

THEORY OF FREE AND SUSTAINED OSCILLATIONS*

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The amplitude of a free oscillation decreases with time because of the dissipation of energy in the circuit. The energy is generally considered as being dissipated by a series resistance or by the equivalent of a series resistance. It is often useful to differentiate between damping due to resistance in series and damping due to conductance in parallel with the capacitance of an oscillatory circuit. The current taken by a detector circuit will produce conductance damping in a receiving oscillatory circuit. In a transmitter the losses thru insulators, corona and dielectric losses cause conductance damping. During each oscillation the conductance loss is a maximum when the potential is a maximum while the resistance loss is greatest when the current is a maximum. The damping due to radiation is generally stated as an equivalent resistance damping altho its exact nature has not been determined experimentally.

FREE OSCILLATIONS

Let Figure 1 represent a circuit having concentrated inductance, capacitance, resistance and conductance. The resistance r is in series with the inductance L and the conductance g is in parallel with the capacitance C .

Assume the direction of the arrows positive.

The equation of potential thru inductance L , resistance r and conductance g is

$$L \frac{di}{dt} + ir - \frac{i_g}{g} = 0 \quad (1)$$

Equating potentials in the circuit formed by capacitance C and conductance g gives

$$\frac{i_g}{g} + \int \frac{i_c dt}{C} = 0 \quad (2)$$

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where g is the reciprocal of the insulation resistance.

The current relation is

$$i_c = i + i_g \quad (3)$$

Eliminate i_c from (2) and (3), and differentiate

$$\frac{1}{g} \cdot \frac{d i_g}{d t} + \frac{i}{C} + \frac{i_g}{C} = 0 \quad (4)$$

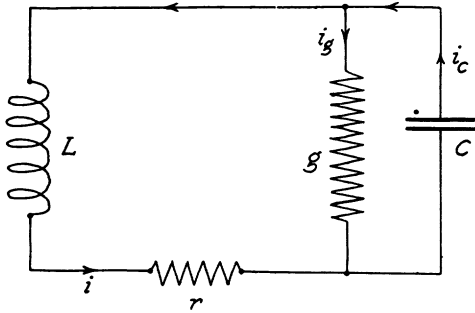


FIGURE 1

Solve (1) for i_g and substitute in (4)

$$\frac{d^2 i}{d t^2} + \left(\frac{r}{L} + \frac{g}{C} \right) \frac{d i}{d t} + \frac{1+g r}{L C} i = 0 \quad (5)$$

The auxiliary equation of (5) is

$$m^2 + \left(\frac{r}{L} + \frac{g}{C} \right) m + \frac{1+g r}{L C} = 0 \quad (6)$$

Solving (6) for m

$$m = - \left(\frac{r}{2L} + \frac{g}{2C} \right) \pm j \sqrt{\frac{1}{LC} - \left(\frac{r}{2L} - \frac{g}{2C} \right)^2}.$$

Let $\alpha = \frac{r}{2L} + \frac{g}{2C}$ and let $\omega = \sqrt{\frac{1}{LC} - \left(\frac{r}{2L} - \frac{g}{2C} \right)^2}.$

When $t=0$, $i=0$, therefore the solution of (5) is

$$i = I e^{-\alpha t} \sin \omega t \quad (7)$$

The damping factor α shows the obvious fact that increasing $\frac{L}{C}$ decreases the resistance damping and that decreasing $\frac{L}{C}$ decreases the conductance damping. Conductance and resistance

damping have the same effect upon the frequency when considered individually but when both are considered present they neutralize each other; so that when the resistance and conductance damping are equal, the frequency is independent of the damping.

A free discharge becomes non-oscillatory when $\left(\frac{r}{2L} - \frac{g}{2C}\right)$ is equal or greater than $\frac{1}{LC}$. It is seen that a non-oscillatory circuit may become oscillatory by the introduction of resistance or conductance damping.

The potential of the capacitance is

$$v = L \frac{di}{dt} + r i. \quad (8)$$

Substitute (7) in (8)

$$v = I \varepsilon^{-\alpha t} [\omega L \cos \omega t + (r - \alpha) \sin \omega t] \quad (9)$$

When $t=0$, $v=E_o$, therefore

$$E_o = \omega L I \quad (10)$$

Substitute (10) in (9)

$$v = E_o \varepsilon^{-\alpha t} \left(\cos \omega t + \frac{r - \alpha}{\omega L} \sin \omega t \right) \quad (11)$$

which may be written in the form

$$v = E \varepsilon^{-\alpha t} \cos (\omega t - \phi) \quad (12)$$

where

$$E_o = E \cos \phi \text{ and } \phi = \tan^{-1} \frac{r - \alpha}{\omega L}.$$

SUSTAINED OSCILLATIONS

An alternating current in a circuit having comparatively large inductance and capacitance and small impedance constitutes a sustained oscillation. The circuit is in resonance when the reactance of the circuit is zero at a given frequency of the impressed e. m. f. The impedance of the circuit is generally stated in terms of resistance, inductance, capacitance, and frequency. The effect of conductance is not always negligible nor can an equivalent resistance always be substituted for it.

