referred to a triangle having as two of its corners the two points common to the three conics,

$$\Delta x^3 + \Delta' y^3 + \Delta'' z^3 + \Theta y^2 z + \ldots\ldots + \Phi xyz,$$

is identical with

$$(fx + f'y + f''z)\{2(gx + g'y + g''z)(hx + h'y + h''z) \\ -(ax + a'y + a''z)(fx + f'y + f''z)\}.$$

7. If three conics have three points in common, their equations may simultaneously be reduced to

$$u \equiv 2f\,yz + 2g\,zx + 2h\,xy = 0,$$
$$v \equiv 2f'yz + 2g'zx + 2h'xy = 0,$$
$$w \equiv 2f''yz + 2g''zx + 2h''xy = 0,$$

so that the invariants reduce to

$$\Delta = 2fgh, \quad \Delta' = 2f'g'h', \quad \Delta'' = 2f''g''h'';$$
$$\Theta = 2(f''g'h' + g''h'f' + h''f'g'), \quad \Theta' = 2(f'g''h'' + g'h''f'' + h'f''g''),$$
$$\Theta_1 = 2(fg''h'' + gh''f'' + hf''g''), \quad \Theta_1' = 2(f''gh + g''hf + h''fg),$$
$$\Theta_2 = 2(f'gh + g'hf + h'fg), \quad \Theta_2' = 2(fg'h' + gh'f' + hf'g');$$
$$\Phi = 2\{f(g'h'' + g''h') + g(h'f'' + h''f') + h(f'g'' + f''g')\};$$

whence
$$\Theta_1'\Theta_2-\Delta\,\Phi = 4\,(f'f''g^2h^2+g'g''h^2f^2+h'h''f^2g^2),$$
$$\Theta\,\Theta_2'-\Delta'\,\Phi = 4\,(f''fg'^2h'^2+g''gh'^2f'^2+h''hf'^2g'^2),$$
$$\Theta'\Theta_1-\Delta''\Phi = 4\,(ff'g''^2h''^2+gg'h''^2f''^2+hh'f''^2g''^2);$$
also
$$\frac{\Theta_2}{\Delta}=\frac{f'}{f}+\frac{g'}{g}+\frac{h'}{h},\quad \frac{\Theta_2'}{\Delta}=\frac{g'h'}{gh}+\frac{h'f'}{hf}+\frac{f'g'}{fg},\quad \frac{\Delta'}{\Delta}=\frac{f'g'h'}{fgh},$$
$$\frac{\Theta_1'}{\Delta}=\frac{f''}{f}+\frac{g''}{g}+\frac{h''}{h},\quad \frac{\Theta_1}{\Delta}=\frac{g''h''}{gh}+\frac{h''f''}{hf}+\frac{f''g''}{fg},\quad \frac{\Delta''}{\Delta}=\frac{f''g''h''}{fgh},$$
$$\frac{\Theta'}{\Delta''}=\frac{f'}{f''}+\frac{g'}{g''}+\frac{h'}{h''},\quad \frac{\Theta}{\Delta''}=\frac{g'h'}{g''h''}+\frac{h'f'}{h''f''}+\frac{f'g'}{f''g''},\quad \frac{\Delta'}{\Delta''}=\frac{f'g'h'}{f''g''h''},$$
$$\frac{\Theta_1'\Theta_2-\Delta\Phi}{\Delta^2}=\frac{f'f''}{f^2}+\frac{g'g''}{g^2}+\frac{h'h''}{h^2},\quad \frac{\Theta\Theta_2'-\Delta'\Phi}{\Delta'^2}=\frac{f''f}{f'^2}+\frac{g''g}{g'^2}+\frac{h''h}{h'^2},$$
$$\frac{\Theta'\Theta_1-\Delta''\Phi}{\Delta''^2}=\frac{ff'}{f''^2}+\frac{gg'}{g''^2}+\frac{hh'}{h''^2}.$$

8. It follows that when the three conics have three points in common, of the three equations,
$$\Delta\,x^3-\Theta_2x^2+\Theta_2'x-\Delta'=0,$$
$$\Delta\,x^3-\Theta_1x^2-\Theta_1x-\Delta''=0,$$
$$\Delta''x^3-\Theta'x^2+\Theta\,x-\Delta\;=0,$$
the roots of any one are the quotients of those of another divided each by one of those of the third. This gives three relations among the coefficients of the three equations; viz., that the coefficients of the last should be proportional to those of the cubic, p. 410.

