THE PRODUCT OF TWO HYPERGEOMETRIC FUNCTIONS

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IT is possible to establish a relation which connects the product of two hypergeometric functions

$$F(\alpha, \beta;\ \gamma;\ z) \times F(\alpha, \beta;\ \gamma;\ Z)$$

with the hypergeometric function of two variables of Appell's fourth type

$$F_4[\alpha, \beta;\ \gamma,\ \alpha+\beta-\gamma+1;\ zZ,\ (1-z)(1-Z)].$$

The reader will remember that the definition* of Appell's function is

$$F_4[\alpha, \beta;\ \gamma, \gamma';\ \xi, \eta] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}\,(\beta)_{m+n}}{(\gamma)_m\,(\gamma')_n\, m!\, n!}\, \xi^m \eta^n,$$

where a symbol of the form $(\alpha)_m$ denotes

$$\alpha(\alpha+1)(\alpha+2) \dots (\alpha+m-1).$$

In the special case in which $z = Z$, the existence of the relation has been indicated to a certain extent by Appell† himself, for he has shown that $\{F(\alpha, \beta;\ \gamma;\ Z)\}^2$ and $F_4[\alpha, \beta;\ \gamma,\ \alpha+\beta-\gamma+1;\ Z^2, (1-Z)^2]$ are solutions of the same linear differential equation of the third order.

The more general case in which z and Z are unequal, which is the subject of this paper, would appear to give the best theorem concerning the expression of functions of the fourth type in terms of hypergeometric functions, just as Appell's theorem‡ that

$$F_1(\alpha;\ \beta,\ \gamma-\beta;\ \gamma;\ X,\ Y) = (1-Y)^{-\alpha} F\left(\alpha, \beta;\ \gamma;\ \frac{X-Y}{1-Y}\right)$$

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* *Comptes Rendus*, t. 90 (1880), pp. 296, 731.

† *Journal de Math.*, Sér. 3, t. 10 (1884), pp. 418–421.

‡ *Journal de Math.*, Sér. 3, t. 8 (1882), p. 175; see also Barnes, *Proc. London Math. Soc.* Ser. 2, Vol. 6 (1908), p. 169.

is, in all probability, the best theorem concerning functions of the first type.

In this paper I propose to establish the general relation with the aid of contour integrals of Barnes' types, after considering the special case of the relation in which α is a negative integer, so that the hypergeometric series reduce to polynomials. The fact that the relation is of a somewhat abstruse character is indicated by the impracticability of proving it in a simple manner in the special case without making use of infinite series.

The importance of the relation arises from its existence, and not from the methods used in proving it, for the proof requires only a certain amount of analytical ingenuity. I may state that the method by which I discovered the relation was a consideration of various types of normal solutions of the wave-equation in four dimensions which have been the subject of a paper by Bateman.*

2. When α is a negative integer $-n$, the relation to be proved assumes the simple form†

$$F(-n, \beta+n;\ \gamma;\ z)\times F(-n, \beta+n;\ \gamma;\ Z)$$
$$= (-)^n \frac{(\beta-\gamma+1)_n}{(\gamma)_n} F_4[-n, \beta+n;\ \gamma, \beta-\gamma+1;\ zZ, (1-z)(1-Z)].$$

To prove the relation, we transform the expression on the right in the following manner, using Vandermonde's theorem in the fifth and sixth lines of the analysis:

$$(1-Z)^{\beta-\gamma} F_4[-n, \beta+n;\ \gamma, \beta-\gamma+1;\ zZ, (1-z)(1-Z)]$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(\beta+n)_{r+s}}{(\gamma)_r(\beta-\gamma+1)_s\, r!\, s!} z^r Z^r (1-z)^s (1-Z)^{\beta-\gamma+s}$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(\beta+n)_{r+s}}{(\gamma)_r\, r!} \sum_{k=0}^{s} \frac{(-)^k z^{r+k}}{k!\,(s-k)!} \sum_{p=0}^{\infty} \frac{(-)^p Z^{r+p}}{p!\,(\beta-\gamma+1)_{s-p}}$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{l=r}^{r+s} \sum_{q=r}^{\infty} \frac{(-n)_{r+s}(\beta+n)_{r+s}(-)^{l+q} z^l Z^q}{(\gamma)_r(\beta-\gamma+1)_{r+s-q}\, r!\,(l-r)!\,(r+s-l)!\,(q-r)!}$$
$$= \sum_{l=0}^{n} \sum_{q=0}^{\infty} \sum_{r=0}^{q} \sum_{s=l-r}^{n-r} \frac{(-n)_{r+s}(\beta+n)_{r+s}(-)^{l+q} z^l Z^q}{(\gamma)_r(\beta-\gamma+1)_{r+s-q}\, r!\,(l-r)!\,(r+s-l)!\,(q-r)!}$$

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 3 (1905), pp. 111–123.

† It is convenient to take $\beta+n$ as the second element in order to retain the usual notation for Jacobi's polynomials.

$$= \sum_{l=0}^{n} \sum_{q=0}^{\infty} \sum_{r=0}^{q} \frac{(-n)_l (\beta+n)_l (\gamma+l+q)_{n-l} (-)^{n+q} z^l Z^q}{(\gamma)_r (\beta-\gamma+1)_{n-q} r! (l-r)! (q-r)!}$$

$$= \sum_{l=0}^{n} \sum_{q=0}^{\infty} \frac{(-n)_l (\beta+n)_l}{(\gamma)_l l!} z^l \frac{(\gamma+q)_n (\gamma-\beta)_{q-n}}{q!} Z^q$$

$$= (-)^n \frac{(\gamma)_n}{(\beta-\gamma+1)_n} F(-n, \beta+n; \gamma; z) \times F(\gamma-\beta-n, \gamma+n; \gamma; Z)$$

$$= (-)^n \frac{(\gamma)_n}{(\beta-\gamma+1)_n} (1-Z)^{\beta-\gamma} F(-n, \beta+n; \gamma; z) \times F(-n, \beta+n; \gamma; Z),$$

by a well-known transformation of hypergeometric functions; and this establishes the stated relation.

3. In order to establish one form of the general relation let us consider the expression

$$(1-Z)^{\alpha+\beta-\gamma} \left(\frac{1}{2\pi i}\right)^2 \int_{-\infty i}^{\infty i} \int_{-\infty i}^{\infty i} \Gamma(\alpha+s+t)\, \Gamma(\beta+s+t)\, \Gamma(1-\gamma-s) \times \Gamma(\gamma-\alpha-\beta-t)\, \Gamma(-s)\, \Gamma(-t) \{zZ\}^s \{(1-z)(1-Z)\}^t \, ds\, dt,$$

which is an absolutely convergent double integral, provided that

$$|\arg\{zZ\}| < 2\pi, \quad |\arg\{(1-z)(1-Z)\}| < 2\pi,$$

and it is supposed* that the contours have loops, if necessary, to ensure that the points $-\alpha, -\alpha-1, -\alpha-2, \ldots, -\beta, -\beta-1, -\beta-2, \ldots$ lie on the left of the s-contour, and the other poles of the integrand lie on the right of the contours.

We now define $-z$ and $-Z$ by the equations

$$-z = ze^{\pm\pi i}, \quad -Z = Ze^{\pm\pi i},$$

where $$|\arg(-z)| < \pi, \quad |\arg(-Z)| < \pi,$$

and then $$|\arg(zZ)| = |\arg(-z)+\arg(-Z)| < 2\pi.$$

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* It simplifies the argument if it is first supposed that α, β, γ, $\gamma-\alpha-\beta$ have positive real parts, and then at the end of the reasoning to use the theory of analytic continuation to remove these restrictions.

