



## Philosophical Magazine Series 6

Publication details, including instructions for authors and subscription information:

<http://www.tandfonline.com/loi/tphm17>

### LIV. Wave propagation over parallel wires: The proximity effect

John R. Carson<sup>a</sup>

<sup>a</sup> Department of Development and Research,  
American Telephone and Telegraph Company

Version of record first published: 08 Apr 2009

To cite this article: John R. Carson (1921): LIV. Wave propagation over parallel wires: The proximity effect , Philosophical Magazine Series 6, 41:244, 607-633

To link to this article: <http://dx.doi.org/10.1080/14786442108636251>

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therefore be such as to satisfy the two equations :

$$\nabla^4 \psi = 0$$

$$\text{and} \quad \nabla^2 \psi = f(\psi),$$

where  $f$  is arbitrary, along with the appropriate boundary conditions. Such problems, of course, give a system of stream-lines in virtue of  $\nabla^4 \psi$ , which are not changed when the direction of motion of the body is reversed. The only known solutions of this type so far obtained are those of the steady motion of a fluid between the walls of a channel and the steady relative rotation of two concentric cylinders.

It may be remarked, in conclusion, that certain classes of problem, although violating the ordinary condition of steadiness, viz.  $\frac{\partial u}{\partial t}$  and  $\frac{\partial y}{\partial t} = 0$  everywhere, can still be reduced to that of a steady-motion case by the superposition of a uniform linear velocity or angular velocity upon the axes of reference. For example, if every point of a cylinder of any given shape describe a circular path inside and concentric with a given circular cylinder, the motion may be reduced to that of a steady case.

LIV. *Wave Propagation over Parallel Wires: The Proximity Effect.* By JOHN R. CARSON, *Department of Development and Research, American Telephone and Telegraph Company*\*.

#### I. Introduction.

THE importance of the problem dealt with in the present paper—wave propagation along a conducting system composed of two similar and equal parallel wires—has been emphasized by modern developments in telephonic transmission such as the carrier wave system of the American Telephone and Telegraph Company, and the utilization of loaded cable circuits in which the wires are in very close juxtaposition. For such systems, where the frequencies employed are relatively high and the wires very close together, considerable theoretical work has been found necessary to reduce the solution to a form available for immediate engineering use, in spite of the previous

\* Communicated by the Author.

valuable researches of such mathematicians as Mie\* and Nicholson†.

In the present paper the analysis of the problem starts with Maxwell's equations, but one simplifying assumption is introduced *ab initio*—namely, that the exponential propagation factor is a small quantity. The approximations involved in this assumption are fully justified in all physical systems which could actually be employed for the transmission of electrical energy; so that from a practical standpoint the restriction thus imposed on the generality of the solution is purely formal. By aid of this simplifying assumption the determination of the current distribution in the wires is essentially reduced to a two-dimensional problem, which is solvable from the boundary conditions satisfied by the *tangential magnetic force* and the *normal magnetic induction* at the surfaces of the wires. With the current distribution in the wires thus determined, the exponential propagation factor  $\gamma$  is solvable by applying the law  $\text{curl } \mathbf{E} = -\mu \frac{d}{dt} \mathbf{H}$  to an appropriate surface bounded by a contour which includes line elements in the surfaces of the wires. By this means it is shown that the propagation factor satisfies an equation of the form

$$\gamma^2 / ipK = 2Z + ipL,$$

where  $K$  is the electrostatic capacity between wires,  $Z$  the "impedance" of the wire per unit length, and  $L$  the inductance corresponding to the magnetic flux between the wires. This equation is of exactly the same form as that derivable from the *telegraph equation*, but differs therefrom in that  $Z$  and  $L$  are both functions of the frequency  $p/2\pi$  and the parameter  $k$  (ratio radius of wire to interaxial separation between wires). As formulated in the present paper, the actual calculation of  $Z$  and  $L$  involves only the computation of Bessel functions.

The method of solution sketched above and worked out

\* G. Mie, *Annalen der Physik*, vol. ii. pp. 201-249 (1900). In this paper the problem is attacked in a fundamental manner. The results arrived at are, however, limited to a restricted range of frequencies and the parameter  $k$  (ratio radius of wire to interaxial separation). Furthermore, Mie's method of attack does not admit of extension to other types of transmission systems in which the surfaces of the conductors are generated by lines parallel to the axis of transmission.

† J. W. Nicholson, *Phil. Mag.* vol. xvii. p. 255 (1909), and vol. xviii. p. 417 (1909). In these papers formulas are derived for the resistance and reactance of parallel wires which are valid for a very wide range of frequencies, but which are applicable only when the ratio of the radius of the wire to the interaxial separation between wires is a relatively small quantity.

in the following section of this paper has one substantial advantage which gives it an interest extending beyond the specific problem: it is quite generally applicable to problems in wave propagation along conducting systems in which the surfaces of the conductors are generated by lines parallel to the axis of propagation. For example, it has been successfully applied by the writer to the problem of wave propagation along a wire parallel to the plane surface of a semi-infinite solid of finite conductivity; the corresponding practical problem is, of course, transmission over a ground return circuit. Again, it has been applied to quantitatively investigate the effect of a concentric ring of iron armour wires on submarine cable transmission.

From an engineering standpoint the most important effect in parallel wire transmission is the dissipation of energy in non-magnetic wires. Consequently formulas for the alternating current resistance of the wire have been worked out in detail, and the functions involved have been computed and graphed, the data being collected in section III of this paper. The a.c. resistance of the wire is expressed in the form

$$R = CR_0,$$

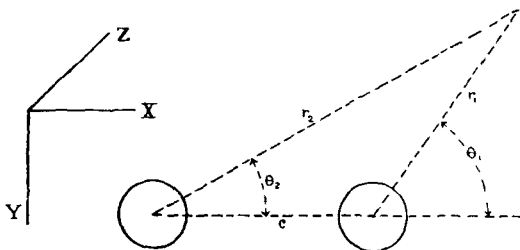
where  $R_0$  is the a.c. resistance of the wire *when the return conductor is concentric* (and is therefore calculable from well-known formulæ and tables), and  $C$  is a correction factor which formulates the modifying effect of the current in the adjacent wire. This is termed the *proximity effect correction factor*, following a usage suggested by Kennelly\*. The correction factor  $C$  approaches an upper limit  $C_m$ , which is a function of the parameter  $k$  only (ratio of radius to inter-axial separation between wires), which it approaches in accordance with an asymptotic formula derived from the asymptotic expansion of the Bessel functions involved. By aid of the data of section III the calculation of  $C$  is reduced to a very simple matter.

## II. Mathematical Analysis and Derivation of Formulae.

The conducting system under consideration, as stated, consists of two long similar and equal parallel wires of circular cross-section, in which equal and opposite currents are flowing. The radius of the wire is denoted by  $a$ , its conductivity and permeability by  $\lambda$  and  $\mu$  respectively, and

\* Kennelly, Laws, and Pierce, Proceedings A.I.E.E. 1915, pp. 1749-1813.

the interaxial separation between wires by  $c$ . The co-ordinates of any point in the system with respect to the axis of one wire are denoted by  $r_1, \theta_1$ , and the co-ordinates of the same point with respect to the axis of the second or return wire by  $r_2, \theta_2$ , as shown in the sketch herewith.