It appears also that if the equation be formed which has for its roots the sums of the products of those of the first of the three cubics above, each multiplied by one of those of the second, the expression $\frac{\Theta_1'\Theta_2-\Delta\Phi}{\Delta^2}$ must satisfy that equation. In the "Quarterly Journal of Mathematics," Vol. VI., pp. 331–333, I have investigated the form* of this derivative (of the sixth degree) of two cubics, and have shown that, assuming
$$K=\Theta_2^2\Theta_1+\Theta_1'^2\Theta_2'-3\Delta\Theta_1\Theta_2',$$
$$2L=2(\Delta''\Theta_2^3+\Delta'\Theta_1'^3-\Theta_1\Theta_2\Theta_1'\Theta_2')+3\,(\Theta_2\Theta_2'-3\Delta\Delta')\,(\Theta_1\Theta_1'-3\Delta\Delta''),$$
$$M=27\Delta^2\Delta'^2+4\Delta\Theta_2'^3+4\Delta'\Theta_2^3-\Theta_2^2\Theta_2'^2-18\Delta\Delta'\Theta_2\Theta_2',$$
$$M'=27\Delta^2\Delta''^2+4\Delta\Theta_1^3+4\Delta''\Theta_1'^3-\Theta_1^2\Theta_1'^2-18\Delta\Delta''\Theta_1\Theta_1',$$
it is
$$4\,(\Delta^4\phi^3-\Delta^2\Theta_2\Theta_1'\phi^2+\Delta K\phi-L)^2-MM'=0.$$

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* Which may be verified *à posteriori* by a much shorter process, by showing that the left side vanishes when two roots of either cubic are equal.

Substituting for ϕ, $\dfrac{\Theta_1'\Theta_2-\Delta\Phi}{\Delta^2}$, the condition then is

$$4\{\Delta^2\Phi^3-2\Delta\Theta_1'\Theta\,\Phi^2+(\Theta_1'^2\Theta_2^2+\Delta K)\,\Phi-\Theta_1'\Theta_2K+L\}^2-\Delta^2MM'=0.$$

It will be remarked that, though Φ, Θ, Θ', considered separately as functions of the roots of

$$\Delta x^3-\Theta_1'x^2+\Theta_1x-\Delta'=0,$$
$$\Delta x^3-\Theta_2x^2+\Theta_2'x-\Delta''=0,$$

have each six values, they are so related that, one being determined, the other two are thereby determined. Consequently, to obtain *sufficient* as well as necessary conditions, relations must be found between Θ and Θ' in the first degree with Φ. These may be obtained as follows.

Consider the identity

$$(\beta-\gamma)(\beta\gamma'+\beta'\gamma)+(\gamma-\alpha)(\gamma\alpha'+\gamma'\alpha)+(\alpha-\beta)(\alpha\beta'+\alpha'\beta)$$
$$-2\{\beta\gamma(\beta'-\gamma')+\gamma\alpha(\gamma'-\alpha')+\alpha\beta(\alpha'-\beta')\}$$
$$+(\alpha+\beta+\gamma)\{(\beta-\gamma)\alpha'+(\gamma-\alpha)\beta'+(\alpha-\beta)\gamma'\}=0,$$

or, what is the same thing,

$$\begin{vmatrix}1&1&1\\ \alpha&\beta&\gamma\\ \beta\gamma'+\beta'\gamma&\gamma\alpha'+\gamma'\alpha&\alpha\beta'+\alpha'\beta\end{vmatrix}-2\begin{vmatrix}1&1&1\\ \alpha'&\beta'&\gamma'\\ \beta\gamma&\gamma\alpha&\alpha\beta\end{vmatrix}+(\alpha+\beta+\gamma)\begin{vmatrix}1&1&1\\ \alpha&\beta&\gamma\\ \alpha'&\beta'&\gamma'\end{vmatrix}=0,$$

which may be transformed into

$$\begin{vmatrix}1&1&1\\ \beta+\gamma&\gamma+\alpha&\alpha+\beta\\ \beta\gamma+\beta'\gamma&\gamma\alpha'+\gamma'\alpha&\alpha\beta'+\alpha'\beta\end{vmatrix}-2\begin{vmatrix}1&1&1\\ \beta'+\gamma'&\gamma'+\alpha'&\alpha'+\beta'\\ \beta\gamma&\gamma\alpha&\alpha\beta\end{vmatrix}$$
$$=(\alpha+\beta+\gamma)\begin{vmatrix}1&1&1\\ \beta+\gamma&\gamma+\alpha&\alpha+\beta\\ \beta'+\gamma'&\gamma'+\alpha'&\alpha'+\beta'\end{vmatrix}.$$