In the double integral we now make use of the formulæ

$$\Gamma(-t)(1-z)^t = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-\phi)\,\Gamma(\phi-t)(-z)^{\phi}\,d\phi,$$

$$\Gamma(\gamma-\alpha-\beta-t)(1-Z)^{\alpha+\beta-\gamma+t} = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-\psi)\Gamma(\psi-\alpha-\beta+\gamma-t)(-Z)^{\psi}\,d\psi,$$

whence it follows that the double integral is equal to

$$\left(\frac{1}{2\pi i}\right)^4 \int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i} \Gamma(\alpha+s+t)\,\Gamma(\beta+s+t)\,\Gamma(1-\gamma-s)\,\Gamma(-s)$$
$$\times\Gamma(-\phi)\,\Gamma(-\psi)\,\Gamma(\phi-t)\,\Gamma(\psi-\alpha-\beta+\gamma-t)(-z)^{s+\phi}\,(-Z)^{s+\psi}\,d\phi\,d\psi\,ds\,dt$$
$$= \left(\frac{1}{2\pi i}\right)^4 \int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i} \Gamma(\alpha+s+t)\,\Gamma(\beta+s+t)\,\Gamma(1-\gamma-s)\,\Gamma(-s)$$
$$\times\Gamma(s-\phi)\Gamma(s-\psi)\Gamma(\phi-s-t)\Gamma(\psi-\alpha-\beta+\gamma-s-t)(-z)^{\phi}(-Z)^{\psi}$$
$$d\phi\,d\psi\,ds\,dt$$
$$= \left(\frac{1}{2\pi i}\right)^3 \int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i} \Gamma(1-\gamma-s)\,\Gamma(-s)\,\Gamma(s-\phi)\,\Gamma(s-\psi)$$
$$\times\frac{\Gamma(\alpha+\phi)\,\Gamma(\beta+\phi)\,\Gamma(\psi+\gamma-\alpha)\,\Gamma(\psi+\gamma-\beta)}{\Gamma(\phi+\psi+\gamma)}(-z)^{\phi}\,(-Z)^{\psi}\,ds\,d\phi\,d\psi$$
$$= \left(\frac{1}{2\pi i}\right)^2 \int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i} \Gamma(-\phi)\,\Gamma(-\psi)\,\Gamma(1-\gamma-\phi)\,\Gamma(1-\gamma-\psi)$$
$$\times\Gamma(\alpha+\phi)\Gamma(\beta+\phi)\,\Gamma(\psi+\gamma-\alpha)\,\Gamma(\psi+\gamma-\beta)$$
$$\times\frac{\sin(\phi+\psi+\gamma)\pi}{\pi}(-z)^{\phi}(-Z)^{\psi}\,d\phi\,d\psi.$$

In each of the last two lines, Barnes' lemma,* that

$$\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(\alpha_1+w)\,\Gamma(\alpha_2+w)\,\Gamma(\beta_1-w)\,\Gamma(\beta_2-w)\,dw = \frac{\Gamma(\alpha_1+\beta_1)\,\Gamma(\alpha_2+\beta_2)\,\Gamma(\alpha_2+\beta_1)\,\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\beta_1+\beta_2)},$$

has been used.

We now evaluate the initial and final integrals by calculating the residues at the poles on the right of the contours, and after dividing by

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 6 (1908), pp. 154, 155.

