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XXXVII. *On the new Method proposed by Dr. YOUNG for calculating the Atmospheric Refraction.* By JAMES IVORY, M.A. F.R.S.

IN the last Number of the Quarterly Journal of Science (No. 22), Dr. Young has reprinted what he calls *A Postscript on Atmospheric Refraction*, which was first published in the Philosophical Transactions for 1819. The problem is a very difficult one, and has been treated of by geometers of the first rank; and, in the new point of view in which it is here presented, it is supposed that the principal difficulties have been evaded or overcome. No apology will therefore be necessary, if we endeavour to appreciate the improvement thus achieved in mathematical science, by candidly inquiring how far the pretensions held out are fulfilled.

The leading idea of Dr. Young's method is to develop the density of the air in a series of terms containing the powers of the refraction sought. By this means the problem is brought to the solution of an equation, or to the reversion of a series.

All the methods for computing the refractions that have gained celebrity among astronomers, if we except that of Laplace, are equivalent to the solution of an equation of the second degree. This is true of the rules of Bradley, of Mayer, of Simpson; which are sufficiently accurate for all altitudes within a few degrees of the horizon. It is therefore certain that the two first terms only of Dr. Young's series, namely, those containing the first and second powers of the unknown quantity, will be sufficient for the greater part of a Table of Refractions. The improvement effected by the new method must therefore consist in enabling the calculator to complete the Table, by carrying the refractions quite down to the horizon; for which purpose all the former methods, except that of the French astronomers, are found to be insufficient. The principal point we have to inquire into will therefore relate to the *convergency* of the new series for low altitudes, and more particularly in the extreme case of the horizontal refraction.

Two different ways may be supposed to have occurred to the author for examining the convergency of his series. The most scientific way was to ascertain the rate of the decrease of the terms, by determining the general law of the coefficients. It may be doubted whether this is practicable in the present case, more particularly in the mode of calculation followed by the author. Another way was to take some example about the accuracy of which no doubt existed; and to compare the known result with that obtained from the same data by the new method.

For this purpose the horizontal refraction, on the supposition of a uniform dispersion of heat in the atmosphere, might have been chosen with great propriety, as a case that had already been determined with the greatest accuracy by the calculations of Laplace and Kramp. This very instance is indeed one of Dr. Young's examples; but he employs data different from what the foreign geometers proceed upon; and, on this account, it is difficult to compare the results obtained by the different methods. Besides, the numerical computations in the article in the *Journal of Science*, are so inaccurate that no conclusion can be drawn from them in which confidence can be placed. Thus, at p. 357, the horizontal refraction on the supposition above alluded to, is brought to an equation of which the solution is said to be $r^2 = \cdot 000121$; but the real value which will be found by actual substitution to satisfy the equation, is $r^2 = \cdot 0001259$; and hence $r = \cdot 011220$, or $38' 34''$, being $44''$ greater than the result in the *Journal of Science*. Again, in a similar calculation, p. 359, the author concludes, $r^2 = \cdot 000103$, and $r = 34' 53''$; but it should be, $r^2 = \cdot 0001066$, and $r = 35' 29''$.

The examples given in the *Quarterly Journal* leave the question of the convergency of the series quite undecided. There can be no doubt that a few of the first terms will in every case enable us to compute the greater part of the quantity sought; but the author has done nothing to determine the precise degree of exactness that will be attained, when only a certain number of the first terms of the series are taken in, and the rest rejected. We shall succeed better in this inquiry if we do not confine ourselves strictly to the mode of calculation imagined by the author. It will conduce greatly to render the theory more accessible, if, by a preliminary investigation, we separate those conditions of the problem that are indispensable from such as are accessory only, and by this means reduce the necessary equations to the least number possible.

Now the fundamental equations of the problem are these two, given § 2, p. 353, viz.

$$u = \frac{S}{1 + pz},$$

$$dr = \frac{du}{v}.$$

In the first of these equations z is the perpendicular falling from the earth's centre upon the direction of a ray of light, or upon the tangent of the trajectory which the ray describes; S , a constant quantity; p , a small fraction expressing the refractive force of the air when the density is unit; and v , the proportional density of the air at the point of the curve from which the tangent is drawn, the density at the surface of the earth being unit.