Multiplied by $\begin{vmatrix}1&1&1\\ \alpha&\beta&\gamma\\ \alpha'&\beta'&\gamma'\end{vmatrix}$, this becomes

$$\begin{vmatrix}3&2(\alpha+\beta+\gamma)&\alpha(\beta'+\gamma')+\ldots\\ \alpha+\beta+\gamma&2(\beta\gamma+\gamma\alpha+\alpha\beta)&2(\alpha'\beta\gamma+\beta'\gamma\alpha+\gamma'\alpha\beta)\\ \alpha'+\beta'+\gamma'&\alpha'(\beta+\gamma)+\ldots&2(\alpha\beta'\gamma'+\beta\gamma'\alpha'+\gamma\alpha'\beta')\end{vmatrix}$$
$$-2\begin{vmatrix}3&2(\alpha'+\beta'+\gamma')&\beta\gamma+\gamma\alpha+\alpha\beta\\ \alpha+\beta+\gamma&\alpha(\beta'+\gamma')+\ldots&3\alpha\beta\gamma\\ \alpha'+\beta'+\gamma'&2(\beta'\gamma'+\gamma'\alpha'+\alpha'\beta')&\alpha'\beta\gamma+\beta'\gamma\alpha+\gamma'\alpha\beta\end{vmatrix}$$
$$=(\alpha+\beta+\gamma)\begin{vmatrix}3&\alpha+\beta+\gamma&\alpha'+\beta'+\gamma'\\ \alpha+\beta+\gamma&2(\beta\gamma+\ldots)&\alpha(\beta'+\gamma')+\ldots\\ \alpha'+\beta'+\gamma'&\alpha'(\beta+\gamma)+\ldots&2(\beta'\gamma'+\ldots)\end{vmatrix}.$$

Now let $\alpha = \frac{f'}{f}$, $\beta = \frac{g'}{g}$, $\gamma = \frac{h'}{h}$, $\alpha' = \frac{f''}{f}$, $\beta' = \frac{g''}{g}$, $\gamma' = \frac{h''}{h}$; then the above identity becomes

$$\Delta \begin{vmatrix} 3\Delta & 2\Theta_1' & \Phi \\ \Theta_1' & 2\Theta_1 & 2\Theta \\ \Theta_2 & \Phi & 2\Theta' \end{vmatrix} - 2\Delta \begin{vmatrix} 3\Delta & 2\Theta_2 & \Theta_1 \\ \Theta_1' & \Phi & 3\Delta' \\ \Theta_2 & 2\Theta_2' & \Theta \end{vmatrix} = \Theta_1' \begin{vmatrix} 3\Delta & 2\Theta_1' & 2\Theta_2 \\ \Theta_1' & 2\Theta_1 & \Phi \\ \Theta_2 & \Phi & 2\Theta_2' \end{vmatrix}.$$

From this equation a similar one may plainly be deduced by interchanging Δ' and Δ'', Θ_1' and Θ_2, Θ_1 and Θ_2', Θ and Θ'; viz.,

$$\Delta \begin{vmatrix} 3\Delta & 2\Theta_2 & \Phi \\ \Theta_2 & 2\Theta_2' & 2\Theta' \\ \Theta_1' & \Phi & 2\Theta \end{vmatrix} - 2\Delta \begin{vmatrix} 3\Delta & 2\Theta_1' & \Theta_2' \\ \Theta_2 & \Phi & 3\Delta'' \\ \Theta_1' & 2\Theta_1 & \Theta' \end{vmatrix} = \Theta_2 \begin{vmatrix} 3\Delta & 2\Theta_2 & 2\Theta_1' \\ \Theta_2 & 2\Theta_2' & \Phi \\ \Theta_1' & \Phi & 2\Theta_1 \end{vmatrix}.$$

Three analogous relations might be obtained from these by interchanging Δ and Δ', Θ and Θ_1', Θ' and Θ_1; and three others by interchanging Δ and Δ'', Θ_2 and Θ', Θ_2' and Θ.