Before proceeding with the analysis of the specific problem, a very brief discussion of the fundamental field equations will be given, in order to indicate the significance of certain important simplifying assumptions employed in the subsequent analysis and the restrictions thus imposed on the generality of the solution. It may be remarked that these simplifying assumptions are quite generally applicable to problems in wave propagation where the surfaces of the conductors are generated by lines parallel to the axis of propagation. The discussion starts with Maxwell's equations in a continuous medium:

$$\left. \begin{aligned} \text{curl } \mathbf{E} &= -\mu i p \mathbf{H}, \\ \text{curl } \mathbf{H} &= (4\pi\lambda + \mathbf{K} i p) \mathbf{E}, \\ \text{div. } \mathbf{E} &= 0, \\ \text{div. } \mathbf{H} &= 0. \end{aligned} \right\} \dots \text{(I.)}$$

In these equations  $\mathbf{E}$  and  $\mathbf{H}$  denote the electric and magnetic forces, while  $\lambda$ ,  $\mu$ , and  $\mathbf{K}$  are the conductivity, permeability, and specific inductive capacity of the medium. It is assumed throughout the following that *elm. c.g.s. units* are employed. The axis of propagation will be taken as the axis of  $Z$ , and it will be assumed that the electric and magnetic forces vary as  $\exp(ipt - \gamma z)$ ; consequently the frequency is  $p/2\pi$ ,  $\gamma$  is the propagation factor, and the operators  $d/dt$  and  $\partial/\partial z$  are replaceable by  $ip$  and  $-\gamma$  respectively. All six vector components ( $E_{xyz}$ ,  $H_{xyz}$ ) satisfy the wave equation

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2) \phi = -(m^2 + \gamma^2) \phi, \dots \text{(II.)}$$

where

$$m^2 = -(4\pi\lambda\mu ip - (p/v)^2) \quad \text{and} \quad v = 1/\sqrt{\mathbf{K}\mu}.$$

It will be found convenient to write the field equations in the form :

$$(m^2 - \gamma^2)H_x = -\frac{m^2}{\mu ip} \frac{\partial}{\partial y} E_z + \gamma \frac{\partial}{\partial x} H_z, \quad . \quad . \quad . \quad (1)$$

$$(m^2 - \gamma^2)H_y = \frac{m^2}{\mu ip} \frac{\partial}{\partial x} E_z + \gamma \frac{\partial}{\partial y} H_z, \quad . \quad . \quad . \quad (2)$$

$$(m^2 - \gamma^2)E_x = \gamma \frac{\partial}{\partial x} E_z + \mu ip \frac{\partial}{\partial y} H_z, \quad . \quad . \quad . \quad (3)$$

$$(m^2 - \gamma^2)E_y = \gamma \frac{\partial}{\partial y} E_z + \mu ip \frac{\partial}{\partial x} H_z. \quad . \quad . \quad . \quad (4)$$

From equations (3) and (4),

$$-\mu ip H_z = \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x. \quad . \quad . \quad . \quad . \quad (5)$$

We now introduce the assumption, essential to the subsequent analysis, that  $\gamma$  and  $p/v$  are very small quantities of comparable orders of magnitude. That is to say, they are very small compared with unity and also compared with the value of  $m$  in the conductors. The justification for these assumptions and their immediate corollaries, introduced *ab initio*, resides in the fact that the solution obtained by their aid actually satisfies the necessary conditions in transmission systems of ordinary dimensions, even if the frequency exceeds a million cycles per second.

From equations (3), (4), and (5) it follows that the electric force in the plane normal to the axis of propagation is of the order of magnitude of  $\gamma/(m^2 - \gamma^2)$  compared with the axial component  $E_z$ . In the conductors this is a very small quantity of the order of magnitude of  $\gamma/4\pi\lambda\mu p$ , while in the dielectric it is a large quantity of the order of magnitude of  $1/\gamma$ . Consequently in the conductors the electric force in the plane normal to the axis of propagation will be ignored in comparison with  $E_z$ ; in the dielectric, however, the former is large compared with the latter. By corresponding considerations the axial magnetic force  $H_z$  is very small compared with the magnetic force in the plane  $XY$ , both in the conductors and in the dielectric.

As a consequence of the foregoing, the magnetic force in the conductors is derivable from

$$\mu ip H_x = -\frac{\partial}{\partial y} E_z, \quad . \quad . \quad . \quad . \quad (6)$$

$$\mu ip H_y = \frac{\partial}{\partial x} E_z, \quad . \quad . \quad . \quad . \quad (7)$$

which replace (1) and (2).

We are now prepared to take up the analysis of the problem of wave propagation along parallel wires; in the course of this analysis the significance and utility of the simplifying assumptions will become more apparent.

From the general solution of the wave equation in polar co-ordinates and the special conditions of symmetry which obtain, the axial electric force in wire #1 is given by the Fourier-Bessel expansion

$$E_z = \sum_0^{\infty} A_n J_n(\rho_1) \cos n\theta_1, \quad . \quad . \quad . \quad (8)$$

and in wire #2 by

$$E_z = -\sum_0^{\infty} (-1)^n A_n J_n(\rho_2) \cos n\theta_2, \quad . \quad . \quad . \quad (9)$$

where

$$\rho_1 = ir_1 \sqrt{4\pi\lambda\mu ip}, \quad \rho_2 = ir_2 \sqrt{4\pi\lambda\mu ip}.$$

In these equations  $J_n(\rho)$  is the Bessel function of order  $n$  and argument  $\rho$ , and the coefficients  $A_0 \dots A_n$  are to be determined from the boundary conditions at the surfaces of the wires. In either wire the magnetic force is then derivable from

$$\left. \begin{aligned} \mu ip H_\theta &= \frac{\partial}{\partial r} E_z, \\ \mu ip H_r &= -\frac{1}{r} \frac{\partial}{\partial \theta} E_z, \end{aligned} \right\} \quad . \quad . \quad . \quad (10)$$

where  $r, \theta$  denote either  $r_1, \theta_1$  or  $r_2, \theta_2$  according as wire #1 or wire #2 is under consideration. From the symmetry of the system, however, the satisfaction of the boundary conditions imposed at the surface of one wire insures their satisfaction at the surface of the other.

In the dielectric the electric and magnetic forces are expressible as Fourier-Bessel expansions, the Bessel functions, however, being of the "external" or second kind. In accordance with the assumption, however, that  $\gamma$  is a small quantity of the order of magnitude of  $p/v$ , it follows that so long as  $pc/v$  (where  $c$  is the separation between wires) is a small quantity compared with unity, the Bessel functions in the neighbourhood of the wires may be replaced by the limiting forms which they assume for vanishingly small arguments. In particular, the magnetic forces  $H_x$  and  $H_y$  in the neighbourhood of the wires



are expressible as

$$H_x = \sum_{n=1}^{\infty} B_n \left\{ \frac{\sin n\theta_1}{r_1^n} + (-1)^n \frac{\sin n\theta_2}{r_2^n} \right\},$$

$$H_y = \sum_{n=1}^{\infty} C_n \left\{ \frac{\cos n\theta_1}{r_1^n} + (-1)^n \frac{\cos n\theta_2}{r_2^n} \right\}.$$

The magnetic force in the dielectric is thus expressed in terms of two symmetrical waves centred on the axes of the two wires respectively.

From the equation  $\text{div. } H = 0$  it follows that

$$\frac{\partial}{\partial x} H_x + \frac{\partial}{\partial y} H_y$$

differs from zero only by  $\gamma H_z$ , which is a very small quantity since both  $\gamma$  and  $H_z$  are small. With very small error we may therefore write

$$\frac{\partial}{\partial x} H_x + \frac{\partial}{\partial y} H_y = 0,$$

which determines the relation between the B and C coefficients of (11) and gives

$$H_x = \sum_{n=1}^{\infty} B_n \left\{ \frac{\sin n\theta_1}{r_1^n} + (-1)^n \frac{\sin n\theta_2}{r_2^n} \right\}, \quad \dots \quad (12)$$

$$H_y = -\sum_{n=1}^{\infty} B_n \left\{ \frac{\cos n\theta_1}{r_1^n} + (-1)^n \frac{\cos n\theta_2}{r_2^n} \right\}. \quad \dots \quad (13)$$

In the dielectric the electric forces satisfy the wave equation II, and are therefore expressible as two Fourier-Bessel expansions oriented on the axes of the two wires. In accordance with our assumption, however, that  $\gamma$  and  $p/v$  are very small quantities, the Bessel functions are replaceable in the neighbourhood of the wires by the limiting forms which they assume for vanishingly small arguments. The same result is arrived at if we take  $E_x$  and  $E_y$  as satisfying the equations