Let

Let a denote the radius of the earth, and A the apparent altitude of the star: it is obvious that, when the trajectory meets the earth's surface, $u = a \cos A$; wherefore, because $z=1$, we

have, $a \cos A = \frac{S}{1+p}$; whence $S = a \cos A \times (1+p)$. Now, this value of s being substituted, the general equation will become

$$u = a \cos A \times \frac{1+p}{1+p^2} = a \cos A \times \frac{1+p}{1+p-p(1-z)} :$$

and from this we readily deduce,

$$\left. \begin{aligned} u &= \frac{a \cos A}{1-\beta\omega}, \\ \beta &= \frac{p}{1+p}, \\ \omega &= 1-z. \end{aligned} \right\} \quad (1)$$

If the light, instead of coming from the star to the spectator, be conceived to proceed in an opposite course from the spectator to the star, u will increase from $a \cos A$ to its ultimate value, while ω increases from zero to unit.

In the second of the fundamental equations, viz. $dr = \frac{du}{v}$;

r stands for the angular refraction, and v denotes the part of the tangent between the curve and the perpendicular u . Hence, if x be put for the height above the earth's surface of the point in the curve from which the tangent is drawn, it is obvious that

$$v = \sqrt{(a+x)^2 - u^2}; \text{ wherefore, } dr = \frac{du}{\sqrt{(a+x)^2 - u^2}}.$$

If we now substitute the value of u before found, we shall get

$$dr = \frac{\beta \cos A}{(1-\beta\omega)^2} \times \frac{d\omega}{\sqrt{\left(1+\frac{x}{a}\right)^2 - \frac{\cos^2 A}{(1-\beta\omega)^2}}};$$

or, which is the same thing,

$$dr = \frac{\beta \cos A}{1-\beta\omega} \times \frac{d\omega}{\sqrt{\left(1+\frac{x}{a}\right)^2 (1-\beta\omega)^2 - \cos^2 A}}.$$

As the refractive force of the air ceases to be sensible at a height which bears a very small proportion to the earth's semi-diameter, $\frac{x}{a}$ will be a very small fraction even at the utmost limits of the atmosphere; wherefore, because β is also very small we may suppose $\left(1+\frac{x}{a}\right)^2 (1-\beta\omega)^2 = 1 + 2\frac{x}{a} - 2\beta\omega$; thus,

$$dr = \frac{\beta \cos A}{1-\beta\omega} \times \frac{d\omega}{\sqrt{\sin^2 A + 2\frac{x}{a} - 2\beta\omega}}.$$

Again,

Again, the factor $\frac{1}{1-\beta\omega}$ is always between the limits 1 and $\frac{1}{1-\beta}$; wherefore, if we put

$$dr = \beta \cos A \times \frac{d\omega}{\sqrt{\sin^2 A + 2\frac{x}{a} - 2\beta\omega}},$$

we may consider r as the exact refraction; for the true value of the refraction will be between the limits r and $\frac{r}{1-\beta}$, quantities which are so near one another that the difference will in no case amount to half a second.

The differential expression of the refraction now contains only two variable quantities; namely, the height above the earth's surface, and the decrease of the density of the air in ascending to that height. These two quantities are not entirely independent of one another. They are connected by a condition which depends on the pressure, and which we must now investigate. Let y denote the pressure of the atmosphere at the height x , measured by a barometer; and y' , the like pressure at the earth's surface. Dr. Young supposes the pressure at the earth's surface to be unit, and uses y to denote the relative pressure at any altitude, equivalent to $\frac{y}{y'}$, when the symbols are taken in the sense here defined. If we suppose x to become $x+dx$, y will become $y-dy$; and the small column of mercury dy will be equivalent in weight to the mass of air $dx \times z$. According to Laplace the elastic force of air at the temperature of melting ice, whatever be the density, is measured by the weight of a homogeneous column equal in altitude to 7974 metres, or 4360.25 fathoms. At any other temperature t reckoned on the centigrade scale; and, allowing that air expands $\frac{1}{250}$ for every centesimal degree of rise of temperature; the length of the homogeneous column that measures the elastic force will be $4360.25 \times (1 + .004 t)$ fathoms. Now, t denoting the temperature at the earth's surface, if we put $l = 4360.25 \times (1 + .004 t)$, it is obvious that the column of mercury y' will be equal in weight to the column of air l ; for each measures the elastic force. Wherefore we shall have this proportion,

$$y' : dy :: l \times 1 : dx \times z;$$

whence, $d. \frac{y}{y'} = - \frac{dx}{l} \times z$. Finally, let $S = \frac{x}{l}$, and $i = \frac{l}{a}$; then $\frac{x}{a} = iS$; and, by substitution, we shall get these two equations which contain all the conditions of the problem, viz,

$$\left. \begin{aligned} \frac{y}{y'} &= f - ds(1-\omega) \\ dr &= \beta \cos A \times \frac{d\omega}{\sqrt{\sin^2 A + 2iS - 2\beta\omega}} \end{aligned} \right\} \quad (2)$$

In these equations i has the same value with $\frac{1}{m}$ in Dr. Young's Postscript.