8. The covariant

$$\Delta x^3 + \Delta' y^3 + \Delta'' z^3 + \Theta y^2 z + \Theta' z^2 y + \Theta_1 z^2 x + \Theta_1' x^2 z + \Theta_2 x^2 y + \Theta_2' x y^2 + \Phi xyz$$

evidently breaks up into three linear factors when the three conics have three points in common; for, referring them to their common inscribed triangle, the discriminant of $\lambda u + \mu v + \nu w$ is

$$(\lambda f + \mu f' + \nu f'')(\lambda g + \mu g' + \nu g'')(\lambda h + \mu h' + \nu h'');$$

but this is $\Theta\lambda^3 + \ldots + \Phi\lambda\mu\nu$. The Jacobian of u, v, w in the same case breaks up into three right lines; viz., those joining the common points. But this takes place also, as Dr. Salmon has pointed out, when the three conics have a common self-conjugate triangle. In fact, as may easily be verified, it takes place even though only one vertex and the opposite side of the common self-conjugate triangle of two of the conics are pole and polar with respect to the third.

9. The condition that the three conics,

$$\begin{aligned} 2f\,yz + 2g\,zx + 2h\,xy &= 0, \\ 2f'\,yz + 2g'\,zx + 2h'\,xy &= 0, \\ 2f''\,yz + 2g''\,zx + 2h''\,xy &= 0, \end{aligned}$$

should have a fourth point common, is found by eliminating the products of the variables linearly from the above three equations, and is therefore

$$\begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = 0.$$

or

$$fgh\begin{vmatrix} 1 & 1 & 1 \\ \frac{f'}{f} & \frac{g'}{g} & \frac{h'}{h} \\ \frac{f''}{f} & \frac{g''}{g} & \frac{h''}{h} \end{vmatrix} = 0,$$

or

$$\Delta\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix} = 0;$$

squaring which and rejecting $\Delta = 0$,

$$\begin{vmatrix} 3\Delta & 2\Theta_1' & 2\Theta_2 \\ \Theta_1' & 2\Theta_1 & \Phi \\ \Theta_2 & \Phi & 2\Theta_2' \end{vmatrix} = 0.$$

In fact, since generally, if the three conics should have three points in common, it has been shown that each of the four determinants,

$$\begin{vmatrix} 3\Delta & 2\Theta_1' & \Phi \\ \Theta_1' & 2\Theta_1 & 2\Theta \\ \Theta_2 & \Phi & 2\Theta' \end{vmatrix} \qquad \begin{vmatrix} 3\Delta & 2\Theta_2 & \Theta_1 \\ \Theta_1' & \Phi & 3\Delta' \\ \Theta_2 & 2\Theta_2' & \Theta \end{vmatrix}$$

$$\begin{vmatrix} 3\Delta & 2\Theta_2 & \Phi \\ \Theta_2 & 2\Theta_2' & 2\Theta' \\ \Theta_1' & \Phi & 2\Theta \end{vmatrix} \qquad \begin{vmatrix} 3\Delta & 2\Theta_1' & \Theta_2 \\ \Theta_2 & \Phi & 3\Delta'' \\ \Theta_1' & 2\Theta_1 & \Theta' \end{vmatrix}$$

has the determinant $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}$ as a factor, as well as the determinant which has been shown to be the square of the preceding, it follows that when the conics have a fourth point also common, each of the above four determinants will also vanish.

Four conditions which are necessary, and sufficient, in order that the three conics should have four points in common are, therefore,

$$4(\Delta^2\Phi^3 - 2\Delta\Theta_2\Theta_1'\Phi^2 + K'\Phi - L')^2 - \Delta^2 MM' = 0,$$

$$\begin{vmatrix} 3\Delta & 2\Theta_1' & 2\Theta_2 \\ \Theta_1' & 2\Theta_1 & \Phi \\ \Theta_2 & \Phi & 2\Theta_2' \end{vmatrix} = 0,$$

together with *any two* of following four equations:

$$\begin{vmatrix} 3\Delta & 2\Theta_2 & \Theta_1 \\ \Theta_1' & \Phi & 3\Delta' \\ \Theta_2 & 2\Theta_2' & \Theta \end{vmatrix} = 0, \qquad \begin{vmatrix} 3\Delta & 2\Theta_1' & \Theta_2' \\ \Theta_2 & \Phi & 3\Delta'' \\ \Theta_1' & 2\Theta_1 & \Theta' \end{vmatrix} = 0,$$

$$\begin{vmatrix} 3\Delta & 2\Theta_1' & \Phi \\ \Theta_1' & 2\Theta_1 & 2\Theta \\ \Theta_2 & \Phi & 2\Theta' \end{vmatrix} = 0, \qquad \begin{vmatrix} 3\Delta & 2\Theta_2 & \Phi \\ \Theta_2 & 2\Theta_2' & 2\Theta' \\ \Theta_1' & \Phi & 2\Theta \end{vmatrix} = 0.$$

But the existence of any three of the last four together with the first, though necessary, would not be *sufficient*, in general, because each of the last four determinants has another factor besides $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}$; nor would the existence of the second and three of the last four in general be sufficient, because the second contains Φ in the second degree, and is therefore satisfied by a value of Φ which does not necessarily also satisfy the first.

It is hardly necessary to remark that in the above enumeration of conditions the same interchange of symbols may be made as before.

10. Since, when the three conics have four points in common, $\begin{vmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{vmatrix} = 0$, it at once appears that in this case the three lines whose equation is $\Delta x^3 + \Delta' y^3 + \Delta'' z^3 + \Theta y^2 z + \ldots\ldots + \Phi xyz = 0$, have a common point of intersection. The Jacobian of the system becomes indeterminate—every one of its coefficients vanishing—as must be the case, since it should represent the right lines joining two and two any three of the four points common to the conics; and at the same time their common self-conjugate triangle.

APPENDIX.

THE following is the Abstract furnished by Mr. W. Barrett Davis of his communication "On Methods used in the construction of Tables of Divisors," (p. 10):—

"The following three distinct numerical calculations were originally undertaken to obtain large series of prime numbers, and to exhibit the status of large numbers with regard to the number of divisors. I do not know that any algebraical means exist of ascertaining how many numbers are composed of two or three prime divisors, say, or binary and ternary &c. composites, between given limits.

"The first Table exhibits the divisors of numbers in the neighbourhood of 10^{xi}. I have calculated the place of all the prime numbers as far as 80,021, and after that as far as the square root, viz., $\sqrt{312000}$ by estimation, those that would enter within my limits. The second is a similar Table for 10^8. An extract from this has been published in

Liouville's 'Journal de Mathématiques' for 1866. The third computation is an attempt to find the factors of the differences of powers, and was suggested by a paper by M. Lamé, in the *Comptes Rendus*, and the facility of the process in the case of cubes and quintics led me to go on. I have succeeded in obtaining the primes and composites of large numbers, that would be quite unapproachable otherwise. These factors are all of one form for each power; thus for cubes they are $6m+1$, for quintics $10m+1$; and this reminds me to ask the members of the Society for an algebraical form that will approximately solve the following problem: 'Required the number of primes of the form $6m+1$ between given limits A and B.'

"In conclusion I beg to say that though the matter is simple multiplication, some mechanical contrivances have been used to shorten the labour, which could not have been carried to the same extent without the use of Col. Oakes' reciprocals and others. Dase, Burckhardt, and Chernac have been constantly used; and I have carefully looked through the 'Commentationes Arithmeticæ' of Euler to find a theorem which I could apply in a few cases only. I should have been glad to use Gauss's Table and Method for detecting a prime number, but I had not his book long enough in my own private use."

Prof. Cayley communicated an article to the "London, Edinburgh, and Dublin Philosophical Magazine" for May, 1872, "On a Bicyclic Chuck." See p. 104.

The remarkable echo at Bedgebury Park (p. 185) became the subject of a correspondence, initiated by Lord Rayleigh in "Nature," under the heading, "Harmonic Echoes." See "Nature," No. 199 (vol. viii., pp. 319, 320, August 21, 1873), Nos. 200, 208, and 210 (vol. ix. p. 6).

With reference to Mr. Macleod's paper (p. 236) it may be noted, as was pointed out to the Secretaries, that the very oldest books on artillery (as Santbech, 1561) appear to use the principle of the hodograph, though in a bungling fashion. It may be of interest to notice that it was thought that it would be desirable to introduce a similar treatment of the question into elementary works on the subject of projectiles.