In Figure 1, let v = the e. m. f. of the capacitance.

Then $i_g = g v.$ (13)

Eliminate i_g and i_c from (2), (3), and (13) by differentiating and rearranging

$$i = -C \frac{dv}{dt} - g v \quad (14)$$

Assume a sinusoidal e. m. f., $e = -E \sin \omega_1 t$, impressed upon the inductance L . Equation (1) will then become

$$L \frac{di}{dt} + r i - v = e \quad (15)$$

Let

$$\omega_2 = \sqrt{\frac{1}{LC} - \left(\frac{r}{2L} - \frac{g}{2C}\right)^2} \quad (16)$$

which may also be written

$$\frac{1+gr}{LC} = \omega_2^2 + a^2 \quad (17)$$

Substitute (14) in (15) and divide by LC

$$\frac{d^2 v}{dt^2} + 2a \frac{dv}{dt} + (\omega_2^2 + a^2) v = \frac{E \sin \omega_1 t}{LC} \quad (18)$$

Differentiate (18) twice, solve for $E \sin \omega_1 t$, and substitute in (18)

$$\frac{d^4 v}{dt^4} + 2a \frac{d^3 v}{dt^3} + (\omega_1^2 + \omega_2^2 + a^2) \frac{d^2 v}{dt^2} + 2a \omega_2^2 \frac{dv}{dt} + \omega_1^2 (\omega_2^2 + a^2) v = 0 \quad (19)$$

The auxiliary equation of (19) is

$$m^4 + 2a m^3 + (\omega_1^2 + \omega_2^2 + a^2) m^2 + 2a \omega_2^2 m + \omega_1^2 (\omega_2^2 + a^2) = 0 \quad (20)$$

which consists of the factors

$$m^2 + \omega_1^2 = 0, \text{ and } m^2 + 2a m + \omega_2^2 + a^2 = 0 \quad (21)$$

Therefore the roots of the above are

$$m = \pm j \omega_1, \text{ and } m = -a \pm j \omega_2 \quad (22)$$

Hence the solution of (19) is

$$v = A \epsilon^{j \omega_1 t} + B \epsilon^{-j \omega_1 t} + C \epsilon^{-a + j \omega_2 t} + D \epsilon^{-a - j \omega_2 t} \quad (23)$$

which may be written in the form

$$v = V \sin (\omega_1 t + \phi) + V_1 \epsilon^{-a t} \sin (\omega_2 t + \theta) \quad (24)$$

The transient component, $V_1 \epsilon^{-a t} \sin (\omega_2 t + \theta)$, disappears in a very short time; therefore the sustained oscillation is represented by

$$v = V \sin (\omega_1 t + \phi) \quad (25)$$

in which V and ϕ are constants of integration to be determined for the sustained component of the oscillation. V is the maximum potential and ϕ represents the phase difference between the impressed e. m. f. and the oscillatory e. m. f.

Substitute (25) in (18) and let $t = 0$.

$$V \omega_1^2 \sin \phi + 2 a V \omega_1 \cos \phi + (\omega_2^2 + a^2) V \sin \phi = 0 \quad (26)$$

Substitute (25) in (18) and let $\omega_1 t + \phi = 0$

$$2 a V \omega_1 + \frac{E \sin \phi}{L C} = 0, \text{ or } \phi = \sin^{-1} \left(-\frac{2 a V \omega_1 L C}{E} \right) \quad (27)$$

Eliminate ϕ from (26) and (27)

$$V L C [\omega_1^2 - (\omega_2^2 + a^2)] = -\sqrt{E^2 - (2 a V \omega_1 L C)^2}$$

or

$$V = \frac{E}{L C \sqrt{(\omega_1^2 - (\omega_2^2 + a^2))^2 + [2 a V \omega_1 L C]^2}} \quad (28)$$

which may also be written

$$V = \frac{E}{\omega_1 C \sqrt{\left(r + \frac{L}{C} \cdot g\right)^2 + \left(\omega_1 L - \frac{1 + g r}{\omega_1 C}\right)^2}} \quad (29)$$

Substitute (25) in (14)

$$i = -V \sqrt{(\omega_1 C)^2 + g^2} \cos \left[\omega_1 t + \phi + \tan^{-1} \left(\frac{-g}{\omega_1 C} \right) \right] \quad (30)$$

Let the maximum value of $i = I$, then

$$I = V \sqrt{(\omega_1 C)^2 + g^2} \quad (31)$$

There are two ways of securing maximum potential or current in an oscillatory circuit. Either the inductance may be varied while the capacitance remains constant or the capacitance may be varied while the inductance remains constant. If ω_1 and C remain constant, and L is varied, an inspection of (31) shows that both I and V are simultaneously a maximum for all values of r and g .