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2) E_x = 0,$$

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2) E_y = 0.$$

Furthermore, from the relative magnitudes of  $E_z$  and the electric force in the plane XY in the dielectric, the equation  $\text{div. } E = 0$  may with very slight error be replaced by

$$\frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y = 0.$$

These equations are satisfied if we introduce a function  $V$  which satisfies the equation

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2)V = 0$$

and then derive  $E_x$  and  $E_y$  from  $V$  in accordance with

$$E_x = -\frac{\partial}{\partial x} V,$$

$$E_y = -\frac{\partial}{\partial y} V.$$

Now, at the surfaces of the wires the tangential electric force in the plane  $XY$ , which is continuous, is very small compared with the normal component. Consequently very small error is introduced if in determining  $V$  it is taken as constant over the circumferences of the wires in the plane  $XY$ . It follows at once that

$$V = V_0 e^{(ipt - \gamma z)}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where  $V_0$  is the electrostatic potential and the surfaces of the wires are equipotential surfaces. The determination of  $E_x$  and  $E_y$  in the dielectric is therefore reduced to a two-dimensional electrostatic problem, in which the surfaces of the two wires are equipotential surfaces.

The solution of our problem—namely, the determination of  $\gamma$  and the coefficients  $A_0 \dots A_n$  and  $B_1 \dots B_n$  of equations (8) and (12)—is obtained by formulating and satisfying the boundary conditions which obtain at the surfaces of the wires. These are that the tangential electric and magnetic forces are continuous. The current distribution in the wire, which carries with it its alternating current resistance, is, however, determinable by a less general statement of the boundary conditions; namely, that *the tangential magnetic forces and the normal magnetic induction are continuous*. With the current distribution in the wire determined, the propagation factor  $\gamma$  is determined without difficulty, as is shown subsequently.

Before proceeding with the determination of the coefficients  $A_0 \dots A_n$  of equation (8), the alternating current resistance of the wire will be formulated. Let the value of  $\rho_1$  at the surface of wire #1 be denoted by

$$\xi = bi\sqrt{i} = ia\sqrt{4\pi\lambda\mu ip},$$

and let

$$A_n = h_n A_0 \quad (n = 1, 2, 3 \dots).$$

Then, omitting the subscript in  $\theta_1$ , the axial electric force at the surface of wire #1 is

$$A_0(J_0(\xi) + h_1 J_1(\xi) \cos \theta + h_2 J_2(\xi) \cos 2\theta + \dots), \quad (15)$$

and the value of the tangential magnetic force  $H_\theta$  at the surface of the wire is by (10):

$$\frac{\xi A_0}{a\mu ip} (J_0'(\xi) + h_1 J_1'(\xi) \cos \theta + h_2 J_2'(\xi) \cos 2\theta + \dots). \quad (16)$$

Since  $4\pi$  times the total current  $I$  flowing in the wire is equal to the line integral of the magnetic force  $H_\theta$  around the circumference of the wire, it follows at once that

$$\frac{\xi J_0'(\xi)}{\mu ip} A_0 = 2I, \quad . . . . . \quad (17)$$

which determines the fundamental coefficient  $A_0$  in terms of the current in the wire.

The resistance  $R$  of the wire per unit length is conveniently defined as the mean dissipation per unit length, divided by the mean square current. The dissipation  $W$  in the wire is very conveniently and simply formulated by Poynting's theory of the energy-flow in the electromagnetic field, which, applied to the present problem, gives

$$W = \frac{a}{4\pi} \int_0^{2\pi} E_z H_\theta d\theta, \quad . . . . . \quad (18)$$

where  $E_z$  and  $H_\theta$  are the values at the surface of the wire, as given by (15) and (16). If these series are substituted for  $E_z$  and  $H_\theta$  in (18) and the value of  $A_0$  is taken from (17), and if the resulting expressions are realized, it follows without difficulty that

$$R = R_0 \left\{ 1 + \frac{1}{2} \sum_{n=1}^{\infty} |h_n|^2 \frac{u_n v_n' - u_n' v_n}{u_0 v_0' - u_0' v_0} \right\}, \quad . . \quad (19)$$

where

$$u_n + iv_n = J_n(\xi) = J_n(bi\sqrt{i}),$$

$$u_n' + iv_n' = \frac{d}{db} J_n(bi\sqrt{i}).$$

$R_0$  denotes the a.c. resistance of the wire when the coefficients  $h_1 \dots h_n$  are all zero; that is,  $R_0$  is the resistance of the wire where the return wire is concentric, which is calculable from well-known formulæ and tables. The functions  $u_n$  and  $v_n$ , it will be observed, correspond precisely with the well-known ber and bei functions, which are

similarly derived from the Bessel function of zero order and complex argument  $bi\sqrt{i}$ . From (19) the *proximity effect correction factor*  $C$  is given by

$$C = 1 + 1/2 \sum_{n=1}^{\infty} |h_n|^2 \frac{u_n v_n' - u_n' v_n}{u_0 v_0' - u_0' v_0} \quad \dots \quad (20)$$

By aid of formulæ (19) and (20) the a.c. resistance of the wire is calculable, once the coefficients  $h_1 \dots h_n$  or  $A_1 \dots A_n$  are determined; to this determination we now proceed.

As stated and discussed above, the harmonic coefficients are determined by the continuity of the tangential magnetic force and the normal magnetic induction at the surfaces of the wires; that is, by the continuity of  $H_{\theta 1}$  and  $\mu H_{r1}$  at  $r_1 = a$ , and of  $H_{\theta 2}$  and  $\mu H_{r2}$  at  $r_2 = a$ . From considerations of symmetry, however, these boundary conditions need be formulated at the surface of one wire only, and their satisfaction at the surface of either wire insures their satisfaction at the surface of the other. To formulate these conditions at the surface of wire #1, we require that the tangential and normal components of the magnetic force at the surface of this wire be expressed in terms of  $r_1$  and  $\theta_1$  only, whereas  $H_x$  and  $H_y$  of formulæ (12) and (13) are expressed in terms of both  $r_1, \theta_1$  and  $r_2, \theta_2$ . As a preliminary, we therefore require the expansion of  $H_x$  and  $H_y$ , as given by equations (12) and (13) in terms of  $r_1$  and  $\theta_1$  alone. This is effected by the following transformations:—

$$\frac{\cos s\theta_2}{r_2^s} = \frac{1}{c^s} \left\{ 1 - \frac{s}{1!} (r_1/c) \cos \theta_1 + \frac{(s)(s+1)}{2!} (r_1/c)^2 \cos 2\theta_1 \right. \\ \left. - \frac{(s)(s+1)(s+2)}{3!} (r_1/c)^3 \cos 3\theta_1 \dots \right\} \quad (21)$$

$$\frac{\sin s\theta_2}{r_2^s} = \frac{1}{c^s} \left\{ \frac{s}{1!} (r_1/c) \sin \theta_1 - \frac{(s)(s+1)}{2!} (r_1/c)^2 \sin 2\theta_1 \right. \\ \left. + \frac{(s)(s+1)(s+2)}{3!} (r_1/c)^3 \sin 3\theta_1 \dots \right\} \quad (21a)$$

(It may be remarked in passing that these transformations may be very advantageously employed in calculating the capacity coefficients of a system of parallel cylinders.)