Every possible hypothesis relating to the density of the atmosphere; or, which is the same thing, every relation that can subsist between S and ω , must be such, that the integral $f - ds(1-\omega)$, taken between the limits $\omega=0$ and $\omega=1$, must itself extend from 1 to zero. This condition being fulfilled, the second formula will determine the refractions in that constitution of the atmosphere.

According to the method of Dr. Young, we must suppose

$$\omega = Br + Cr^2 + Dr^3 + Er^4 + \&c.$$

Now, if we write Δ for $\sqrt{\sin^2 A + 2\frac{x}{a} - 2\beta\omega}$, we shall get from the last equations, $\frac{d\omega}{dr} = \frac{\Delta}{\beta \cos A}$, by which the coefficient

B will be determined, viz. $B = \frac{\sin A}{\beta \cos A}$, because r, s, ω are all evanescent together. Again, take the fluxions of the equation

$$\frac{d\omega}{dr} = \frac{\Delta}{\beta \cos A}; \text{ thus, } \frac{d d\omega}{dr^2} = \frac{1}{\beta \cos A} \times \left\{ i \frac{ds}{d\omega} - \beta \right\} \times \frac{d\omega}{dr} \times \frac{1}{\Delta}; \text{ but } \frac{d\omega}{dr} \times \frac{1}{\Delta} = \frac{1}{\beta \cos A}; \text{ wherefore,}$$

$$\left. \begin{aligned} \frac{d d\omega}{dr^2} &= \frac{i}{\beta^2 \cos^2 A} \times \left\{ \frac{dS}{d\omega} - \lambda \right\}, \\ \lambda &= \frac{\beta}{i}. \end{aligned} \right\} \quad (3)$$

As we must suppose an equation between S and ω from which the value of $\frac{ds}{d\omega}$ will be found, the last formula will determine C , the second coefficient of the series. If we take the fluxions again, we shall get, $\frac{d^3 \omega}{dr^3} = \frac{i}{\beta^2 \cos^2 A} \times \frac{d dS}{d\omega^2} \times \frac{d\omega}{dr}$; by which the third coefficient D will be determined. And by continuing the like operations all the coefficients of the series may be found. We may also proceed in another way that will bring the determination of the series more immediately under the ordinary rules of analysis. For having an equation between S and ω , we may thence find a value of S , and likewise one of $\frac{dS}{d\omega}$, in terms of

ω ;

ω ; by which means the foregoing equation (3) will be converted into one containing only two variable quantities.

In the case of the horizontal refraction we have $\sin A = 0$, $\cos A = 1$; and, the series for ω containing only the even powers of r , it will be determined by the single equation,

$$\frac{d\omega}{dr^2} = \frac{i}{\beta^2} \cdot \left\{ \frac{dS}{d\omega} - \lambda \right\}.$$

The calculation will be rendered more simple by putting $r = g \times \frac{\beta}{\sqrt{i}}$; for then,

$$\left. \begin{aligned} \frac{d\omega}{d\varrho^2} &= \frac{dS}{d\omega} - \lambda \\ r &= g \times \frac{\beta}{\sqrt{i}} \end{aligned} \right\} \quad (4)$$

and we have now to determine the series,

$$\omega = C\varrho^2 + E\varrho^4 + G\varrho^6 + \&c.$$

In order to bring the question of the convergency to a decision, the best way will be to examine the case of the horizontal refraction in a particular hypothesis; for instance, in that of a uniform temperature prevailing in the atmosphere. In this hypothesis, the densities are proportional to the pressures; that is $\frac{y}{y'} = z = 1 - \omega$. Wherefore the first of the equations (2) will become $(1 - \omega) = f - ds(1 - \omega)$; whence $\frac{dS}{d\omega} = \frac{1}{1 - \omega}$. The equation (4) will therefore become,

$$\frac{d\omega}{d\varrho^2} = \frac{1}{1 - \omega} - \lambda = (1 - \lambda) + \omega + \omega^2 + \&c.$$

The coefficients of the series for ω will be determined by substitution, as usual, viz.

$$\begin{aligned} C &= \frac{1 - \lambda}{2}, \\ E &= \frac{1 - \lambda}{2 \cdot 12}, \\ G &= \frac{1 - \lambda}{2 \cdot 12 \cdot 30} + \frac{1}{30} \cdot \left(\frac{1 - \lambda}{2} \right)^2, \\ &\&c. \end{aligned}$$