$(1-Z)^{a+\beta-\gamma}$, we find that

$$\Gamma(a)\,\Gamma(\beta)\,(1-\gamma)\,\Gamma(\gamma-a-\beta)\,F_4[a, \beta;\ \gamma,\ a+\beta-\gamma+1;\ zZ,\ (1-z)(1-Z)]$$
$$+(zZ)^{1-\gamma}\,\Gamma(a-\gamma+1)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma-1)\,\Gamma(\gamma-a-\beta)$$
$$\times F_4[a-\gamma+1,\ \beta-\gamma+1;\ 2-\gamma,\ a+\beta-\gamma+1;\ zZ,\ (1-z)(1-Z)]$$
$$+\{(1-z)(1-Z)\}^{\gamma-a-\beta}\,\Gamma(\gamma-\beta)\,\Gamma(\gamma-a)\,\Gamma(1-\gamma)\,\Gamma(a+\beta-\gamma)$$
$$\times F_4[\gamma-\beta,\ \gamma-a;\ \gamma,\ \gamma-a-\beta+1;\ zZ,\ (1-z)(1-Z)]$$
$$+(zZ)^{1-\gamma}\{(1-z)(1-Z)\}^{\gamma-a-\beta}\,\Gamma(1-\beta)\,\Gamma(1-a)\,\Gamma(\gamma-1)\,\Gamma(a+\beta-\gamma)$$
$$\times F_4[1-\beta,\ 1-a;\ 2-\gamma,\ \gamma-a-\beta+1;\ zZ,\ (1-z)(1-Z)]$$
$$=\frac{\Gamma(a)\,\Gamma(\beta)}{\Gamma(\gamma)}\,\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)\,\Gamma(1-\gamma)\,F(a, \beta;\ \gamma;\ z)$$
$$\times(1-Z)^{\gamma-a-\beta}\,F(\gamma-\beta,\ \gamma-a;\ \gamma;\ Z)$$
$$+(zZ)^{1-\gamma}\,\frac{\Gamma(a-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(2-\gamma)}\,\Gamma(1-a)\,\Gamma(1-\beta)\,\Gamma(\gamma-1)$$
$$\times F(a-\gamma+1,\ \beta-\gamma+1;\ 2-\gamma;\ z)\,(1-Z)^{\gamma-a-\beta}\,F(1-\beta,\ 1-a;\ 2-\gamma;\ Z)$$
$$=\frac{\Gamma(a)\,\Gamma(\beta)}{\Gamma(\gamma)}\,\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)\,\Gamma(1-\gamma)\,F(a, \beta;\ \gamma;\ z)\,F(a, \beta;\ \gamma;\ Z)$$
$$+(zZ)^{1-\gamma}\,\frac{\Gamma(a-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(2-\gamma)}\,\Gamma(1-a)\,\Gamma(1-\beta)\,\Gamma(\gamma-1)$$
$$\times F(a-\gamma+1, \beta-\gamma+1;\ 2-\gamma;\ z)\,F(a-\gamma+1, \beta-\gamma+1;\ 2-\gamma;\ Z),$$

and this is an equation of the specified type.

4. If we had dealt in a similar manner with the integral

$$(1-Z)^{a+\beta-\gamma}\left(\frac{1}{2\pi i}\right)^2\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\frac{\Gamma(a+s+t)\,\Gamma(\beta+s+t)}{\Gamma(\gamma+s)}\,\Gamma(-s)\,z^s\,(-Z)^s$$
$$\times\Gamma(-t)\,\Gamma(\gamma-a-\beta-t)\,\{(1-z)(1-Z)\}^t\,ds\,dt,$$

which is convergent when

$$|\arg z+\arg(-Z)| < \pi, \quad |\arg(1-z)+\arg(1-Z)| < 2\pi,$$

we should have found it equal to

$$\left(\frac{1}{2\pi i}\right)^3\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\frac{\Gamma(-s)\,e^{\mp s\pi i}}{\Gamma(\gamma+s)}\,\Gamma(s-\phi)\,\Gamma(s-\psi)$$
$$\times\frac{\Gamma(a+\phi)\,\Gamma(\beta+\phi)\,\Gamma(\psi+\gamma-a)\,\Gamma(\psi+\gamma-\beta)}{\Gamma(\phi+\psi+\gamma)}\,(-z)^{\phi}\,(-Z)^{\psi}\,ds\,d\phi\,d\psi,$$

and, when $R(\gamma)$ is positive, this is equal to

$$\left(\frac{1}{2\pi i}\right)^2 \int_{-\infty i}^{\infty i} \int_{-\infty i}^{\infty i} \frac{\Gamma(-\phi)\,\Gamma(-\psi)}{\Gamma(\gamma+\phi)\,\Gamma(\gamma+\psi)}\,\Gamma(a+\phi)\,\Gamma(\beta+\phi)$$

$$\times\Gamma(\psi+\gamma-a)\,\Gamma(\psi+\gamma-\beta)(-z)^{\phi}\,(-Z)^{\psi}\,d\phi\,d\psi.$$