If these transformations are substituted in (12) and (13),  $H_x$  and  $H_y$  in the dielectric are expressed entirely in terms of  $r_1$  and  $\theta_1$ , or, omitting subscripts, in terms of  $r$  and  $\theta$ . If we now employ the relations

$$H_r = H_x \cos \theta + H_y \sin \theta,$$

$$H_\theta = H_y \cos \theta - H_x \sin \theta,$$

we get, after rearrangement and simplification, the following infinite series for the tangential and normal components of the magnetic force in the dielectric at the surface of wire #1 :—

$$\begin{aligned} H_{\theta} = & -B_1/a - \cos \theta (B_2/a^2 - \Sigma_0) \\ & - \cos 2\theta \left( B_3/a^3 + \frac{1}{1!} (a/c) \Sigma_1 \right) \\ & - \cos 3\theta \left( B_4/a^4 - \frac{1}{2!} (a/c)^2 \Sigma_2 \right) \\ & - \dots \dots \dots (22) \end{aligned}$$

$$\begin{aligned} H_r = & \sin \theta (B_2/a^2 + \Sigma_0) \\ & + \sin 2\theta \left( B_3/a^3 - \frac{1}{1!} (a/c) \Sigma_1 \right) \\ & + \sin 3\theta \left( B_4/a^4 + \frac{1}{2!} (a/c)^2 \Sigma_2 \right) \dots (23) \end{aligned}$$

In these expressions the  $\Sigma$ 's denote the following infinite series :—

$$\left. \begin{aligned} \Sigma_0 &= B_1/c - B_2/c^2 + B_3/c^3 - \dots, \\ \Sigma_1 &= B_1/c - 2B_2/c^2 + 3B_3/c^3 - 4B_4/c^4 - \dots, \\ \Sigma_2 &= 1.2.B_1/c - 2.3.B_2/c^2 + 3.4.B_3/c^3 - \dots, \\ &\dots \dots \dots \\ \Sigma_n &= n! B_1/c - \frac{(n+1)!}{1!} B_2/c^2 + \frac{(n+2)!}{2!} B_3/c^3 - \dots \end{aligned} \right\} (24)$$

From (10), (15), and (17) the tangential and normal components of the magnetic force at the surface of the wire are, in terms of the internal solution and the current  $I$  in the wire :

$$H_{\theta} = (2I/a) \left( 1 + \frac{J_1'}{J_0'} h_1 \cos \theta + \frac{J_2'}{J_0'} h_2 \cos 2\theta + \dots \right), \quad (25)$$

$$H_r = (2I/a) (1/\xi J_0') (J_1 h_1 \sin \theta + 2J_2 h_2 \sin 2\theta + \dots), \quad (26)$$

where the argument of the Bessel functions  $J_0 \dots J_n$  and  $J_0' \dots J_n'$  is  $\xi = ai\sqrt{4\pi\lambda\mu ip}$ .

The boundary condition of the continuity of  $H_{\theta}$  at the

surface of the wire gives by direct equation of corresponding terms of (22) and (25) :

$$\left. \begin{aligned} 2I/a &= -B_1/a, \\ B_1 &= -2I, \end{aligned} \right\} \cdot \cdot \cdot \cdot \cdot \quad (27)$$

and

$$(2I/a) \frac{J_n'}{J_0'} h_n = -B_{n+1}/a^{n+1} + \frac{(-1)^{n-1}}{(n-1)!} (a/c)^{n-1} \Sigma_{n-1}, \quad (28)$$

$$n = 1, 2, 3, \dots$$

Similarly, the boundary condition of the continuity of the normal magnetic induction applied to (23) and (26) gives :

$$(2I/a) \frac{n\mu J_n}{\xi J_0'} h_n = B_{n+1}/a^{n+1} + \frac{(-1)^{n-1}}{(n-1)!} (a/c)^{n-1} \Sigma_{n-1}, \quad (29)$$

$$n = 1, 2, 3, \dots$$

From (28) and (29) :

$$B_{n+1} = -a^n I h_n \frac{\xi J_n' - n\mu J_n}{\xi J_0'} \cdot \cdot \cdot \cdot \cdot \quad (30)$$

and

$$(I/a) \frac{\xi J_n' + n\mu J_n}{\xi J_0'} h_n = \frac{(-1)^{n-1}}{(n-1)!} (a/c)^{n-1} \Sigma_{n-1} \cdot \cdot \cdot \quad (31)$$

It is now convenient to introduce the following notation :

$$\left. \begin{aligned} \sigma_n &= (\xi J_n' - n\mu J_n) / \xi J_0', \\ \rho_n &= (\xi J_n' - n\mu J_n) / (\xi J_n' + n\mu J_n), \\ q_n &= \sigma_n h_n, \\ a/c &= k. \end{aligned} \right\} \cdot \cdot \cdot \quad (32)$$

In terms of this notation it follows from (30) that

$$B_{n+1} = -a^n q_n I \cdot \cdot \cdot \cdot \cdot \quad (33)$$

If the  $B$  coefficients in the  $\Sigma$  functions as defined by (24) are replaced by their values as given by (27) and (33), it is easy to show that equations (31) may be written as

$$q_n = (-1)^n 2\rho_n k^n - \frac{(-1)^n}{(n-1)!} \rho_n k^n \left( \frac{n!}{1!} k q_1 - \frac{(n+1)!}{2!} k^2 q_2 + \dots \right),$$

which may conveniently be written as

$$q_n = (-1)^n 2\rho_n k^n - \frac{(-1)^n}{(n-1)!} \rho_n k^n \Sigma_n(q), \quad \cdot \cdot \cdot \quad (34)$$

$$n = 1, 2, 3, \dots$$

Equations (34) constitute an infinite system of equations in the infinitely many variables  $q_1 \dots q_n$ , and on their solution depends the determination of the harmonic coefficients  $h_1 \dots h_n$ .

The solution of (34) is to be obtained by some process of successive approximation. For example, a formal solution is gotten by taking  $q_1 \dots q_n$  as the limit of the sequences :

$$\begin{array}{ccccccc} q_1^{(0)}, & q_1^{(1)}, & q_1^{(2)}, & q_1^{(3)}, & \dots & q_1^{(s)} & \dots \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_n^{(0)}, & q_n^{(1)}, & q_n^{(2)}, & q_n^{(3)}, & \dots & q_n^{(s)} & \dots \end{array}$$

where the successive terms of the sequences are defined by the relations :

$$q_n^{(0)} = (-1)^n 2\rho_n k^n, \quad n = 1, 2, 3 \dots$$

$$\text{and} \quad q_n^{(s+1)} = (-1)^n 2\rho_n k^n - \frac{(-1)^n}{(n-1)!} \rho_n k^n \sum_n (q^{(s)}).$$

The method of solution results in a convergent sequence provided the parameter  $k$  is less than its limiting value  $1/2$ , and for values of  $k$  likely to be encountered in practice a very rapidly convergent sequence.

Another method of successive approximations which may often be advantageously employed may be termed the method of successive ignorations. This consists in first ignoring all the variables except  $q_1$  and determining its first approximate value  $-2\rho_1 k$  from the first equation of the system. A second and higher approximation is then gotten by retaining  $q_1$  and  $q_2$  and evaluating them from the first two equations. A third and still higher approximation results from retaining  $q_1$ ,  $q_2$ , and  $q_3$  and solving for them from the first three equations. This process is to be continued until the convergence of the sequence is evident. This latter method of solution likewise results in a convergent sequence, and works very well in practice unless the parameter is too close to its limiting value.

While some such process of approximation is to be employed in the general case, and indeed has been successfully applied by the writer to several similar problems, a simpler method of solution fortunately suggests itself in the special case of greatest practical importance—namely, when the wires are composed of non-magnetic metals and, in consequence, the permeability  $\mu$  is equal to unity. The resulting formulæ have the added advantage of being

asymptotic in character, and consequently give the values of  $q_1 \dots q_n$  with increasing precision in the practically important range of values. It should be remarked that the formulæ now to be derived constitute asymptotic solutions also when  $\mu$  is greater than unity; they cannot, however, be safely applied when the permeability is large unless the frequency is very high.