There is an analysis given in the June (1873) number of the "London, Edinburgh, and Dublin Philosophical Magazine" (pp. 450—454), which bears upon Mr. Hayward's communication (p. 289) in little more than name only. The title of the article in question is, "The first extension of the term *Area* to the case of an autotomic plane circuit; by Thomas Muir, MA., Assistant-Professor of Mathematics in Glasgow University." The writer quotes De Morgan's words, "No such extension of the word has been made that I ever met with;" and then he directs attention to a paper by Alb. Ludov. Fried. Meister in the Göttingen Commentaries for 1769-1770, entitled, "De genesi figurarum planarum et inde pendentibus earum affectionibus," which, he says, contains almost all that even yet can be said on the subject.

A Member of the Council supplies the following note upon Prof. Wolstenholme's paper (p. 321): "Eckardt, in Schlömilch's Zeitschrift for 1870, has shown that if P and Q move in the same circle with uniform velocities, a point dividing the chord PQ in a constant ratio describes an epitrochoid."

Lord Rayleigh communicated to the November (1873) number of the London, Edinburgh, and Dublin Philosophical Magazine a "Note on Vibrations of approximately Simple Systems," which bears upon his paper on page 357 *supra*.

Prof. Wolstenholme's Note on p. 330 does away with the necessity of inserting *in extenso* his paper read before the Society at the June (1873) meeting (p. 356), but a few of its results may be given here. Designating by a', b', c' the quantities given in his Note (p. 330):

If we take $\alpha=\frac{3}{2}\pi$, $a=b$, we get the well-known case of the cardioid $a'=0$, $b'=0$, $c'=3a$.

If we take $\alpha=\frac{1}{2}\pi$, $a=b$, or $\alpha=\frac{5}{2}\pi$, we get in each case the same locus, as was shown in the Author's former paper.

Again, if we take $a=-3b$, $\alpha=\frac{1}{2}\pi$, we have again $a'=b'=0$, $c'=b$; or the locus of the intersection of perpendicular tangents to a tricusped hypocycloid is the inscribed circle.

The locus will in general be a circle, if either $\frac{a+b}{a+2b}\,\alpha = r\pi$, or if $\frac{b}{a+2b}\,\alpha=r\pi$, r being any integer. Thus if we take $2a=-3b$ and $\alpha=\frac{1}{2}\pi$, we again get the tricusped hypocycloid under its alternative method of generation, and obtain the circle by making $c'=0$.

If $a=-4b$, and $\alpha=\frac{2}{3}\pi$ or $\frac{4}{3}\pi$, we have $a'=b'=0$, $c'=2b$, giving the inscribed circle of the four-cusped hypocycloid; and it has been shown by the Author, in his paper on epicycloids, that three of the tangents from any point of this circle are so inclined.

In general, if we make $a'=b'=0$, we get

$$c'=(a+2b)\sin\frac{b}{a+b}\,.\,r\pi\div\sin\frac{a+2b}{a+b}\,r\pi=(-1)^r(a+2b),$$

giving the circle through the vertices of the epicycloid (or hypocycloid).

Also, if we make $c'=0$, we have

$$\frac{a'}{a}=\frac{b'}{b}=(-1)^r\frac{a+2b}{a+b},$$

therefore $$a'+b'=(-1)^r(a+2b),$$

giving again the circle through the vertices.

It appears, then, that the only cases in which the locus is a circle are those discussed in the paper on epicycloids, p. 321.

The locus can become an epicycloid only by taking $c'=b'$ on to $-b'$, which gives the equation

$$b\sin\frac{a+b}{a+2b}\,\alpha=\pm(a+b)\sin\frac{b}{a+2b}\,\alpha,$$

or with the notation of the paper referred to,

$$m \sin n\theta = \pm n \sin m\theta.$$

The form of the equation shows that if C be the centre of the fixed circle, P, P′ the points of contact of two moving circles when the tangents are inclined at the given angle, Q the corresponding point of contact of the moving circle in the epitrochoid; then CQ bisects the angle PCP′.