For maximum V by varying L , let $\frac{dV}{dL} = 0$; this gives

$$\omega_1 = \sqrt{\frac{1}{L C} - \left(\frac{g}{C}\right)^2} \quad (32)$$

For maximum V by varying C , let $\frac{dV}{dC} = 0$; this gives

$$\omega_1 = \sqrt{\frac{1}{L C} - \left(\frac{r}{L}\right)^2} \quad (33)$$

It is seen from (32) and (33) that with large conductance and small resistance in the oscillatory circuit, it is necessary to vary the capacitance to obtain maximum V at resonance; and, with large resistance and small conductance the inductance must be varied to get maximum V at resonance.

The potential energy, $\frac{1}{2} C V^2$, is always a maximum at resonance unless the conductance is large, then the maximum is obtained by varying the capacitance when

$$\omega_1 = \sqrt{\frac{1+gr}{LC - \frac{gL(rC+gL)}{1+gr}}} \quad (34)$$

and is obtained by varying the inductance when

$$\omega_1 = \sqrt{\frac{1}{LC} - \left(\frac{g}{C}\right)^2}. \quad (35)$$

Substituting (29) in (31), the maximum current amplitude

$$I = \frac{E \sqrt{\omega_1^2 C^2 + g^2}}{\omega_1 C \sqrt{\left(r + \frac{L}{C}g\right)^2 + \left(\omega_1 L - \frac{1+gr}{\omega_1 C}\right)^2}}. \quad (36)$$

The condition for resonance is that $\omega_1 = \sqrt{\frac{1+gr}{LC}}$. It will be noted that the reactance depends to a small extent upon the resistance and conductance when both are considered present. In (36) g is generally negligible compared to $\omega_1 C$ and the product gr is negligible compared to unity, therefore (36) may be written

$$I = \frac{E}{\sqrt{\left(r + \frac{L}{C}g\right)^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2}} \quad (37)$$

When $g=0$ this reduces to the familiar expression

$$I = \frac{E}{\sqrt{r^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2}} \quad (38)$$

And when $r=0$ equation (37) becomes

$$I = \frac{E}{\frac{L}{C} \sqrt{g^2 + \left(\omega_1 C - \frac{1}{\omega_1 L}\right)^2}} \quad (39)$$

The product of $\frac{L}{C}$ times a conductance or susceptance has the dimensions of a resistance or reactance.

To determine the maximum I of equation (36) by varying C , let $\frac{dI}{dC} = 0$; this gives

$$\omega_1 = \sqrt{\frac{1+2gr}{LC} + \left(\frac{g}{C}\right)^2} \quad (40)$$

Equation (32) applies to both potential and current; but equations (33) and (40) show that by varying the capacitance, the potential and current do not attain a maximum value simultaneously; the potential reaches a maximum at a frequency less than resonance and the current at a frequency greater than resonance frequency. When the conductance is negligible, all maximum values are coincident with resonance except maximum potential when the capacitance is varied for a maximum.

The radiation resistance is not large enough to change appreciably the frequency for maximum values of current and potential, otherwise equations (32), (33), and (40) could be used to determine experimentally whether radiation causes resistance or conductance damping; furthermore, maximum radiation probably takes place when the product of instantaneous potential and current, vi , is a maximum; that is, when the transfer of energy in the oscillatory circuit is a maximum. Maximum vi is displaced 45° from both maximum potential and maximum current and occurs at double the frequency.

The expressions derived for maximum potential and current are based upon a constant frequency of the impressed e. m. f. A slight fluctuation in the frequency introduces a reactance into the oscillatory circuit. The detrimental effect of this reactance increases as the ratio $\frac{L}{C}$ increases and as the effective resistance, $r + \frac{L}{C}g$, decreases.

In equation (37), let $\omega_1 L = \frac{1}{\omega_1 C}$ and let ρ represent $r + \frac{gL}{C}$, then the resonance current

$$I_r = \frac{E}{\rho}. \quad (41)$$

Let the fluctuation in frequency be equivalent to changing the frequency by a factor a . Substituting $a\omega_1$ for ω_1 in (37) then (41) will become

$$I_a = \frac{E}{\sqrt{\rho^2 + \left[\frac{1}{\omega_1 C} \left(\frac{a^2 - 1}{a} \right) \right]^2}} \quad (42)$$

where I_a is the measured resonance current.