Restricting attention, therefore, to the case where  $\mu=1$ , we observe that the functions  $\rho_n$  and  $\sigma_n$  of (32) may be written as

$$\begin{aligned}\sigma_n &= J_{n+1}/J_1, \\ \rho_n &= -J_{n+1}/J_{n-1}.\end{aligned}$$

These identities follow from the definitions of equations (32) and well-known properties of Bessel functions. We know also that when the argument  $\xi$  is large compared with the order  $n$ , the function  $\rho_n$  becomes closely equal to its limiting value unity. We are therefore led to consider the auxiliary system of equations in the auxiliary variables  $p_1 \dots p_n$ , which is obtained from (34) by replacing the functions  $\rho_1 \dots \rho_n$  therein by their common limit unity. That is, we define the auxiliary variables  $p_1 \dots p_n$  by the following system of equations:—

$$p_n = (-1)^n 2k^n - \frac{(-1)^n}{(n-1)!} k^n \Sigma_n(p), \quad . . . \quad (35)$$

$$(n = 1, 2, 3, \dots).$$

Now, since the functions  $\rho_1 \dots \rho_n$  approach the common limit unity as the argument  $\xi$  approaches infinity, it is evident that  $p_1 \dots p_n$  are simply the limiting values assumed by the variables  $q_1 \dots q_n$  when the wire is of infinite conductivity. From the known surface distribution of magnetic force in this case the following solution of equations (35) at once suggests itself, and may be readily verified; the variables  $p_1 \dots p_n$  are simply the Fourier coefficients of the expansion

$$\frac{1}{1+2k \cos \theta} = K(1+p_1 \cos \theta + p_2 \cos 2\theta + \dots). \quad (36)$$

From (36) it is easy to show that

$$\left. \begin{aligned} p_1 &= -2ks, \\ p_n &= (-1)^n 2k^n s^n, \end{aligned} \right\} . . . . . \quad (37)$$



where  $s$  is the series ratio

$$s = \frac{1 + \frac{3}{1!}k^2 + \frac{4 \cdot 5}{2!}k^4 + \frac{6 \cdot 7 \cdot 8}{3!}k^6 + \dots}{1 + \frac{2}{1!}k^2 + \frac{3 \cdot 4}{2!}k^4 + \frac{5 \cdot 6 \cdot 7}{3!}k^6 + \dots} \\ = 2 \frac{1 - \sqrt{1 - (2k)^2}}{(2k)^2} \dots \dots \dots (38)$$

Having thus solved equations (35) for the variables  $p_1 \dots p_n$ , it is easy to show from (34) and (35) that if we write

$$q_n = p_n + d_n,$$

the variables  $d_1 \dots d_n$  satisfy the system of equations:

$$d_n = (\rho_n - 1)p_n - \frac{(-1)^n}{(n-1)!} \rho_n k^n \Sigma_n(d). \quad \dots \quad (39)$$

The system of equations (39) in  $d_1 \dots d_n$  admit of solution by successive approximations, as discussed in connexion with the solution of the corresponding system of equations (34) in  $q_1 \dots q_n$ . For the important case of non-magnetic conductors, however, a very close approximation to the exact solution is obtained by replacing (39) by the approximations:

$$d_n = (\rho_n - 1)p_n + (-1)^n \rho_n k^{n+1} d_1. \quad \dots \quad (40)$$

This gives to the same order of approximation

$$q_n = \rho_n p_n + (-1)^n \rho_n (1 - \rho_1) k^{n+1} p_1. \quad \dots \quad (41)$$

Since by (32)  $h_n = q_n / \sigma_n$ , this gives for non-magnetic conductors

$$h_n = p_n \frac{J_1}{J_{n-1}} (1 - n(1 - \rho_1) k^2 / s^{n-1}) \\ = p_n \frac{J_1}{J_{n-1}} \left( 1 - 2n \frac{k^2}{s^{n-1}} \frac{J_1}{\xi J_0} \right). \quad \dots \quad (42)$$

We are now in a position to formulate the proximity effect correction factor  $C$  of equation (20), which involves the harmonic coefficient  $h_1 \dots h_n$ . From (42) to the same order of approximation as (40)

$$|h_n|^2 = p_n^2 \frac{u_1^2 + v_1^2}{u_{n-1}^2 + v_{n-1}^2} (1 + 2ngk^2 / s^{n-1}),$$

where  $g$  denotes the function

$$g = \frac{\sqrt{2}}{b} \frac{u_1(u_0 + v_0) - v_1(u_0 - v_0)}{u_0^2 + v_0^2}. \quad \dots \quad (43)$$

It will be remembered, of course, that

$$b = a\sqrt{4\pi\lambda\mu p}$$

$$\text{and } J_n(i\sqrt{i}b) = u_n + iv_n.$$

If the foregoing is substituted in (20), some easy simplifications give

$$C = 1 + \frac{2}{aR_0} \sqrt{\frac{\mu p}{\pi\lambda}} (S_1 - 2gk^2S_2), \quad . \quad . \quad . \quad (44)$$

where

$$S_1 = \sum_{n=1}^{\infty} w_n k^{2n} s^{2n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$S_2 = \sum_{n=1}^{\infty} n w_n k^{2n} s^{2n} s^{n+1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$R_0 = \frac{1}{a} \sqrt{\frac{\mu p}{\pi\lambda}} \frac{u_0 v_0' - u_0' v_0}{u_1^2 + v_1^2}, \quad . \quad . \quad . \quad . \quad . \quad (47)$$

= resistance of wire with concentric return,

and  $w_n$  is the auxiliary function

$$w_n = \frac{u_n v_n' - u_n' v_n}{u_{n-1}^2 - v_{n-1}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Equation (44) is the formula for the correction factor  $C$  for non-magnetic conductors, the evaluation of which is discussed in section III. For the purposes of numerical calculation, the following asymptotic expressions are useful. For values of the argument  $b$  equal to or greater than 5,

$$g \sim \sqrt{2}/b, \\ \frac{1}{aR_0} \sqrt{\frac{\mu p}{\pi\lambda}} \sim \sqrt{2} - 1/b. \quad . \quad . \quad . \quad . \quad (49)$$

Consequently for  $b \geq 5$ ,

$$C \doteq 1 + 2(\sqrt{2} - 1/b) \left( S_1 - \frac{2\sqrt{2}}{b} k^2 S_2 \right). \quad . \quad . \quad (50)$$

From the asymptotic expansions of  $J_n$  it is easy to show that for  $b \geq n^2$

$$w_n \sim 1/\sqrt{2} - (2n-1)/2b. \quad . \quad . \quad . \quad . \quad (51)$$

If this is substituted for  $w_n$  in  $S_1$  and  $S_2$  of equation (44), some easy simplifications give

$$C \sim C_m(1 - A/b), \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where

$$C_m = \frac{1 + k^2 s^2}{1 - k^2 s^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

and

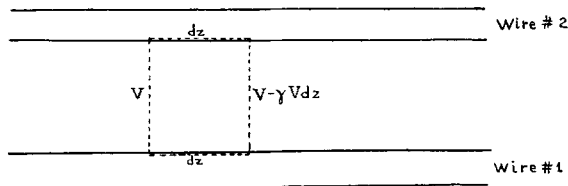
$$A = 2\sqrt{2} \frac{k^2 s^2}{1 - k^4 s^4} \left[ 1 + 2k^2 \left( \frac{1 - k^2 s^2}{1 - k^2 s^2} \right)^2 \right] \quad . \quad (54)$$

The limiting value of the correction factor is therefore  $C_m$ , and this is a function only of the parameter  $k = a/c$ . The asymptotic formula (54) can be used under the following conditions and in accordance with the following rule:—If the series  $\sum k^{2n} s^{2n}$  converges to a required order of approximation in a finite number of terms  $n$ , then the correction factor  $C$  may be calculated from (53) provided the argument  $b$  is such that

$$b > n^2 > 5.$$

The correction-factor formulæ need not be further considered here, as they are fully discussed in section III. We shall therefore now proceed to complete the solution of the problem by formulating the propagation factor  $\gamma$ . In this discussion it will be assumed that  $h_1 \dots h_n$  and  $q_1 \dots q_n$  have been evaluated in accordance with the methods fully discussed above. For non-magnetic wires they may be calculated from (41) and (42), while in other cases any of the methods of successive approximations discussed above may be applied to equations (34).