In the February number (1873, p. 98) of the "Philosophical Magazine" appears a "Note on the History of certain Formulæ in Spherical Trigonometry" (see the Appendix to Vol. iii. of the Proceedings of the London Mathematical Society, pp. 320, 321). In this Mr. Todhunter combats the view taken in the Appendix. He remarks that "the writer (Klügel) states correctly the positions of Gauss and Mollweide; and then he adds that Delambre published the formulæ in the *Connaissance des Tems* for 1808, and so French writers usually call them after him. But these few words relating to Delambre seem to me to fall below the usual standard of German accuracy. For, in the first place, the erroneous date (1808) must have been borrowed without verification, although there is nothing to warn us of this. And in the next place, the writer apparently puts the claims of Mollweide and Delambre as equal, by ascribing to both the date 1808, overlooking the fact that the *Connaissance des Tems* for an assigned year is published in advance of that year. Thus, finally, although Mollweide has priority over Gauss, yet he comes about a year and a half after Delambre; and therefore, until any other person can be shown to have published the formulæ before April, 1807, they must be justly ascribed to Delambre."

Further on, Mr. Todhunter points out that Delambre's second mode of proof is substantially the same as that independently discovered and printed in the "Proceedings," vol. iii., p. 13. One step, however, in Prof. Crofton's proof he states is simpler than the corresponding step in Delambre's, viz., the proof of the equality of the angles MVA and CVP.

A Mathematical Society of Paris has been founded on the plan of the similar Societies of London, Moscow, and Berlin, having for its object to encourage mathematical studies, and increase mathematical knowledge, and to form a bond of union of those interested in the mathematical sciences. The Society publishes a bulletin of its Proceedings (*Nature*, August 1, 1872; and for a more detailed account see *Nouvelles Annales*, August, 1872).

Want of leisure has precluded Prof. Henrici from drawing up a full obituary notice of the late Dr. Clebsch. We shall content ourselves with the following brief sketch he has outlined.

Rudolf Friedrich Alfred Clebsch was born 19th January, 1833, at Königsberg in Prussia, where his father was a medical officer in the army. He was a pupil at the Gymnasium, and at the early age of seventeen he became a student at the University of Königsberg. Here he attended the lectures of Neumann, Richelot, and Hesse on Mathematical Physics and Mathematics. The influence of each of these three men is perceptible in his writings. He took his degree in 1854, and went to Berlin, where he was for several years a teacher of mathematics at different schools, developing, in a short time, his extraordinary talents in this capacity. In his teaching we are told that it was not his aim to commence with a series of "difficult abstract definitions, but to start from concrete presentation, and rouse the pupil's interest by intuition. This was subsequently the characteristic of his university lectures; the subject-matter of the lecture grows up organically before his audience." We learn further that "it was, in the highest sense of the word, an æsthetic pleasure to listen to his exposition." Clebsch accepted, in 1858, the appointment of Professor of Theoretical Mechanics at the Polytechnic School at Carlsruhe; this he resigned for the chair of Pure Mathematics at the University of Giessen. Here he found, for the first time, the proper sphere for the full and simultaneous development of the gifts which he possessed, both for teaching and for original research; and consequently here it is that his scientific work rapidly increases. In the autumn of 1868 he left Giessen for Göttingen, where he died four years afterwards, on the 7th November, 1872.

His writings are too numerous to be specially considered in the brief space at our command; they may, however, be divided, according to contents and succession, into six groups.

First, he wrote on Mathematical Physics, Hydrostatics, Elasticity, and Optics; next on the Calculus of Variations, and on Partial Differential Equations of the first order. There follow his first geometrical papers on general Theory of Curves and Surfaces. In connection with these comes a period characterised by his study of Abelian Functions and their use in Geometry. In his last years he was occupied at the same time by Plane-Representation of Surfaces and Theory of Invariants.*

A full report of his works is given by a number of his pupils in the Mathem. Ann., vol. vii., p. 1.

Dr. Clebsch was elected an Honorary Foreign Member of the Society on December 14th, 1871. R. T.

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* This account has been chiefly drawn up from the account in the "Mathematische Annalen," vol. vi., p. 1, which is itself a reprint from the "Nachrichten der Königl. Gesellschaft der Wissenschaften in Göttingen." Translations are given in the "Nouvelles Annales," and also in the "Giornale di Matematiche" (Gennaio e Febbraio, 1874). In this latter journal also appeared, in Jan.-Feb. 1873, "Commemorazione di R. F. A. Clebsch indirizzata all' Accademia di Göttingen dal Prof. Carlo Neumann," from the "Göttingen Nachrichten."