Without carrying the calculation further, we may observe that the series will contain the part,

$$\frac{1}{3 \cdot 4} \left(\frac{1 - \lambda}{2} \right) \varrho^4 + \frac{1}{5 \cdot 6} \left(\frac{1 - \lambda}{2} \right)^2 \varrho^6 + \frac{1}{7 \cdot 8} \left(\frac{1 - \lambda}{2} \right) \varrho^8 + \&c.$$

which, as ϱ^2 is about 2, will converge very slowly. We may therefore conclude with certainty, that the method of calculation proposed by Dr. Young, is deficient in convergency, that is, a few
of

of the first terms of the series, or even a considerable number of them, are not sufficient for computing the refractions with the requisite exactness. It appears therefore that no great improvement of the theory of refraction is to be expected from the new way of considering the subject.

For greater illustration we may apply the foregoing method to the actual calculation of the horizontal refraction, taking the data as they are given in the *Mécanique Céleste*; that is, the mean pressure of the atmosphere being 0.76 metres, and the temperature at the earth's surface, that of melting ice. Then,

$$\beta = .000293876$$

$$i = .00125254$$

$$\lambda = \frac{\beta}{i} = .234625$$

$$C = .0.382625$$

$$E = .0.0318906$$

$$G = .0.0059446$$

&c.

and we have this equation for finding ϱ , viz.

$$1 = .382625 \varrho^3 + .0318906 \varrho^4 + .0059446 \varrho^6;$$

the solution of which is $\varrho^3 = 2.10117$; and $\varrho = 1.44964$. Hence

$$r = \frac{\beta}{\sqrt{i}} \times \varrho = .0120365, \text{ or } 41' 22''; \text{ which is } 88'' \text{ too much,}$$

the true quantity being $2394''.6$ according to the calculation of Laplace. This great excess arises from the terms of the series that are left out; and, although the error would be lessened, yet, on account of the slow convergency, it would by no means be quite corrected by taking in two or three more terms. There can be no doubt that the calculations, § vii. pp. 357 and 359, likewise bring out results considerably above the truth.

The observations that have been made relate only to Dr. Young's Theory, and do not bear at all upon the Table of Refractions published in the Nautical Almanac 1822. In the explanation annexed to the Table, we are told indeed that it is constructed upon principles explained by Dr. Young in the Philosophical Transactions; but the truth is, that the formula and the Table have very little reference to any theoretical principles, and must both be considered as entirely empirical. The real authority of the Table, or the ground on which its estimation with astronomers must rest, is the manner in which the coefficients have been determined; and upon this point we have very little satisfactory information.

We may suppose that the author of the Table employed two ways for finding the numeral coefficients of his formula. He

may

may have adjusted them to represent some good Table of Refractions, as that of the French astronomers: he may have employed for the same purpose a great number of accurate observations; he may have had recourse to both these methods.

The Table in the Nautical Almanac is easily compared with that in the *Connaissance des Temps*. Both suppose the same mean temperature, 50° of Fahrenheit, and 10° of the centigrade scale. In the English Table the mean pressure of the atmosphere is taken at 30 inches; in the French Table, at 0.76 metres, or 29.92 inches. The numbers in the two tables will therefore be brought to the same circumstances, if those in the French Table be increased by the $\frac{8}{3000}$ or $\frac{1}{375}$ part. When this is done the tables will stand as below:

Altitudes.			<i>Conn. des Temps.</i>			Naut. Alm.
0°	33' 51"	33' 51"
0½	28 37	28 37
1	24 25	24 25
1½	21 5	21 7
2	18 25	18 29
2½	16 16	16 21
3	14 31	14 35
3½	13 3	13 7
4	11 50	11 52
4½	10 49	10 50
5	9 56	9 58

In the remaining parts the two tables agree perfectly with one another. It appears therefore that the French Table is very accurately represented by Dr. Young's formula, the greatest difference being no more than 4" or 5" at low altitudes between 1 and 4 degrees. And in like manner, there can be no doubt, a similar formula may be so adjusted as to represent with equal exactness the Table of Bradley, or any other Table of Refractions.

It would be extremely important to be informed, whether a great number of good and original astronomical observations has been employed in constructing the English Table, and what those observations are. If this has actually been the case; if the English Table has a real and solid foundation different from the Table in the *Connaissance des Temps*; it must be allowed that no greater or more honourable testimony can be given in favour of the accuracy of the labours of the French astronomers.

Sept. 4, 1821.

J. IVORY.