The propagation factor  $\gamma$  is determined by applying the law  $\text{curl } \mathbf{E} = -\mu i p \mathbf{H}$  to any appropriate surface, the contour of which includes a line segment  $dz$  in the surface of each wire and two lines in the dielectric joining their corresponding ends. The most convenient surface to take is a plane surface in the plane of the axes bounded by the



elements  $dz$  in the inner or adjacent surfaces of the wires and the straight lines in the dielectric joining the corresponding ends of the elements  $dz$ , as shown in the cross-section sketch herewith.

To apply the law  $\text{curl } \mathbf{E} = -\mu i p \mathbf{H}$  to this surface, it is only necessary to calculate the magnetic flux through the surface and the line integral of the electric force around the contour. The contribution to the line integral from the elements  $dz$  is, by (15) and (17),

$$dz \frac{2\mu i p J_0}{\xi J_0'} (2I) \left( 1 - h_1 \frac{J_1}{J_0} + h_2 \frac{J_2}{J_0} - \dots \right),$$

which may be written as

$$2IZdz = 2Z_0 I dz \left( 1 - h_1 \frac{J_1}{J_0} + h_2 \frac{J_2}{J_0} - \dots \right), \quad \dots \quad (55)$$

where  $Z_0 = 2\mu i p J_0 / \xi J_0'$  and the argument of the Bessel functions is  $\xi = ia \sqrt{4\pi\lambda\mu i p}$ .  $Z_0$  is the "internal" or "self-impedance" of the wire when the return wire is either concentric or at such a distance as to make the proximity effect negligible.

To calculate the contribution to the contour integral of the lines in the dielectric joining the corresponding ends of the segments  $dz$ , it will be recalled that the electric force in the dielectric in the plane  $XY$  is derivable as the gradient of a scalar or electrostatic potential, as given in equation (14). Consequently the contributions of these lines are simply

$$\left( V_0 + \frac{\partial}{\partial z} V_0 - V_0 \right) dz = -\gamma V_0 dz,$$

where  $V_0$  is the *electrostatic potential between the two wires*. If  $K$  denote the electrostatic capacity between the two wires, then

$$V_0 = \frac{\gamma}{ipK} I \quad \dots \quad (56)$$

and

$$-\gamma V_0 dz = -\frac{\gamma^2}{ipK} I dz,$$

and the total line integral of electric force around the contour is

$$\left( 2Z - \frac{\gamma^2}{ipK} \right) I dz. \quad \dots \quad (57)$$

The calculation of the electrostatic capacity  $K$  involves merely the solution of the two-dimensional potential problem in which the surfaces of the wires are equipotential.

We have now to calculate the magnetic flux through the surface; it is

$$\Phi = - \int_a^{c-a} H_y dr,$$

which by reference to equation (13) becomes

$$\begin{aligned} & -2dz \int_a^{c-a} dy (B_1/y - B_2/y^2 + B_3/y^3 - \dots) \\ & = -2dz \left[ B_1 \log \left( \frac{c-a}{a} \right) - B_2 \left( \frac{1}{a} - \frac{1}{c-a} \right) \right. \\ & \quad \left. + \frac{1}{2} B_3 \left( \frac{1}{a^2} - \frac{1}{(c-a)^2} \right) - \dots \right]. \end{aligned}$$

From (27) and (33) this reduces to

$$\begin{aligned} \Phi = 2Idz \left\{ 2 \log \left( \frac{c-a}{a} \right) - q_1 \left[ 1 - \frac{a}{c-a} \right] \right. \\ \left. + \frac{1}{2} q_2 \left[ 1 - \left( \frac{a}{c-a} \right)^2 \right] + \dots \right\} \quad (58) \end{aligned}$$

The law  $\text{curl } \mathbf{E} = -\mu i p \mathbf{H}$  now gives at once

$$\frac{\gamma^2}{i p K} = 2Z + i p L, \quad (59)$$

where

$$Z = Z_0 (1 - h_1 J_1/J_0 + h_2 J_2/J_0 - \dots), \quad (60)$$

$$L = L_0 \left( 1 - 2q_1 \frac{1 - \frac{k}{1-k}}{L_0} + q_2 \frac{1 - \left( \frac{k}{1-k} \right)^2}{L_0} + \dots \right), \quad (61)$$

and

$$Z_0 = 2\mu i p J_0 / \xi J_0', \quad (62)$$

$$L_0 = 4 \log \left( \frac{1-k}{k} \right). \quad (63)$$

$Z$  may therefore be regarded as the impedance of the wire, and  $L$  the inductance corresponding to the magnetic flux between the wires;  $Z_0$  and  $L_0$  are their limiting values when the parameter  $k$  is vanishingly small—that is, when the proximity effect is negligibly small. While it is convenient from this standpoint to regard  $L$  as the inductance per unit length of the circuit, it must be carefully borne in mind that both  $Z$  and  $L$  are complex. Consequently the

true resistance  $R^*$  and reactance  $X$  of the circuit are defined by the relation

$$R + iX = 2Z + ipL. \quad \dots \quad (64)$$

Having calculated  $q_1 \dots q_n$  and  $h_1 \dots h_n$  in accordance with methods discussed above, it is a straightforward process to calculate  $Z$ ,  $L$ ,  $R$ , and  $X$  from (60), ... (64), the only operations involved being the evaluation of the Bessel functions appearing in the formulas. For very low frequencies,  $Z$  and  $L$  approach the limit  $Z_0$  and  $L_0$  respectively. On the other hand, when the frequency is sufficiently high they approach upper limits corresponding to

$$q_n \sim p_n = (-1)^n 2k^n s^n$$

$$h_n J_n / J_0 \sim 2k^n s^n \quad \text{when } n \text{ is even.}$$

Consequently,

$$(1 + h_2 J_2 / J_0 + h_4 J_4 / J_0 + \dots) \sim \frac{1 + k^2 s^2}{1 - k^2 s^2} = C_m,$$

where  $C_m$  is the upper limit of the correction factor  $C$ . It may also be shown that

$$2Z + ipL \sim 2C_m Z_0 + 4ip \log \left( \frac{1}{ks} \right).$$

The limiting values of  $Z$  and  $L$  correspond to the surface distribution of currents which would exist if the wires were of infinite conductivity.

The calculation of  $Z$  and  $L$  from the foregoing formulas and tables of Bessel functions is not a difficult matter. The writer, however, hopes when time permits to prepare numerical tables and the theoretical data for the computation of  $Z$  and  $L$ , similar to those given in section III for the correction factor  $C$ . The latter function is, however, of much the greater engineering importance.

### III. FORMULÆ FOR CORRECTION FACTOR $C$ FOR NON-MAGNETIC CONDUCTORS.

#### *List of Symbols.*

- $a$  = radius of wire in cm.
- $c$  = interaxial separation between wires in cm.
- $k$  = ratio  $a/c$ .
- $\lambda$  = conductivity of wire in elm. c.g.s. units.
- $\mu$  = permeability       "       "       "
- $p$  =  $2\pi$  times frequency in cycles per second.
- $i$  =  $\sqrt{-1}$ .
- $b$  =  $a \sqrt{4\pi\lambda\mu p}$ .

\* The circuit resistance is, of course, twice that of the wire.

$J_n(bi\sqrt{i}) = u_n + iv_n$   
 = Bessel function of order  $n$  and argument  $bi\sqrt{i}$ .  
 $R$  = resistance of wire per unit length.  
 $R_0$  = „ „ „ „ with concentric return.  
 $C$  = Proximity Effect Correction Factor  
 $R = CR_0$ . . . . . (I.)