The term "frequency factor" may be applied to the ratio

$$\frac{I_a}{I_r} = \frac{\rho}{\sqrt{\rho^2 + \left(\frac{a^2 - 1}{a \omega_1 C} \right)^2}} \quad (43)$$

which is the power factor of a circuit in which all of the reactance is due to frequency fluctuations of the impressed e. m. f.

In a receiving antenna circuit, damping is due to the losses in the circuit and to the energy withdrawn for useful work. The former consists of re-radiation, resistance, and conductance losses; and the latter consists of the energy withdrawn by the detector circuit or its equivalent. To obtain maximum energy in the detector circuit, the familiar principle applies, viz., that the damping due to useful energy withdrawn must equal the damping due to energy loss.

APPLICATIONS AND NUMERICAL EXAMPLES

Some of the equations will be stated in a form required for the substitution of practical units. Numerical examples will refer, unless otherwise noted, to a standard antenna of 0.002 microfarad capacitance, 10^5 cycles frequency (that is, of wave length 3,000 meters), and 5 ohms effective resistance. Let C_m = capacitance in microfarads, L = inductance in cm., r_o = the resistance in ohms, ρ_o = the effective resistance in ohms, and g_m = conductance in mhos.

To estimate the effect of conductance upon damping, the conductance may be expressed as an equivalent resistance by the relation $\frac{r}{2L} = \frac{g}{2C}$, or, $r = \frac{L}{C} g = \frac{\rho}{\omega^2 C^2}$, which shows that an equivalent resistance is directly proportional to the conductance. In practical units

$$r_o = \frac{10^{12}}{4 \pi^2} \cdot \frac{g_m}{f^2 C_m^2} \quad (46)$$

When $g_m = 10^{-6}$ mhos, $r_o = 0.62$ ohms; i. e., one megohm antenna insulation is equivalent to $\frac{5}{8}$ ohm antenna resistance in its damping effect.

The ratio of resonance frequency to the frequency of a free oscillation is expressed by

$$\frac{\sqrt{\frac{1+gr}{LC}}}{\sqrt{\frac{1+gr}{LC} - \frac{\rho^2}{4L^2}}} = \sqrt{\frac{1}{1 - \left(\frac{\delta}{2\pi}\right)^2}} \quad (47)$$

where the logarithmic decrement, $\delta = \frac{2\pi^2}{10^6} \rho_o C_m f$. When $\delta = 0.2$, the ratio (47) = 1.0005, or a variation of 50 cycles at a frequency of 10^5 cycles. The decrement of the standard antenna is 0.02 for which the ratio (47) is practically unity.

The maximum potential in practical units neglecting g and in (32) and (33) is

$$V = \frac{10^6}{2\pi} \cdot \frac{E}{\rho_o f C_m}. \quad (48)$$

Let $E = 100$ volts, then $V = 16,000$ volts.

The frequency factor may be expressed, with sufficient approximation, in practical units by

$$\frac{\rho_o}{\sqrt{\rho_o^2 + \frac{1}{10} \left(\frac{10^4 h}{f C_m} \right)^2}} \quad (49)$$

where $h = 100 (a - 1)$ = the per cent. frequency variation.

The per cent. frequency variation (h) and the corresponding frequency factor are tabulated below for a standard antenna, and also for the same antenna with 10 ohms effective resistance.

TABLE I

Frequency Variation (%)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Impedance (ohms)	5	5.25	5.9	6.9	8.1	9.4	10.7	12.1	13.6	15.0	16.6
Frequency Factor (%)	100	95	85	72.5	62	53	46	41	37	33	30
Impedance (ohms)	10	10.1	10.5	11.1	11.8	12.7	13.8	14.9	16.1	17.4	18.7
Frequency Factor (%)	100	98.8	95	90	85	79	73	67	62	57	53

The ratio $\frac{L}{C}$ is naturally high in a receiving antenna circuit while the efficiency is greatly increased by reducing the effective resistance. The re-radiation resistance, which may be considered equal to the radiation resistance, cannot be reduced, but all joulean loss should be made as small as practicable.

The effect of resistance and conductance upon the frequency is generally only of theoretical interest, but becomes appreciable in a sustained oscillation receiving or measuring circuit which is highly damped or sharply tuned.

The expressions derived are based upon Ohm's law; conductances, such as detector current and corona current, do not follow this law but approach more nearly to it than does an equivalent resistance.

SUMMARY: The free oscillations produced on a circuit having capacity, inductance, resistance, and conductance (leakance) are studied. The transient and permanent conditions with sustained oscillations are similarly treated. The resonance and energy relations of such circuits are carefully considered, together with the influence of conductance and resistance on decrement of the circuit and period thereof.