If we calculate the residues of the initial and final integrals at the poles on the right of the contours, and then divide by $(1-Z)^{a+\beta-\gamma}$, we find that

$$\frac{\Gamma(a)\,\Gamma(\beta)}{\Gamma(\gamma)}\,\Gamma(\gamma-a-\beta)\,F_4[a,\beta;\,\gamma,\,a+\beta-\gamma+1;\,zZ,\,(1-z)(1-Z)]$$

$$+\{(1-z)(1-Z)\}^{\gamma-a-\beta}\,\frac{\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)}\,\Gamma(a+\beta-\gamma)$$

$$\times F_4[\gamma-\beta,\,\gamma-a;\,\gamma,\,\gamma-a-\beta+1;\,zZ,\,(1-z)(1-Z)]$$

$$=\frac{\Gamma(a)\,\Gamma(\beta)\,\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)}{\{\Gamma(\gamma)\}^2}\,F(a,\beta;\,\gamma;\,z)\,F(a,\beta;\,\gamma;\,Z),$$

and the restriction that $R(\gamma)>0$ may now be removed by the theory of analytic continuation.

Since

$$F_4(a,\beta;\,\gamma,\gamma';\,\xi,\eta)=\sum_{m=0}^{\infty}\frac{(a)_m\,(\beta)_m}{(\gamma)_m\,m!}\,\xi^m F(a+m,\,\beta+m;\,\gamma';\,\eta),$$

the last result may be written in the form

$$\frac{\Gamma(a)\,\Gamma(\beta)\,\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)}{\{\Gamma(\gamma)\}^2}\,F(a,\beta;\,\gamma;\,z)\,F(a,\beta;\,\gamma;\,Z)$$

$$=\Gamma(\gamma-a-\beta)\sum_{m=0}^{\infty}\frac{\Gamma(a+m)\,\Gamma(\beta+m)}{\Gamma(\gamma+m)\,m!}(zZ)^m\,F[a+m,\,\beta+m,\,a+\beta-\gamma+1;\,(1-z)(1-Z)]$$

$$+\Gamma(a+\beta-\gamma)\sum_{m=0}^{\infty}\frac{\Gamma(\gamma-\beta+m)\,\Gamma(\gamma-a+m)}{\Gamma(\gamma+m)\,m!}(zZ)^m\,\{(1-z)(1-Z)\}^{\gamma-a-\beta}$$

$$\times F[\gamma-\beta+m,\,\gamma-a+m,\,\gamma-a-\beta+1;\,(1-z)(1-Z)].$$

If we combine corresponding terms of the series on the right, we find that they are expressible in terms of

$$F(a+m,\,\beta+m,\,\gamma+2m,\,z+Z-zZ),$$

so that we finally get

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}{\{\Gamma(\gamma)\}^2}\,F(\alpha,\,\beta;\;\gamma;\;z)\,F(\alpha,\,\beta;\;\gamma;\;Z)$$

$$=\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+m)\,\Gamma(\beta+m)\,\Gamma(\gamma-\alpha+m)\,\Gamma(\gamma-\beta+m)}{\Gamma(\gamma+m)\,\Gamma(\gamma+2m)\,m!}\,(zZ)^m$$

$$\times F(\alpha+m,\,\beta+m\,;\;\gamma+2m\,;\;z+Z-zZ).$$

We can therefore express the product as a double series, thus

$$F(\alpha,\,\beta\,;\;\gamma\,;\;z)\times F(\alpha,\,\beta\,;\;\gamma\,;\;Z)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\beta)_{m+n}\,(\gamma-\alpha)_m\,(\gamma-\beta)_m}{(\gamma)_m\,(\gamma)_{2m+n}\,m!\,n!}\,(zZ)^m\,(z+Z-zZ)^n.$$

The series on the right is not one of Appell's functions as it stands, but, as we have seen, it is expressible in terms of two functions of Appell's fourth type.