The auxiliary functions involved are :

$$R_0 = \frac{1}{a} \sqrt{\frac{p}{\pi\lambda}} \frac{u_0 v_0' - u_0' v_0}{u_1^2 + v_1^2}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$g = \frac{\sqrt{2}}{b} \frac{u_1(u_0 + v_0) - v_1(u_0 - v_0)}{u_0^2 + v_0^2}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$w_n = \frac{u_n v_n' - u_n' v_n}{u_{n-1}^2 + v_{n-1}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$s = 2 \frac{1 - \sqrt{1 - (2k)^2}}{(2k)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The formula for the correction factor  $C$  is then

$$C = 1 + \frac{2}{aR_0} \sqrt{\frac{p}{\pi\lambda}} (S_1 - 2gk^2 S_2), \quad . \quad . \quad (II.)$$

where

$$S_1 = \sum_{n=1}^{\infty} w_n k^{2n} s^{2n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$S_2 = \sum_{n=1}^{\infty} n w_n k^{2n} s^{n+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For values of the argument  $b \geq 5$ ,

$$C \doteq 1 + 2(\sqrt{2} - 1/b) \left( S_1 - \frac{2\sqrt{2}}{b} k^2 S_2 \right). \quad . \quad . \quad (III.)$$

For larger values of the argument  $b$ , the correction factor  $C$  approaches an upper limit

$$C_m = \frac{1 + k^2 s^2}{1 - k^2 s^2} \quad . \quad . \quad . \quad . \quad . \quad (IV.)$$

in accordance with the asymptotic formula

$$C \sim C_m (1 - A/b), \quad . \quad . \quad . \quad . \quad . \quad (V.)$$

where

$$A = 2\sqrt{2} \frac{k^2 s^2}{1 - k^4 s^4} \left[ 1 + 2k^2 \left( \frac{1 - k^2 s^2}{1 - k^2 s^2} \right)^2 \right]. \quad . \quad . \quad (7)$$

Fig. 1.

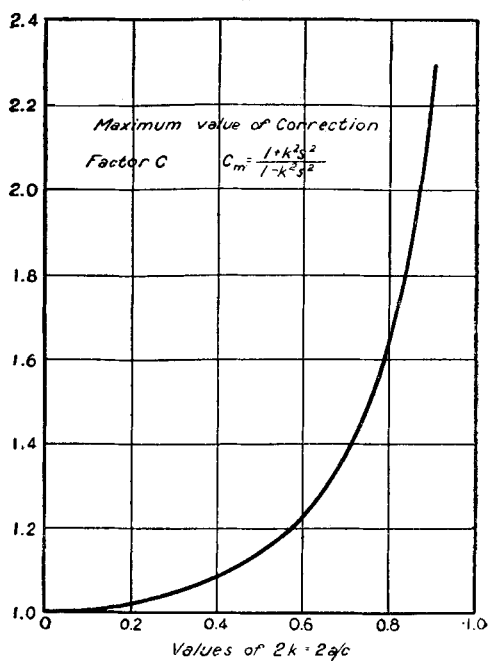
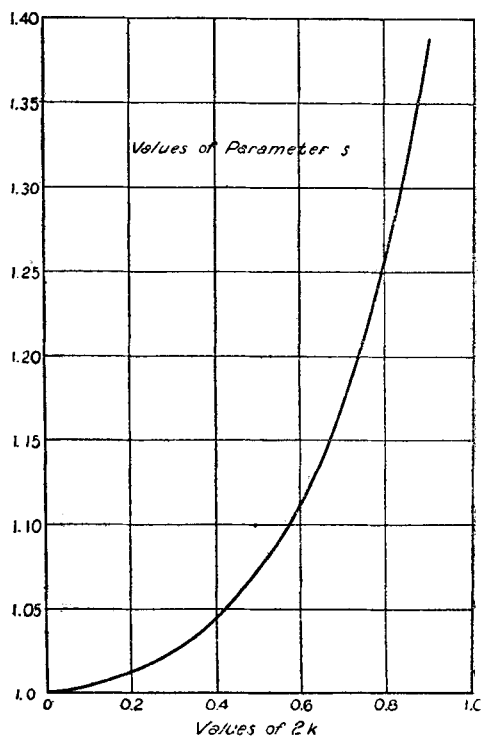


Fig. 2.

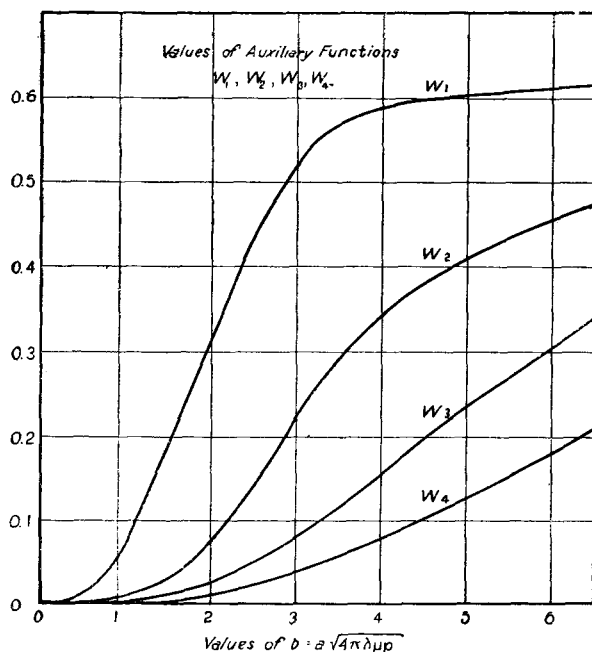




The correction factor  $C$  may be calculated from the asymptotic formula V instead of II or III under the following condition and in accordance with the following rule:—If the series  $1+k^2s^2+k^4s^4+\dots$  converges to a required order of approximation in a finite number of terms  $n$ , then formula V may be employed, provided that  $b \geq n^2 \geq 5$ .

The auxiliary functions involved in the foregoing formulæ have been computed and are plotted in the accompanying curves, the accuracy of which is believed to be sufficient for

Fig. 3.



all engineering purposes. An example of their use will now be given in calculating the correction factor  $C$  for the following representative case :  $b=5$  and  $2k=2a/c=0.75$ .

From the curves of fig. 3,

$$w_1 = 0.605$$

$$w_2 = 0.410$$

$$w_3 = 0.235$$

$$w_4 = 0.130.$$

Fig. 4.

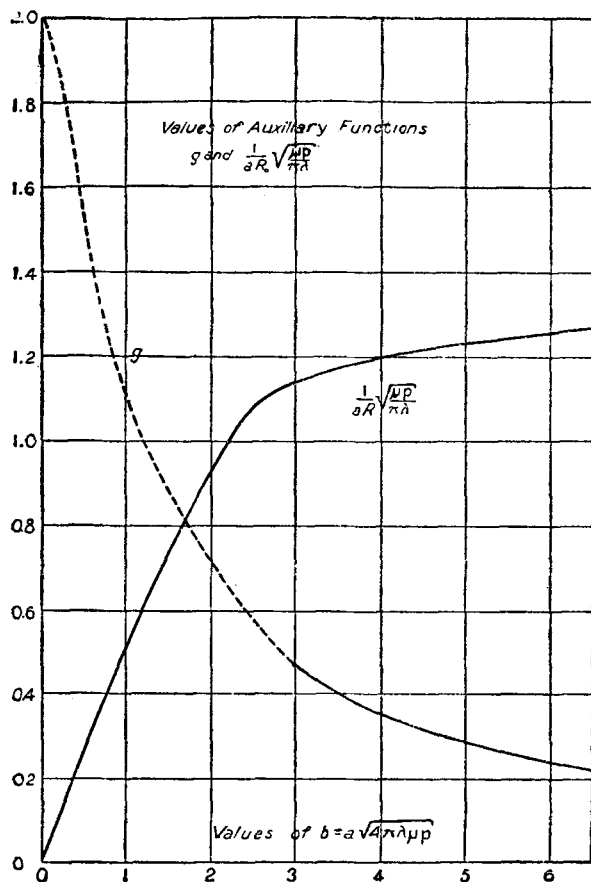
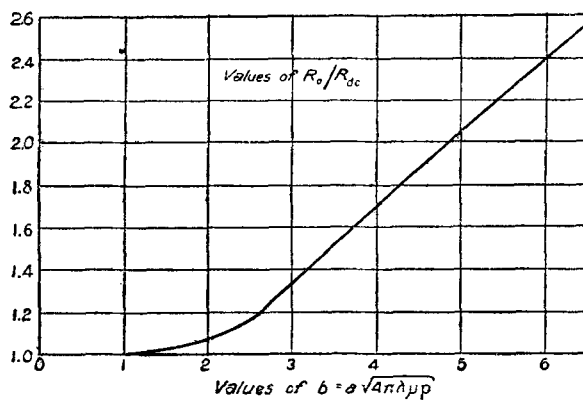


Fig. 5.



From the curves \* of fig. 4,

$$g = 0.285 \quad \text{and} \quad \frac{1}{aR_0} \sqrt{\frac{p}{\pi\lambda}} = 1.225.$$

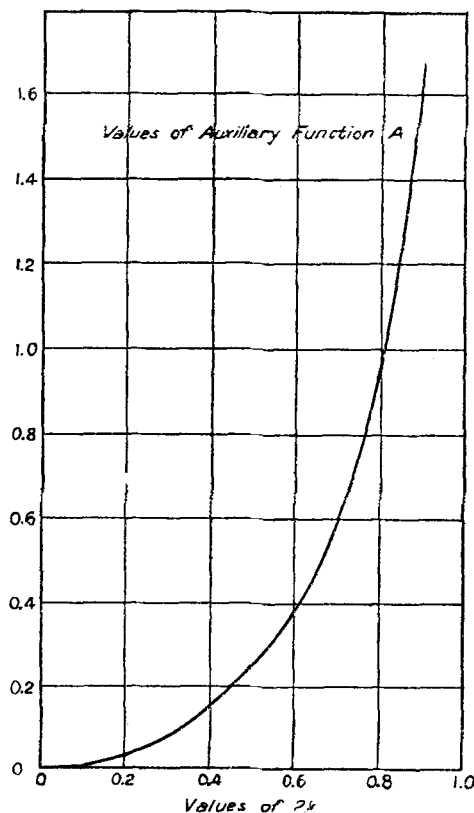
From the curve of fig. 2,

$$s = 1.20.$$

Consequently,

$w_1 k^2 s^2 = .1225$	$w_1 k^2 s^2 = .1225$
$w_2 k^4 s^4 = .01681$	$2w_2 k^4 s^4 = .02801$
$w_1 k^6 s^6 = .001952$	$3w_3 k^6 s^6 = .004065$
$w_4 k^8 s^8 = .0002187$	$4w_4 k^8 s^8 = .0005058$
$S_1 = .1415$	$S_2 = .1551$

Fig. 6.



\* The first four orders of  $w_n$  are sufficient for practical purposes. Higher orders can be calculated from the Bessel function recurrence formulæ.

Substitution in formula II gives

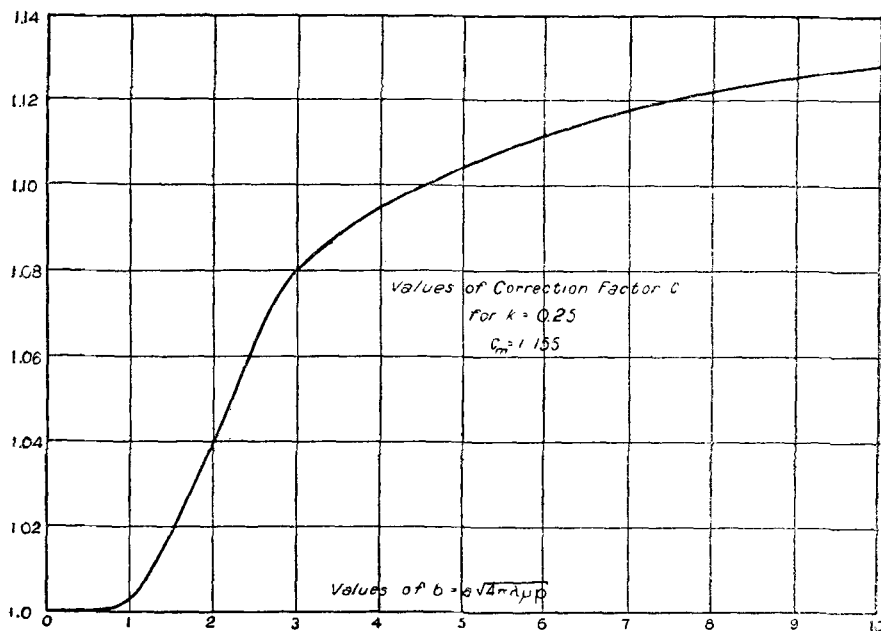
$$\begin{aligned} C &= 1 + 2.45 (.1415 - .01223) \\ &= 1.317. \end{aligned}$$

From the curve of fig. 1 the maximum value  $C_m$  of the correction factor for this case is 1.51, and from the curve of fig. 6 the value of the auxiliary function  $A$  is 0.76. The asymptotic formula V therefore gives :

$$C \sim 1.51 (1 - 0.76/5) = 1.28.$$

As would be expected from the magnitudes involved, the asymptotic formula therefore gives a result which is in error by a small amount.

Fig. 7.



To give an idea of the variation of the correction factor  $C$  with frequency, it has been computed for  $k=0.25$  and plotted in the curve of fig. 7 as a function of the argument  $b$ . This case is of practical interest, since the corresponding conductor spacing is that of telephone cable circuits. For this ratio of  $a/c$  the value of  $C_m=1.55$ ; this limiting value

is however approached quite slowly, as is evident from an inspection of fig. 7.

To facilitate the calculation of  $R_0$ , the ratio  $R_0/R_{dc}$  has been plotted in fig. 5.  $R_d$  denotes, of course, the d.c. resistance of the wire. For values of the argument  $b \geq 5$ ,

$$R_0/R_{dc} \sim b/(2\sqrt{2}-2/b),$$

which makes the calculation very simple.

April 17th, 1920.

LIV. *The Separation of Miscible Liquids by Distillation.*  
By A. F. DUFFON, B.A., *Frecheville Research Fellow,*  
*Royal School of Mines* \*.

1. **I**N the search for a perfect apparatus for the separation of mixtures by distillation the greatest advances have been made in industrial practice. M. Sorel, one of the leading French authorities upon the distillation and the rectification of alcohol, in reviewing the principles underlying the construction of stills, writes:—

“La plus grande partie des données dont nous avons besoin peuvent être déterminées dans la laboratoire du physicien. Malheureusement bien peu de savants s’en sont occupés, soit que le sujet leur parût peu important, soit que l’impossibilité jusqu’ici reconnue d’arriver à des lois mathématiques les ait rebutés. Il faut donc que le constructeur se transforme en expérimentateur...”

“C’est l’aveu franc et net,” to quote M. Chenard †, “d’un empirisme certain.”

The extent to which laboratory practice has been outstripped may be seen by comparing the still designed by Coffey ‡, of Dublin, in 1832, or Derosne’s still, which give in continuous distillation on the large scale the strongest spirit which can be obtained, with the various laboratory still-heads examined by Dr. Young § in 1899. The only continuous laboratory still appears to be an experimental one devised by Lord Rayleigh ||, which consisted of a long length (12 metres) of copper tubing 15 mm. in diameter, and a similar one described by Carveth ¶.

\* Communicated by Sir E. Rutherford, F.R.S.

† Chenard, *Bulletin de l’Association des Chimistes de Sucrierie et de Distillerie de France*, 1915.

‡ ‘Chemistry as applied to the Arts and Manufactures.’ Vol. I. Alcohol.

§ Young, *Journ. Chem. Soc.* 1892, p. 679.

|| Rayleigh, *Phil. Mag.* (4) 1902, p. 536.

¶ Carveth, *J. Phys. Chem.* vi. p. 253 (1902).